Identification and Estimation of Dynamic Games when Players' Beliefs Are Not in Equilibrium

Victor Aguirregabiria*
University of Toronto and CEPR

Arvind Magesan*
University of Calgary

This version: January 20, 2016

Abstract

This paper deals with the identification and estimation of dynamic games when players' beliefs about other players' actions are biased, i.e., beliefs do not represent the probability distribution of the actual behavior of other players conditional on the information available. First, we show that a exclusion restriction, typically used to identify empirical games, provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs. Second, we prove that this exclusion restriction, together with consistent estimates of beliefs at several points in the support of the *special* state variable (i.e., the variable involved in the exclusion restriction), is sufficient for nonparametric point-identification of players' payoff and belief functions. The consistent estimates of beliefs at some points of support may come either from an assumption of unbiased beliefs at these points in the state space, or from available data on elicited beliefs for some values of the state variables. Third, we propose a simple two-step estimation method. We illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains. The key conditions for the identification of beliefs and payoffs in our application are the following: (a) the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit because the existence of sunk entry costs and economies of density in these costs; and (b) firms' beliefs are unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores, though beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs. Furthermore, imposing the restriction of unbiased beliefs generates a substantial attenuation bias in the estimate of competition effects.

Keywords: Dynamic games; Rational behavior; Rationalizability; Identification; Estimation; Market entry-exit.

Corresponding Author: Victor Aguirregabiria. Department of Economics. University of Toronto. 150 St. George Street. Toronto, Ontatio. victor.aguirregabiria@utoronto.ca

Arvind Magesan. Department of Economics. University of Calgary. 2500 University Drive, N.W. Calgary, Alberta arvindmagesan@gmail.com

*We would like to thank comments from Andres Aradillas-Lopez, John Asker, Chris Auld, Dan Bernhardt, Teck Hua Ho, Pedro Mira, Stephen Morris, Sridhar Moorthy, Stephen J. Redding, Philipp Schmidt-Dengler, Katsumi Shimotsu, Otto Toivanen, Jean-Francois Wen and participants in seminars and conferences at Calgary, Carlos III-Madrid, New York University, Penn State, Princeton, Urbana-Champaign, CEPR IO conference in Tel-Aviv, the Canadian Econometric Study Group, and the Econometric Society World Congress. The first author thanks the Social Science and Humanities Research Council of Canada (SSHRCC) for financial support.

1 Introduction

The principle of *revealed preference* (Samuelson, 1938) is a cornerstone in the empirical analysis of decision models, either static or dynamic, single-agent problems or games. Under the principle of *revealed preference*, agents maximize expected payoffs and their actions reveal information on the structure of payoff functions. This simple but powerful concept has allowed econometricians to use data on agents' decisions to identify important structural parameters for which there is very limited information from other sources. Examples of parameters and functions that have been estimated using the principle of revealed preference include, among others, consumer willingness to pay for a product, agents' degree of risk aversion, intertemporal rates of substitution, market entry costs, adjustment costs and switching costs, preference for a political party, or the benefits of a merger. In the context of empirical games, where players' expected payoffs depend on their beliefs about the behavior of other players, most applications combine the principle of revealed preference with the assumption that players' beliefs about the behavior of other players are *in equilibrium*, in the sense that these beliefs represent the probability distribution of the actual behavior of other players conditional on the information available. The assumption of equilibrium beliefs plays an important role in the identification and estimation of games, and as such, is a mainstay in the empirical game literature. Equilibrium restrictions have identification power even in models with multiple equilibria (Tamer, 2003, Aradillas-Lopez and Tamer, 2008, Bajari, Hong, and Ryan, 2010). Imposing these restrictions contributes to improved asymptotic and finite sample properties of game estimators. Moreover, the assumption of equilibrium beliefs is very useful for evaluating counterfactual policies in a strategic environment. Models where agents' beliefs are endogenously determined in equilibrium not only take into account the direct effect of the new policy on agents' behavior through their payoff functions, but also through an endogenous change in agents' beliefs.

Despite the clear benefit that the assumption of equilibrium beliefs delivers to an applied researcher, there are situations and empirical applications where the assumption is not realistic and it is of interest to relax it. There are multiple reasons why players may have biased beliefs about the behavior of other players in a game. For instance, in games with multiple equilibria, players can be perfectly rational in the sense that they take actions to maximize expected payoffs given their beliefs, but they may have different beliefs about the equilibrium that has been selected. This situation corresponds to the concept of *strategic uncertainty* as defined in Van Huyk, Battalio, and Beil (1990) and Crawford and Haller (1990), and applied by Morris and Shin (2002, 2004), and Heinemann, Nagel, and Ockenfels (2009), among others. For instance, competition in oligopoly industries is often prone to strategic uncertainty (Besanko et al., 2010). Dynamic games of oligopoly competition are typically characterized by multiple equilibria, and the selection between two possible equilibria implies that some firms are better off but others are worse off. Firm managers do not have incentives to coordinate their beliefs in the same equilibrium. They can be very secretive about their own strategies and face significant uncertainty about the strategies of their competitors.[1] Strategic uncertainty may also be an important consideration in the evaluation of a policy change in a strategic environment. Suppose that to evaluate a policy change we estimate an empirical game using data before and after a new policy is implemented. After the implementation of the new policy, some players may believe that others' market behavior will continue according to the same type of equilibrium as before the policy change, while others believe the policy change has triggered the selection of a different type of equilibrium.[2] Thus, at least for some period of time, players' beliefs will be out of equilibrium, and imposing the restriction of equilibrium beliefs may bias the estimates of the effects of the new policy. Another example comes from the structural estimation of games using data generated by laboratory experiments. Studies in the literature of experimental games commonly find significant heterogeneity in players' elicited beliefs, and that this heterogeneity is often one of the most important factors in explaining heterogeneity in observed behavior in the laboratory.[3] Imposing the assumption of equilibrium beliefs in these applications does not seem reasonable. Interestingly, however, recent empirical papers establish a significant divergence between stated or elicited beliefs and the beliefs inferred from players' actions using, for example, revealed preference-based methods (see Costa-Gomes and Weizsäcker, 2008, and Rutström and Wilcox, 2009). The results in our paper can be applied to estimate beliefs and payoffs, using either observational or laboratory data, when the researcher wants to allow for the possibility of biased beliefs but she does not have data on elicited beliefs, or data on elicited beliefs is limited to only a few states of the world.

In this paper we study nonparametric identification, estimation, and inference in dynamic discrete games of incomplete information when we assume that players are rational, in the sense that each player takes an action that maximizes his expected payoff given some beliefs, but we relax the assumption that these beliefs are in equilibrium. In the class of models that we consider, a player's belief is a probability distribution over the space of other players' actions conditional on some state variables, or the player's information set. Beliefs are biased, or not in equilibrium, if they are different from the actual probability distribution of other players' actions conditional on the state variables of the model. We consider a nonparametric specification of beliefs and treat these probability distributions as *incidental parameters* that, together with the structural parameters in payoff functions and transition probabilities, determine the stochastic process followed by players' actions. Our framework includes as a particular case games where the source of biased beliefs is strategic uncertainty, i.e., every player has beliefs that correspond to an equilibrium of the game

[1] See Morris and Shin (2002) for examples of models with strategic uncertainty and related experimental evidence.

[2] For example in a game of investment there may be high investment or low investment equilibria. Prior to the policy change the game may be in the high investment equilibrium, and after the policy change one player believes that this equilibrium will continue to prevail while the other player switches to behavior according to the low equilibrium.

[3] See Camerer (2003) and recent papers by Costa-Gomes and Weizsäcker (2008), and Palfrey and Wang (2009).

but their beliefs are not 'coordinated'. However, our identification and estimation results do not rely on this restriction and our approach is therefore not restricted to this case.

The recent literature on identification of games of incomplete information is based on two main assumptions: (i) players' beliefs are in equilibrium such that they can be identified, or consistently estimated, by simply using a nonparametric estimator of the distribution of players' actions conditional on the state variables; and (ii) there is an exclusion restriction in the payoff function such that there is a player-specific state variable which enters the payoff of the player and is excluded from the payoffs of other players, but is known to other players and thus influences their beliefs (Bajari, Hong, and Ryan, 2010, Bajari et al., 2010). When players beliefs are not in equilibrium, or when the exclusion restriction is not satisfied, the model is not identified. In this context, this paper presents two main identification results. First, we show that the exclusion restriction alone provides testable nonparametric restrictions of the null hypothesis of equilibrium beliefs. Under this type of exclusion restriction, the observed behavior of a player can identify a function that depends only on her beliefs about the behavior of other player, and not on her preferences. Under the null hypothesis of equilibrium beliefs, this identified function of beliefs should be equal to the same function but where we replace beliefs by the actual expected behavior of the other player. Second, we prove that this exclusion restriction, together with consistent estimates of beliefs at two points in the support of the *special* state variable (i.e., the variable that satisfies the exclusion restriction), is sufficient for nonparametric point-identification of players' payoff and belief functions. This is the core result of our paper, and also serves to highlight the fact that empirical games are typically over-identified, as they assume consistent beliefs at every point in the support. The consistent estimates of beliefs at two points of the support may come either from an assumption of unbiased beliefs at these points in the state space, or from data on elicited beliefs for some values of the state variables. We also discuss three different approaches to select the values of the special state variable where we impose the restriction of unbiased beliefs: (a) minimization of beliefs bias; (b) most visited states; and (c) testing for the monotonicity of beliefs and using this restriction. Third, we propose a simple two-step estimation method. The two-step method has an analogy to instrumental variables estimation in regression models, and we use this analogy to discuss the identification power of our restrictions and the potential problem of weak instruments in applications. Finally, we illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by retail chains.

We use Monte Carlo experiments to further understand the trade-off a researcher faces when deciding whether or not to impose the assumption of equilibrium beliefs in an application, and how this trade off depends on properties of the underlying data generating process. We find that while there is a loss in precision when we drop the assumption of equilibrium beliefs, the estimates are still accurate enough to provide meaningful results. We provide evidence on how identification of payoffs and beliefs is affected by how useful the “special" excluded variable is as an instrument. This variable acts to shift one player's payoffs exogenously, while only affecting the other player's behavior through his beliefs. As the quality of this instrument improves, the mean-squared error of estimates is significantly reduced. We also find that the bias associated with incorrectly assuming equilibrium beliefs is very significant. Estimates of payoff parameters are in some cases biased by over 60% of their true value.

To illustrate our model and methods in the context of an empirical application, we consider a dynamic game of store location between McDonalds and Burger King. There has been very little work on the bounded rationality of firms, as most empirical studies on bounded rationality have concentrated on individual behavior.[4] The key conditions for the identification of beliefs and payoffs in our application are the following. The first condition is an exclusion restriction in a firm's profit function that establishes that the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit because of the existence of sunk entry costs and economies of density in these costs. The second condition restricts firms' beliefs to be unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores. However, beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs for Burger King. More specifically, we find that this firm underestimated the probability of entry of McDonalds in markets that were relatively far away from McDonalds' network of stores. Furthermore, imposing the restriction of unbiased beliefs generates a substantial attenuation bias in the estimate of competition effects.

This paper builds on the recent literature on estimation of dynamic games of incomplete information (see Aguirregabiria and Mira, 2007, Bajari, Benkard and Levin, 2007, Pakes, Ostrovsky and Berry, 2007, and Pesendorfer and Schmidt-Dengler, 2008). All the papers in this literature assume that the data come from a Markov Perfect Equilibrium. We relax that assumption. Our research also builds upon the work of Aradillas-Lopez and Tamer (2008) who study the identification power of the assumption of equilibrium beliefs in simple static games using the notion of level-k rationalizability to construct informative bounds around beliefs.[5] Our approach is different in several ways. First, we study dynamic games, including static games as a particular case. The implications of dropping the assumption of equilibrium beliefs in dynamic games, with respect to identification in particular, are quite different from those of static games. As we show in this paper, the characterization and derivation of bounds on choice probabilities in dynamic games is significantly more complicated, and the key identification results in Aradillas-Lopez and Tamer

[4] An exception is the recent paper by Goldfarb and Xiao (2011) that studies entry decisions in the US local telephone industry and finds significant heterogeneity in firms' beliefs about other firms' strategic behavior.

[5] Other papers on the estimation of static games under rationalizability are Kline and Tamer (2012), Uetake and Watanabe (2013), An (2010), and Gillen (2010).

cannot be directly extended. Therefore, we propose a different approach that does not rely on the notion of level-k rationalizability. Instead, we concentrate on level-1 rationalizability and develop methods for nonparametric point identification and estimation of preferences and beliefs.[6] Second, while their study is focused primarily on identification, we propose and implement new tests and estimators and study their properties. And third, they consider identification of parametrically specified models, while our point of departure is nonparametric identification of payoffs and beliefs.

Our paper also complements the growing literature on the use of data on subjective expectations in microeconometric decision models, especially the contributions of Walker (2003), Manski (2004), Delavande (2008), and Van der Klaauw and Wolpin (2008). It is commonly the case that data on elicited beliefs has the form of unconditional probabilities, or probabilities that are conditional only on a strict subset of the state variables in the postulated model. In this context, the framework that we propose in this paper can be combined with the incomplete data on elicited beliefs in order to obtain nonparametric estimates of the complete conditional probability distribution describing an individual's beliefs. Most of these previous empirical papers on biased beliefs consider dynamic single-agent models and beliefs about exogenous future events. We extend that literature by looking at dynamic games and biased beliefs about other players' behavior.

The rest of the paper includes the following sections. Section 2 presents the model and basic assumptions. In section 3, we present our identification results. Section 4 describes estimation methods and testing procedures. Section 5 presents our Monte Carlo experiments. The empirical application is described in section 6. We summarize and conclude in section 7.

2 Model

2.1 Basic framework

This section presents a dynamic game of incomplete information where N players make discrete choices over T periods. We use indexes $i, j \in \{1, 2, ...N\}$ to represent players, and the index $-i$ to represent all players other than i. Time is finite and discrete, and is indexed by $t \in \{1, 2, ..., T\}$. Every period t, players choose simultaneously one out of A alternatives from the choice set $\mathcal{Y} = \{0, 1, ..., A-1\}$. Let $Y_{it} \in \mathcal{Y}$ represent the choice of player i at period t. Each player makes this decision to maximize his expected and discounted payoff, $\mathbb{E}_t(\sum_{s=0}^{T} \beta^s \Pi_{i,t+s})$, where $\beta \in (0, 1)$ is the discount factor, and Π_{it} is his payoff at period t. The one-period payoff function has the following

[6] In relaxing the assumption of Nash equilibrium, they assume that players are *level-k rational* with respect to their beliefs about their opponents' behavior, a concept which derives from the notion of rationalizability (Bernheim, 1984, and Pearce, 1984). Their approach is especially useful in the context of static games with binary or ordered decision variables, as, under the condition that players' payoffs are monotone in the decision of their opponents, it yields a sequence of closed form bounds on players' beliefs that grow tighter as the level of rationality k gets larger. Unfortunately, in the case of dynamic games, the assumptions of Aradillas-Lopez and Tamer do not yield a representation of bounds on players' beliefs that is practical to implement, even for simple dynamic games.

structure:

$$\Pi_{it} = \pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) \tag{1}$$

$\pi_{it}(.)$ is a real-valued function. $\mathbf{Y}_{-it}$ represents the current action of the other players. $\mathbf{X}_t$ is a vector of state variables which are common knowledge for all players. $\boldsymbol{\varepsilon}_{it} \equiv (\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(A))$ is a vector of private information variables for firm i at period t, and we assume that $\boldsymbol{\varepsilon}_{it}$ is independent across i and t.

The vector of common knowledge state variables is $\mathbf{X}_t$, and it evolves over time according to the transition probability function $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$ where $\mathbf{Y}_t \equiv (Y_{1t}, Y_{2t}, ..., Y_{Nt})$. The vector of private information shocks $\boldsymbol{\varepsilon}_{it}$ is independent of $\mathbf{X}_t$ and independently distributed over time and players. Without loss of generality, these private information shocks have zero mean. The distribution function of $\boldsymbol{\varepsilon}_{it}$ is given by G_{it}, which is absolutely continuous and strictly increasing with respect to the Lebesgue measure on $\mathbb{R}^A$.

EXAMPLE 1: Dynamic game of market entry and exit. Consider N firms competing in a market. Each firm sells a differentiated product. Every period, firms decide whether or not to be active in the market. Then, incumbent firms compete in prices. Let $Y_{it} \in \{0, 1\}$ represent the decision of firm i to be active in the market at period t. The profit of firm i at period t has the structure of equation (1), $\Pi_{it} = \pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it})$. We now describe the specific form of the payoff function π_{it} and the state variables $\mathbf{X}_t$ and ε_{it}. The average profit of an inactive firm, $\pi_{it}(0, \mathbf{Y}_{-it}, \mathbf{X}_t)$, is normalized to zero, such that $\Pi_{it} = \varepsilon_{it}(0)$. The profit of an active firm is $\pi_{it}(1, \mathbf{Y}_{-it}, \mathbf{X}_t) + \varepsilon_{it}(1)$ where:

$$\pi_{it}(1, \mathbf{Y}_{-it}, \mathbf{X}_t) \quad = \quad H_t \left(\theta_i^M - \theta_i^D \sum_{j \neq i} Y_{jt} \right) - \theta_{i0}^{FC} - \theta_{i1}^{FC} Z_i - (1 - Y_{it-1}) \theta_i^{EC} \tag{2}$$

The term $H_t \left(\theta_i^M - \theta_i^D \sum_{j \neq i} Y_{jt} \right)$ is the variable profit of firm i. H_t represents market size (e.g., market population) and it is an exogenous state variable. θ_i^M is a parameter that represents the per capita variable profit of firm i when the firm is a monopolist. The parameter θ_i^D captures the effect of the number of competing firms on the profit of firm i.[7] The term $\theta_{i0}^{FC} + \theta_{i1}^{FC}\ Z_i$ is the fixed cost of firm i, where θ_{i0}^{FC} and θ_{i1}^{FC} are parameters, and Z_i is an exogenous, time-invariant, firm characteristic affecting the fixed cost of the firm. The term $1\{Y_{it-1} = 0\}\ \theta_i^{EC}$ represents sunk entry costs, where $1\{.\}$ is the binary indicator function and θ_i^{EC} is a parameter. Entry costs are paid only if the firm was not active in the market at previous period. The vector of common knowledge state variables of the game is $\mathbf{X}_t = (H_t, Z_i, Y_{it-1} : i = 1, 2, ..., N)$. ■

Most previous literature on estimation of dynamic discrete games assumes that the data comes

[7] A more flexible specification allows for each firm j to have a different impact on the variable profit of firm i, i.e., $H_t \left(\theta_i^M - \sum_{j \neq i} \theta_{ij}^D\ Y_{jt} \right)$.

from a Markov Perfect Equilibrium (MPE). This equilibrium concept incorporates four main assumptions.

ASSUMPTION MOD-1 (Payoff relevant state variables): Players' strategy functions depend only on payoff relevant state variables: $\mathbf{X}_t$ *and* $\boldsymbol{\varepsilon}_{it}$*. Also, a player's belief about the strategy of other player is a function only of the payoff relevant state variables of the other player.*

ASSUMPTION MOD-2 (Maximization of expected payoffs): Players are forward looking and maximize expected intertemporal payoffs.

ASSUMPTION MOD-3 (Unbiased beliefs on own future behavior): A player's beliefs about his own actions in the future are unbiased expectations of his actual actions in the future.

ASSUMPTION 'EQUIL' (Unbiased or equilibrium beliefs on other players' behavior): Strategy functions are common knowledge, and players' have rational expectations on the current and future behavior of other players. That is, players' beliefs about other players' actions are unbiased expectations of the actual actions of other players.

First, let us examine the implications of imposing only Assumption MOD-1.[8] The payoff-relevant information set of player i is $\{\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}\}$. The space of $\mathbf{X}_t$ is $\mathcal{X}$. At period t, players observe $\mathbf{X}_t$ and choose their respective actions. Let the function $\sigma_{it}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}) : \mathcal{X} \times \mathbb{R}^A \to \mathcal{Y}$ represent a strategy function for player i at period t. Given any strategy function σ_{it}, we can define a choice probability function $P_{it}(y|\mathbf{X}_t)$ that represents the probability that $Y_{it} = y$ conditional on $\mathbf{X}_t$ given that player i follows strategy σ_{it}. That is,

$$P_{it}(y|\mathbf{X}_t) \equiv \int 1\left\{\sigma_{it}\left(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}\right) = y\right\} \ dG_{it}\left(\boldsymbol{\varepsilon}_{it}\right) \tag{3}$$

It is convenient to represent players' behavior using these *Conditional Choice Probability* (CCP) functions. When the variables in $\mathbf{X}_t$ have a discrete support, we can represent the CCP function $P_{it}(.)$ using a finite-dimensional vector $\mathbf{P}_{it} \equiv \{P_{it}(y|\mathbf{X}_t) : y \in \mathcal{Y},\, \mathbf{X}_t \in \mathcal{X}\} \in [0,1]^{A|\mathcal{X}|}$. Throughout the paper we use either the function $P_{it}(.)$ or the vector $\mathbf{P}_{it}$ to represent the *actual behavior* of player i at period t.

Without imposing Assumption 'Equil' ('Equilibrium Beliefs'), a player's beliefs about the behavior of other players do not necessarily represent the actual behavior of the other players. Therefore, we need functions other than $\sigma_{jt}(.)$ and $P_{jt}(.)$ to represent players i's beliefs about the strategy of other players. Let $b_{it}^{(t_0)}\left(\mathbf{X}_t, \boldsymbol{\varepsilon}_{-it}\right)$ be a function from $\mathcal{X} \times \mathbb{R}^{(N-1)A}$ into $\mathcal{Y}^{N-1}$ that represents player i's belief at period t_0 about the strategy function of all the other players at period t. In principle, this function may vary with t_0 due to players' learning and forgetting, or to other factors

[8] Fershtman and Pakes (2012) study dynamic games where private information is serially correlated. In this context, the concept of Markov Perfect equilibrium implies that the whole past history of players' decisions is payoff relevant. They propose a framework and a new equilibrium concept (Experience-based equilibrium) to deal with this dimensionality problem.

that cause players' beliefs to change over time. Let $B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X}_t)$ be the choice probability associated with $b_{it}^{(t_0)}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{-it})$, i.e., $B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X}_t) \equiv \int 1\{b_{it}^{(t_0)}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{-it}) = \mathbf{y}_{-i}\}\ dG_{-it}(\boldsymbol{\varepsilon}_{-it})$. When $\mathcal{X}$ is a discrete and finite space, we can represent function $B_{it}^{(t_0)}(.)$ using a finite-dimensional vector $\mathbf{B}_{it}^{(t_0)} \equiv \{B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X}) : \mathbf{y}_{-i} \in \mathcal{Y}^{N-1}, \mathbf{X} \in \mathcal{X}\} \in [0,1]^{A^{N-1}|\mathcal{X}|}$. Using this notation, Assumption 'Equil' can be represented in vector form as $\mathbf{B}_{it}^{(t_0)} = \Pi_{j\neq i}\mathbf{P}_{jt}$ for every player i, every t_0, and $t \geq t_0$.

The following assumption replaces the assumption of 'Equilibrium Beliefs' and summarizes our minimum conditions on players' beliefs.

ASSUMPTION MOD-4: A player's belief function $B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X})$ satisfies the following conditions.

> *(A) It is common knowledge that players' private information $\boldsymbol{\varepsilon}_{it}$ is independently distributed across players. This condition implies that a player's beliefs should satisfy the restriction that other players' actions are independent conditional on common knowledge state variables: $B_{it}^{(t_0)}(\mathbf{y}_{-i}|\mathbf{X}) = \Pi_{j\neq i}\ B_{ijt}^{(t_0)}(y_j|\mathbf{X})$, where $B_{ijt}^{(t_0)}(y_j|\mathbf{X})$ represents the beliefs of player i on the behavior of player j.*
>
> *(B) The belief function $B_{ijt}^{(t_0)}$ may vary, in a non-restricted form, over the time period of the opponent's behavior, t, but it is not revised or updated over t_0, i.e., $B_{ijt}^{(t_0)} = B_{ijt}$ for any period $t_0 \leq t$.*

Assumption MOD-4(A) can be seen as natural implication of Assumption MOD-1 and the assumption that private information variables are independent across players. If a player knows that other players' strategy functions depend only on payoff relevant state variables $\mathbf{X}_t$ and $\boldsymbol{\varepsilon}_{it}$ (i.e., Assumption MOD-1) and that private information variables $\boldsymbol{\varepsilon}_{it}$ are independent across players, then this player's beliefs should satisfy the independence condition $B_{it}(\mathbf{y}_{-i}|\mathbf{X}) = \Pi_{j\neq i}\ B_{ijt}(y_j|\mathbf{X})$. This assumption reduces substantially the dimension of the beliefs function in games with more than two players. For a given player and a given value of $\mathbf{X}$, the number of free parameters in beliefs becomes $(N-1)(A-1)$ instead of a number $A^{N-1}-1$ of parameters in the model without Assumption MOD-4(A).[9]

Assumption MOD-4(B) imposes restrictions on the time pattern of beliefs. Using Table 1, we can describe this assumption by saying that beliefs are constant across rows. This implies that each player believes his opponents' behavior may change over time, but beliefs about opponents' behavior at a given period are constant over the entire game and they are not revised as time goes by. Therefore, we do not allow for updating of beliefs. While this assumption seems restrictive, it

[9] Assumptions MOD-1 and MOD-4(A) establish independence between players' private information in a single market. While the model may potentially have multiple equilibria, which is a source of biased beliefs, coordination on an equilibrium can not generate correlation in actions because behavior is conditional on the equilibrium being played. In other words, if players are playing the same equilibrium, once we condition on that equilibrium and the state variables, their actions remain independent. When we look at data from multiple markets, players' actions and beliefs can be correlated across markets. Our model with unobserved market-specific heterogeneity (section 3.2.8) allows for this correlation."

keeps the identification problem tractable. Note that rational or unbiased beliefs also imply this restriction.

ASSUMPTION MOD-5: The state space $\mathcal{X}$ is discrete and finite, and $|\mathcal{X}|$ represents its dimension or number of elements.

For the rest of the paper, we maintain Assumptions MOD-1 to MOD-5 but we do not impose the restriction of 'Equilibrium Beliefs'. We assume that players are 'rational', in the sense that they maximize expected and discounted payoff given their beliefs on other players' behavior. Our approach is agnostic about the formation of players' beliefs.

2.2 Best response mappings

We say that a strategy function σ_{it} (and the associated CCP function P_{it}) is *rational* if for every possible value of $(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}) \in \mathcal{X} \times \mathbb{R}^A$ the action $\sigma_{it}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it})$ maximizes player i's expected and discounted value given his beliefs on the opponent's strategy. Given his beliefs, player i's best response at period t is the optimal solution of a single-agent dynamic programming (DP) problem. This DP problem can be described in terms of: (i) a discount factor, β; (ii) a sequence of expected one-period payoff functions, $\{\pi_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) : t = 1, 2, ..., T\}$, where

$$\pi_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) \equiv \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} \pi_{it}(Y_{it}, \mathbf{y}_{-i}, \mathbf{X}_t) \; B_{it}(\mathbf{y}_{-i}|\mathbf{X}_t) \; ; \tag{4}$$

and (iii) a sequence of transition probability functions $\{f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) : t = 1, 2, ..., T\}$, where

$$f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) \quad = \quad \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} f_t(\mathbf{X}_{t+1}|Y_{it}, y_{-i}, \mathbf{X}_t) \; B_{it}(\mathbf{y}_{-i}|\mathbf{X}_t) \tag{5}$$

Let $V_{it}^{\mathbf{B}}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it})$ be the value function for player i's DP problem given his beliefs. By Bellman's principle, the sequence of value functions $\{V_{it}^{\mathbf{B}} : t = 1, 2, ..., T\}$ can be obtained recursively using backwards induction in the following *Bellman equation*:

$$V_{it}^{\mathbf{B}}(\mathbf{X}_t, \boldsymbol{\varepsilon}_{it}) \quad = \quad \max_{Y_{it} \in \mathcal{Y}} \left\{ \; v_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) + \varepsilon_{it}(Y_{it}) \; \right\} \tag{6}$$

where $v_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t)$ is the *conditional choice value function*

$$v_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) \quad \equiv \quad \pi_{it}^{\mathbf{B}}(Y_{it}, \mathbf{X}_t) + \beta \sum_{\mathbf{X}_{t+1}} \int V_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1}, \boldsymbol{\varepsilon}_{it+1}) \; dG_{it}(\boldsymbol{\varepsilon}_{it+1}) \; f_{it}^{\mathbf{B}}(\mathbf{X}_{t+1}|Y_{it}, \mathbf{X}_t) \tag{7}$$

Given his beliefs, the best response function of player i at period t is the optimal decision rule of this DP problem. This best response function can be represented using the following threshold condition:

$$\{Y_{it} = y\} \text{ iff } \left\{\varepsilon_{it}(y') - \varepsilon_{it}(y) \leq v_{it}^{\mathbf{B}}(y, \mathbf{X}_t) - v_{it}^{\mathbf{B}}(y', \mathbf{X}_t) \;\; \text{for any } y' \neq y\right\} \tag{8}$$

The *best response probability* (*BRP*) *function* is a probabilistic representation of the best response function. More precisely, it is the best response function integrated over the distribution of $\boldsymbol{\varepsilon}_{it}$. In this model, the *BRP* function is:

$$\begin{aligned}\Pr(Y_{it} = y|\mathbf{X}_t) &= \int 1\left\{\varepsilon_{it}(y') - \varepsilon_{it}(y) \leq v_{it}^{\mathbf{B}}(y, \mathbf{X}_t) - v_{it}^{\mathbf{B}}(y', \mathbf{X}_t) \text{ for any } y' \neq y\right\} \, dG_{it}(\boldsymbol{\varepsilon}_{it}) \\ &= \Lambda_{it}\left(y;\ \widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t)\right)\end{aligned}$$

where $\Lambda_{it}(y;\ .)$ is the CDF of the vector $\{\varepsilon_{it}(y') - \varepsilon_{it}(y) : y' \neq y\}$ and $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t)$ is the $(A-1)\times 1$ vector of value differences $\{\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X}_t) : y = 1, 2, ..., A-1\}$ with $\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X}_t) \equiv v_{it}^{\mathbf{B}}(y, \mathbf{X}_t) - v_{it}^{\mathbf{B}}(0, \mathbf{X}_t)$. For instance, if $\varepsilon_{it}(y)$'s are iid Extreme Value type 1, the best response function has the well–known logit form:

$$\frac{\exp\left\{\widetilde{v}_{it}^{\mathbf{B}}(y, \mathbf{X}_t)\right\}}{\sum_{y' \in \mathcal{Y}} \exp\left\{\widetilde{v}_{it}^{\mathbf{B}}(y', \mathbf{X}_t)\right\}} \tag{9}$$

Therefore, under Assumptions MOD-1 to MOD-3 the actual behavior of player i, represented by the CCP function $P_{it}(.)$, satisfies the following condition:

$$P_{it}(y|\mathbf{X}_t) = \Lambda_{it}\left(y;\ \widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X}_t)\right) \tag{10}$$

This equation summarizes all the restrictions that Assumptions MOD-1 to MOD-3 impose on players' choice probabilities. The right hand side of equation (10) is the best response function of a rational player. The concept of Markov Perfect Equilibrium (MPE) is completed with assumption 'Equil' ('Equilibrium Beliefs'). Under this assumption, players' beliefs are in equilibrium, i.e., $\mathbf{B}_{it} = \Pi_{j \neq i} \mathbf{P}_{jt}$ for every player i and every period t. A MPE can be described as a sequence of CCP vectors, $\{\mathbf{P}_{it} : i = 1, 2, ..., N;\ t = 1, 2, ..., T\}$ such that for every player i and time period t, we have that $P_{it}(y|\mathbf{X}_t) = \Lambda_{it}\left(y;\ \widetilde{\mathbf{v}}_{it}^{\mathbf{P}}(\mathbf{X}_t)\right)$. In this paper, we do not impose this equilibrium restriction.

Aradillas-Lopez and Tamer (2008) study the identification power of equilibrium restrictions in different game theoretic models. They consider a static, two-player, binary-choice game of incomplete information that is a specific case of our framework. Under the assumption that players' payoffs are submodular in players' decisions, they derive informative bounds around players' conditional choice probabilities when players are level-k rational, and show that the bounds become tighter as k increases. In their setup, the monotonicity of players' payoffs in the decisions of other players implies monotonicity of players' best response probability functions in the beliefs about other players actions. This type of monotonicity is very convenient in their approach, not only from the perspective of identification, but also because it yields a very simple approach to calculate upper and lower bounds on conditional choice probabilities. However, this property does not extend to dynamic games, even the simpler ones. We discuss the method of Aradillas-Lopez and Tamer (2008) and how it relates to our approach in more detail in the appendix.

Much of the complication associated with extending the bounds approach of Aradillas-Lopez and Tamer (2008) to the estimation of dynamic games is due to the fact that the notion of rationalizability, well-defined as a solution concept in static games, has no counterpart in the solution of dynamic games. Although Pearce (1984) provides an extension of the notion of rationalizability in static games to extensive form games, two problems with this notion exist: (a) the rationalizable outcome may not be a sequential equilibrium (see the example on pg 1044 of Pearce (1984)); and (b) as Battigalli (1997) shows (see page 44) in some extensive form games, allowing for the possibility that rationality is not common knowledge provides an incentive for players to strategically manipulate the beliefs of other players. This lack of solution concept in the absence of equilibrium does not preclude our primary motivation, the presence of multiple equilibria. Dynamic games are especially prone to multiplicity of equilibria, and our approach is well-suited to handle strategic uncertainty generated by multiple equilibria.

3 Identification

3.1 Conditions on Data Generating Process

Suppose that the researcher has panel data with realizations of the game over multiple geographic *locations* and time periods.[10] We use the letter m to index locations. The researcher observes a random sample of M locations with information on $\{Y_{imt}, \mathbf{X}_{mt}\}$ for every player $i \in \{1, 2, ..., N\}$ and every period $t \in \{1, 2, ..., T_{data}\}$. The number of periods in the data, T_{data}, can be either smaller or equal to the number of periods in the game, T. When $T_{data} < T$, we assume that players' beliefs are in equilibrium at every period $t \geq T_{data}$, such that the continuation values at period $t = T_{data}$ are identified under standard conditions in dynamic games with equilibrium beliefs, and we study identification with biased beliefs for periods $t < T_{data}$. Under this condition, identification with $T_{data} < T$ is equivalent to identification with $T_{data} = T$. Therefore, for notational simplicity and without loss of generality, we concentrate on the case with $T_{data} = T$

We assume that T is small and the number of local markets, M, is large. For the identification results in this section we assume that M is infinite. We first study identification in a model where the only unobservable variables for the researcher are the private information shocks $\{\varepsilon_{imt}\}$, which are assumed to be independently and identically distributed across players, markets, and over time. We relax this assumption in section 3.4 where we allow for time-invariant market-specific state variables that are common knowledge to all the players but unobservable to the researcher.

We want to use this sample to estimate the structural 'parameters' or functions of the model: i.e., payoffs $\{\pi_{it}, \beta\}$; transition probabilities $\{f_t\}$; distribution of unobservables $\{\Lambda_{it}\}$; and beliefs $\{B_{it}\}$. For primitives other than players' beliefs, we make some assumptions that are standard in

[10] In the context of empirical applications of games in Industrial Organization, a geographic location is a *local market.*

previous research on identification of static games and of dynamic structural models with rational or equilibrium beliefs.[11] We assume that the distribution of the unobservables, Λ_{it}, is known to the researcher up to a scale parameter. We study identification of the payoff functions π_{it} up to scale, but for notational convenience we omit the scale parameter.[12] Following the standard approach in dynamic decision models, we assume that the discount factor, β, is known to the researcher. Finally, note that the transition probability functions $\{f_t\}$ are nonparametrically identified.[13] Therefore, we concentrate on the identification of the payoff functions π_{it} and belief functions B_{it} and assume that $\{f_t, \Lambda_{it}, \beta\}$ are known.

Let $\mathbf{P}^0_{imt}$ be the vector of CCPs with the true (population) conditional probabilities $\Pr(Y_{imt} = y|i, m, t, \mathbf{X}_{mt} = \mathbf{X})$ for player i in market m at period t. Similarly, let $\mathbf{B}^0_{imt}$ be the vector of probabilities with the true values of player i's beliefs in market m at period t. And let $\boldsymbol{\pi}^0 \equiv \{\pi^0_{it} : i = 1, 2;\ t = 1, 2, ..., T\}$ be the true payoff functions in the population. Assumption *ID-1* summarizes our conditions on the Data Generating Process.

ASSUMPTION ID-1. (A) For every player i, $\mathbf{P}^0_{imt}$ *is the best response of player i given his beliefs* $\mathbf{B}^0_{im}$ *and the payoff functions* $\boldsymbol{\pi}^0$*. (B) A player has the same beliefs in two markets with the same observable characteristics* $\mathbf{X}$*, i.e., for every market m with* $\mathbf{X}_{mt} = \mathbf{X}$*,* $B_{imt}(\mathbf{y}_{-i}|\mathbf{X}) = B_{it}(\mathbf{y}_{-i}|\mathbf{X})$.

Assumption *ID-1*(A) establishes that players are rational in the sense that their actual behavior is the best response given their beliefs. Assumption *ID-1*(B) establishes that a player has the same beliefs in two markets with the same state variables and at the same period of time. This assumption is common in the literature of estimation of games under the restriction of equilibrium beliefs (e.g., Bajari, Benkard, and Levin, 2007, or Bajari et al, 2010). Note that beliefs can vary across markets according to the state variables in $\mathbf{X}_{mt}$. This assumption allows players to have different belief functions in different markets as long as these markets have different values of time-invariant observable exogenous characteristics. For instance, beliefs could be a function of "market type," which are determined by some market specific time-invariant observable characteristics. If the number of market types is small (more precisely, if it does not increase with M), then we can allow players' beliefs to be completely different in each market type. In section 3.3, we relax Assumption ID-1(B) by introducing time-invariant common- knowledge state variables that are unobservable to the researcher. In that extended version of the model, players' beliefs can vary across markets that are observationally equivalent to the researcher.

[11] See Bajari and Hong (2005), or Bajari et al (2010), among others.

[12] Aguirregabiria (2010) provides conditions for the nonparametric identification of the distribution of the unobservables in single-agent binary-choice finite-horizon dynamic structural models. Those conditions can be applied to identify the distribution of the unobservables in our model.

[13] Note that $f_t(\mathbf{X}'|\mathbf{Y}, \mathbf{X}) = \Pr(\mathbf{X}_{mt+1} = \mathbf{X}' \mid \mathbf{Y}_{mt} = \mathbf{Y}, \mathbf{X}_{mt} = \mathbf{X}) = \mathbb{E}\left(1\left\{\mathbf{X}_{mt+1} = \mathbf{X}', \mathbf{Y}_{mt} = \mathbf{Y}, \mathbf{X}_{mt} = \mathbf{X}\right\}\right) / \mathbb{E}\left(1\left\{\mathbf{Y}_{mt} = \mathbf{Y}, \mathbf{X}_{mt} = \mathbf{X}\right\}\right)$, where $1\{.\}$ is the indicator function. Given the discrete support of the variables in the vectors $\mathbf{Y}$ and $\mathbf{X}$, a root-M consistent estimator of this transition probability is, for instance, the frequency estimator $\left(\sum_{m=1}^{M} 1\left\{\mathbf{X}_{mt+1} = \mathbf{X}', \mathbf{Y}_{mt} = \mathbf{Y}, \mathbf{X}_{mt} = \mathbf{X}\right\}\right) / \left(\sum_{m=1}^{M} 1\left\{\mathbf{Y}_{mt} = \mathbf{Y}, \mathbf{X}_{mt} = \mathbf{X}\right\}\right)$.

In dynamic games where beliefs are in equilibrium, Assumption *ID-1* effectively allows the researcher to identify player beliefs. Under this assumption, conditional choice probabilities are identified, and if beliefs are in equilibrium, the belief of player i about the behavior of player j is equal to the conditional choice probability function of player j. When beliefs are not in equilibrium, Assumption *ID-1* is not sufficient for the identification of beliefs. However, assumption *ID-1* still implies that CCPs are identified from the data. This assumption implies that for any player i, any period t, and any value of $(y, \mathbf{X})$, we have that $P^0_{imt}(y|\mathbf{X}) = P^0_{it}(y|\mathbf{X}) = \Pr(Y_{imt} = y|\mathbf{X}_{mt} = \mathbf{X})$, and this conditional probability can be estimated consistently using the M observations of $\{Y_{imt}, \mathbf{X}_{mt}\}$ in our random sample of these variables. This in turn, as we will show, is important for the identification of beliefs themselves.

For notational simplicity, we omit the market subindex m for the rest of this section.

ASSUMPTION ID-2 ('Normalization' of payoff function): The one-period payoff function π_{it} is 'normalized' to zero for $Y_{it} = 0$, i.e., $\pi_{it}(0, \mathbf{Y}_{-it}, \mathbf{X}_t) = 0$ for any value of $(\mathbf{Y}_{-it}, \mathbf{X}_t)$.

Assumption *ID-2* establishes a 'normalization' of the payoff that is commonly adopted in many discrete choice models: the payoff to one of the choice alternatives, say alternative 0, is normalized to zero.[14] The particular form of normalization of payoffs does not affect our identification results as long as the normalization imposes $A^{N-1}|\mathcal{X}|$ restrictions on each payoff function π_{it}.

3.2 Identification of payoff and belief functions

In this subsection we examine different types of restrictions on payoffs and beliefs that can be used to identify dynamic games. The main point that we want to emphasize here is that restrictions that apply either only to beliefs or only to payoffs are not sufficient to identify this class of models. For instance, the assumption of equilibrium beliefs alone can identify beliefs but it is not enough to identify the payoff function. We also show that a exclusion restriction that has been commonly used to identify the payoff function can be exploited to relax the assumption of equilibrium beliefs.

3.2.1 Identification of value differences from choice probabilities

Let $\mathbf{P}_{it}(\mathbf{X})$ be the $(A-1) \times 1$ vector of CCPs $(P_{it}(1|\mathbf{X}), ..., P_{it}(A-1|\mathbf{X}))$, and let $\widetilde{\mathbf{v}}^{\mathbf{B}}_{it}(\mathbf{X})$ be the $(A-1) \times 1$ vector of differential values $(\widetilde{v}_{it}(1, \mathbf{X}), ..., \widetilde{v}_{it}(A-1, \mathbf{X}))$. The model restrictions can be represented using the best response conditions $\mathbf{P}_{it}(\mathbf{X}) = \mathbf{\Lambda}\left(\widetilde{\mathbf{v}}^{\mathbf{B}}_{it}(\mathbf{X})\right)$, where $\mathbf{\Lambda}(\mathbf{v})$ is the vector-valued function $(\Lambda(1|\mathbf{v}), \Lambda(2|\mathbf{v}), ..., \Lambda(A-1|\mathbf{v}))$. Given these conditions, and our normalization assumption ID-2, we want to identify payoffs and beliefs.

For all our identification results, a necessary first step consists of the identification of the vector of value differences $\widetilde{\mathbf{v}}^{\mathbf{B}}_{it}(\mathbf{X})$ from the vector of CCPs $\mathbf{P}_{it}(\mathbf{X})$. The following Theorem, due to Hotz and Miller (1993, Proposition 3), establishes this identification result.

[14] As is well-known, in discrete choice models preferences can be identified only up to an affine transformation.

THEOREM (Hotz-Miller inversion Theorem). If the distribution function $G_{it}(\boldsymbol{\varepsilon})$ is continuously differentiable over the whole Euclidean space, then, for any $(i,t,\mathbf{X})$, the mapping $\mathbf{P}_{it}(\mathbf{X}) = \boldsymbol{\Lambda}\left(\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X})\right)$ is invertible such that there is a one-to-one relationship between the $(A-1)\times 1$ vector of CCPs $\mathbf{P}_{it}(\mathbf{X})$ and the $(A-1)\times 1$ vector of value differences $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X})$.

Let $\mathbf{q}(\mathbf{P}) \equiv (q(1,\mathbf{P}), q(2,\mathbf{P}), ..., q(A-1,\mathbf{P}))$ be the inverse mapping of $\boldsymbol{\Lambda}$ such that if $\mathbf{P} = \boldsymbol{\Lambda}(\mathbf{v})$ then $\mathbf{v} = \mathbf{q}(\mathbf{P})$. Therefore, $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X}) = \mathbf{q}(\mathbf{P}_{it}(\mathbf{X}))$. For instance, for the multinomial logit case with $\Lambda(y|\mathbf{v}) = \exp\{v(y)\}/\sum_{y'\in\mathcal{Y}}\exp\{v(y')\}$, the inverse function $\mathbf{q}(\mathbf{P}_{it}(\mathbf{X}))$ is $q(y,\mathbf{P}_{it}(\mathbf{X})) = \ln(P_{it}(y|\mathbf{X})) - \ln(P_{it}(0|\mathbf{X}))$.

We assume the researcher knows the distribution of private information and so identification is not fully nonparametric in nature. However, the assumption that Λ is known to the researcher can be relaxed to achieve full nonparametric identification. This has been proved before by Aguirregabiria (2010) and Norets and Tang (2014) in the context of single-agent dynamic structural models based on previous results by Matzkin (1992) for the binary choice case, and Matzkin (1993) for the multinomial case.[15] As we do not consider this to be a focus of this paper, for the sake of simplicity, we have assume that the distribution Λ is known.

Given that CCPs are identified and that the distribution function Λ and the inverse mapping $\mathbf{q}(.)$ are known (up to scale) to the researcher, we have that the differential values $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ are identified. Then, hereinafter, we treat $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ as an identified object. To underline the identification of the value differences from inverting CCPs, we will often use $q(y,\mathbf{P}_{it}(\mathbf{X}))$, or with some abuse of notation $q_{it}(y,\mathbf{X})$, instead of $\widetilde{v}_{it}^{\mathbf{B}}(y,\mathbf{X})$.

3.2.2 Identification of payoffs and beliefs without exclusion restrictions

The identification problem can be described in terms of the identification of payoffs π_{it} and beliefs B_{it} given differential values q_{it}. We can represent the relationship between differential values and payoffs and beliefs using a recursive system of linear equations. For every period t and $(y_i,\mathbf{X}) \in [\mathcal{Y}-\{0\}]\times\mathcal{X}$, the following equation holds:

$$q_{it}(y_i,\mathbf{X}) \quad = \mathbf{B}_{it}(\mathbf{X})' \; \left[\boldsymbol{\pi}_{it}(y_i,\mathbf{X}) + \widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i,\mathbf{X})\right] \tag{11}$$

where $\mathbf{B}_{it}(\mathbf{X})$, $\boldsymbol{\pi}_{it}(y_i,\mathbf{X})$, and $\widetilde{\mathbf{c}}_{it}(y_i,\mathbf{X})$ are vectors with dimension $A^{N-1}\times 1$. $\mathbf{B}_{it}(\mathbf{X})$ is the vector of beliefs $\{B_{it}(\mathbf{y}_{-i}|\mathbf{X}) : \mathbf{y}_{-i}\in\mathcal{Y}^{N-1}\}$; $\boldsymbol{\pi}_{it}(y_i,\mathbf{X})$ is a vector of payoffs $\{\pi_{it}(y_i,\mathbf{y}_{-i},\mathbf{X}) : \mathbf{y}_{-i}\in\mathcal{Y}^{N-1}\}$; $\widetilde{\mathbf{c}}_{it}^{\mathbf{B}}(y_i,\mathbf{X})$ is a vector of continuation value differences $\{c_{it}^{\mathbf{B}}(y_i,\mathbf{y}_{-i},\mathbf{X}) - c_{it}^{\mathbf{B}}(0,\mathbf{y}_{-i},\mathbf{X}) : \mathbf{y}_{-i}\in\mathcal{Y}^{N-1}\}$, and $c_{it}^{\mathbf{B}}(\mathbf{Y}_t,\mathbf{X}_t)$ is the *continuation value function* that provides the expected and discounted value of *future* payoffs given future beliefs, current state, and current choices of *all* players:

$$c_{it}^{\mathbf{B}}(\mathbf{Y}_t,\mathbf{X}_t) \equiv \beta\int V_{it+1}^{\mathbf{B}}(\mathbf{X}_{t+1},\boldsymbol{\varepsilon}_{it+1})\; dG_{it}(\boldsymbol{\varepsilon}_{it+1})\; f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t,\mathbf{X}_t) \tag{12}$$

[15] This identification result is based on the assumption that there is a special state variable(s) that enters additively in the index $\widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X})$ and that has full support variation over the Euclidean space.

By definition, continuation values at the last period T are zero, $c^{\mathbf{B}}_{iT}(\mathbf{Y}, \mathbf{X}) = 0$.

The system of equations (11) summarizes all the restrictions of the model. These systems of equations have a recursive nature such that the continuation values in $\widetilde{\mathbf{c}}^{\mathbf{B}}_{it}(y_i, \mathbf{X})$ are determined by payoffs and beliefs at periods after t. Therefore, following a backwards induction argument, for every player i and period t we have $(A-1)|\mathcal{X}|$ restrictions (i.e., as many restrictions as there are free values $q_{it}(y_i, \mathbf{X})$), and the number of unknown parameters is $(A-1)A^{N-1}|\mathcal{X}|$ in the payoff function π_{it}, and $(N-1)(A-1)|\mathcal{X}|$ in the beliefs functions $\{B_{ijt} : j \neq i\}$.

Table 2 presents the number of parameters, restrictions, and over- or under- identifying restrictions for two versions of the model: a model where beliefs are unrestricted, in column A; and a model where beliefs are restricted to be unbiased, in column B. In both models, there are not restrictions in payoffs, other than the normalization restrictions in Assumption ID-2. The table presents these numbers for one player and time period. In both models, there are as many as $(A-1)A^{N-1}|\mathcal{X}|$ parameters in the payoff function $\pi_{it}(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t)$, and $(N-1)(A-1)|\mathcal{X}|$ parameters in the beliefs functions $\{B_{jt}(y_j|\mathbf{X}) : j \neq i\}$. The two models have also the same number of restrictions from observed players' behavior: the $(A-1)$ $|\mathcal{X}|$ value differences $q_{it}(y_i, \mathbf{X})$ in the system of equations (11). The difference between two models is in their restrictions on beliefs. The model with unbiased beliefs imposes $(N-1)$ $(A-1)$ $|\mathcal{X}|$ restrictions on beliefs, as many as beliefs parameters, i.e., $B_{ijt}(y_j|\mathbf{X}) = P_{jt}(y_j|\mathbf{X})$ for every player j, and any value of $(y_j, \mathbf{X})$. However, neither the unrestricted nor the restricted model are identified. They do not satisfy the order condition for identification, i.e., for any number of players N greater or equal to two, we have that the number of over(under) identifying restrictions $(A-1)$ $|\mathcal{X}|$ $\left[1 - A^{N-1}\right]$ is negative. In particular, while the restriction of unbiased beliefs does identify the beliefs function, the payoff function remains under-identified. Therefore, even if a researcher is willing to assume equilibrium beliefs, he still has to impose restrictions on the payoff function in order to get identification.

3.2.3 Identification with exclusion restrictions

Assumption *ID-3* presents nonparametric restrictions on the payoff function that, combined with the assumption of equilibrium beliefs, are typically used for identification in games with equilibrium beliefs.[16]

ASSUMPTION ID-3 (Exclusion Restriction): (i) The vector of state variables $\mathbf{X}_t$ can be partitioned in two subvectors, $\mathbf{X}_t = (\mathbf{S}_t, \mathbf{W}_t)$, such that the vector $\mathbf{W}_t \in \mathcal{W}$ includes variables that enter in the payoff function of more than one player (or even all the players), and $\mathbf{S}_t = (S_{1t}, S_{2t}, ..., S_{Nt}) \in \mathcal{S}^N$ where S_{it} represents state variables that enter into the payoff function of player i but not the payoff function of any of the other players. Therefore, the payoff function π_{it} depends on $(S_{it}, \mathbf{W}_t)$ but

[16] See Aguirregabiria and Mira (2002), Pesendorfer and Schmidt-Dengler (2003), and Bajari et al. (2010).

not on $\mathbf{S}_{-it}$*,*

$$\pi_{it}\left(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{S}_{-it}, \mathbf{W}_t\right) = \pi_{it}\left(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{S}'_{-it}, \mathbf{W}_t\right) \quad \text{for any } \mathbf{S}'_{-it} \neq \mathbf{S}_{-it} \tag{13}$$

(ii) The number of states in the support set $\mathcal{S}$ *is greater or equal than the number of actions* A*, i.e.,* $|\mathcal{S}| \geq A$*. (iii) Conditional on* $(\mathbf{S}_{-it}, \mathbf{W}_t)$*, the probability distribution of* S_{it} *has positive probability at every point in its support* $\mathcal{S}$*.*

With some abuse of notation we use $\pi_{it}\left(Y_{it}, \mathbf{Y}_{-it}, S_{it}, \mathbf{W}_t\right)$, instead of $\pi_{it}\left(Y_{it}, \mathbf{Y}_{-it}, \mathbf{X}_t\right)$, to represent the payoff function under assumption ID-3. Furthermore, the vector of common state variables $\mathbf{W}_t$ does not play any role in the identification of the model, and then we will omit it in some of our expressions.

The exclusion restriction in assumption *ID-3* is common in empirical applications of dynamic games.

EXAMPLE 2: Consider the *dynamic game of market entry and exit* that we introduced in Example 1. The vector of common knowledge state variables of the game is $\mathbf{X}_t = (H_t, Z_i, Y_{it-1} : i = 1, 2, ..., N)$. The specification of the model implies that market size H_t enters in the payoff of every firm. However, a firm's own incumbency status at previous period, Y_{it-1}, and the time-invariant characteristic affecting its fixed cost, Z_i, enter only into the profit function of firm i but not in the profits of the other firms. Therefore, in this example, $S_{it} = (Z_i, Y_{it-1})$ and $\mathbf{W}_t = H_t$. ■

Table 3 describes the order condition of identification for two models that impose the exclusion restriction in Assumption ID-3. Similarly as in Table 2, we have that column A represents the model with unrestricted beliefs, and column B deal with the model where we impose the restriction of unbiased beliefs. Under assumption ID-3, the state space $\mathcal{X}$ is equal to $\mathcal{W} \times \mathcal{S}^N$, and the number of points in the state space is equal to $|\mathcal{W}||\mathcal{S}|^N$. In table 2, for simplicity and without loss of generality, we omit $|\mathcal{W}|$ and use $|\mathcal{S}|^N$ to represent the number of points in the state space. In the model with exclusion restrictions and unbiased beliefs (column B in table 3), the order condition of identification is satisfied as long as the number of points in the space of the *special variable(s)* in the exclusion restriction, $|\mathcal{S}|$, is greater or equal than the number of choice alternatives A, i.e., condition (ii) in assumption ID-3. The rank condition for identification is also satisfied under the condition of full support variation of S_{it} conditional on $(\mathbf{S}_{-it}, \mathbf{W}_t)$, i.e., condition (iii) in assumption ID-3. Therefore, equilibrium beliefs and a exclusion restriction in payoffs can fully identify dynamic games with any number of players. In fact, when the number of states in the set $\mathcal{S}$ is strictly greater than the number of possible actions, the restrictions implied by equilibrium conditions overidentify payoffs. That is the case in the game in Example 1. The dimension of the space of $S_{it} = (Z_i, Y_{it-1})$ is $|\mathcal{Z}|A$ that is greater than the number of actions. In section 3.3, we will return to these overidentifying restrictions to construct a test of the null hypothesis of unbiased beliefs.

Column A in table 3 shows that the exclusion restriction alone, without any restriction on beliefs, is not enough to identify the model.

3.2.4 Tests of equilibrium beliefs and of monotonicity in beliefs

Though Assumptions ID-1 to ID-3 are not sufficient for the identification of payoffs and beliefs, they provide enough restrictions to test the null hypothesis of unbiased beliefs. We present here our test in a game with two players but the test can be extended to any number of players.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ is $(S_i, S_j, \mathbf{W})$, and players' actions are y_i and y_j. Let s_j^0 be an arbitrary value of in the set $\mathcal{S}$. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s_j^0\}$ such that they satisfy two conditions: (1) each of the sets has $A-1$ elements; and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A+1$, it is always possible to construct two subsets that satisfy these conditions. Given these sets, we can define the $(A-1) \times (A-1)$ matrices of beliefs $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W})$ and $\Delta\mathbf{B}_{it}^{(b)}(S_i, \mathbf{W})$, where $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W})$ has elements $\{B_{it}(y_j, S_i, S_j, \mathbf{W}) - B_{it}(y_j, S_i, s_j^0, \mathbf{W}) :$ for $y_j \in \mathcal{Y} - \{0\}$ and $S_j \in \mathcal{S}^{(a)}\}$, and $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W})$ has the same definition but for subset $\mathcal{S}^{(b)}$. Similarly, we can define matrices $\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W})$ and $\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})$, with elements $\{q_{it}(y_i, S_i, S_j, \mathbf{W}) - q_{it}(y_i, S_i, s_j^0, \mathbf{W}) :$ for $y_i \in \mathcal{Y} - \{0\}$ and $S_j \in \mathcal{S}^{(a)}\}$, and matrices $\Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W})$ and $\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})$, with elements $\{P_{jt}(y_j|S_i, S_j, \mathbf{W}) - P_{jt}(y_j|S_i, s_j^0, \mathbf{W}) :$ for $y_j \in \mathcal{Y} - \{0\}$ and $S_j \in \mathcal{S}^{(a)}\}$.

PROPOSITION 1: Suppose that assumptions MOD1 - MOD5 and ID-1 and ID-3 hold and the model is such that $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t) = f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{W}_t)$. Then:

(a) The $(A-1) \times (A-1)$ matrix of beliefs $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{B}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1}$ is identified from the CCPs of player i as $\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1}$;
(b) Under the assumption of unbiased beliefs, $\Delta\mathbf{B}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{B}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1}$ is also identified from the CCPs of the other player, j, as $\Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})\right]^{-1}$;
(c) Combining (a) and (b), the assumption of unbiased beliefs for player i implies the following $(A-1)^2$ restrictions between CCPs of players i and j:

$$\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W}) \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})\right]^{-1} = \mathbf{0} \qquad \blacksquare$$

Proof. In the Appendix.

Under the conditions of Proposition 1, for every value of $(S_i, \mathbf{W})$ we can use player i's CCPs to construct $(A-1)^2$ values that according to the model depend only on the beliefs of player i and not on payoffs, i.e., the observed behavior of player i identifies these functions of beliefs. Of course, if we assume that beliefs are unbiased, we know that these beliefs are equal to the choice probabilities of the other player, and therefore we have a completely different form, with different data, to identify these functions of beliefs. If the hypothesis of equilibrium beliefs is correct, then both approaches should give us the same result. Therefore, the restriction provides a natural approach to test for the null hypothesis of equilibrium or unbiased beliefs.

EXAMPLE 3: Suppose that the dynamic game has two players making binary choices: $N = 2$ and $A = 2$. Then, subsets $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have only one element each: $\mathcal{S}^{(a)} = \{s^{(a)}\}$ and $\mathcal{S}^{(b)} = \{s^{(b)}\}$ with $s^{(a)} \neq s^0$, $s^{(b)} \neq s^0$, and $s^{(a)} \neq s^{(b)}$. By Proposition 1, for a given selection of $(s^0, s^{(a)}, s^{(b)})$, and a given value of $(S_i, \mathbf{W})$, the hypothesis of unbiased beliefs implies one testable restriction. The restriction has this form:

$$\frac{q_{it}(1, S_i, s^{(a)}, \mathbf{W}) - q_{it}(1, S_i, s^0, \mathbf{W})}{q_{it}(1, S_i, s^{(b)}, \mathbf{W}) - q_{it}(1, S_i, s^0, \mathbf{W})} - \frac{P_{jt}(1|S_i, s^{(a)}, \mathbf{W}) - P_{jt}(1|S_i, s^0, \mathbf{W})}{P_{jt}(1|S_i, s^{(b)}, \mathbf{W}) - P_{jt}(1|S_i, s^0, \mathbf{W})} = 0 \tag{14}$$

It is clear that we can estimate nonparametrically all the components of this expression and implement a test. In section 4, we describe a test of the null hypothesis of unbiased beliefs based on this result. ■

In addition to knowledge of whether beliefs are in equilibrium or not, in many applications it is of interest to the researcher to understand properties of the beliefs function. For example, in some applications it is economically interesting to know whether beliefs are monotone in the special variable, especially given that we can identify the CCP function of a player j and compare its properties to those of the belief function of player i about the behavior of player j. Without making any further assumptions, we can in fact test whether beliefs functions are monotone. To see this, consider again the case with $A = 2$. For a given value of $(S_i, \mathbf{W})$ and a given selection of values $\{s^{(1)}, s^{(2)}, s^{(3)}\}$ such that $s^{(1)} < s^{(2)} < s^{(3)}$, define:

$$\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)}) \equiv \frac{q_{it}(1, S_i, s^{(3)}, \mathbf{W}) - q_{it}(1, S_i, s^{(2)}, \mathbf{W})}{q_{it}(1, S_i, s^{(2)}, \mathbf{W}) - q_{it}(1, S_i, s^{(1)}, \mathbf{W})} \tag{15}$$

By Proposition 1 we know that $\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)})$ is identified and is a function of player i's beliefs about player j:

$$\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)}) = \frac{B_{ijt}(1, S_i, s^{(3)}, \mathbf{W}) - B_{ijt}(1, S_i, s^{(2)}, \mathbf{W})}{B_{ijt}(1, S_i, s^{(2)}, \mathbf{W}) - B_{ijt}(1, S_i, s^{(1)}, \mathbf{W})} \tag{16}$$

Moreover, it is clearly the case that $\delta_{it}(s^{(1)}, s^{(2)}, s^{(3)}) \geq 0$ if and only if the beliefs function B_{ijt} is monotonic (either increasing or decreasing) in S_j. Therefore, in addition to a test of equilibrium beliefs, we also have a test of monotonicity versus non-monotonicity of the beliefs function.

3.2.5 Identification with exclusion restrictions and partially unbiased beliefs

The following assumption presents a restriction on beliefs that is weaker than the assumption of equilibrium beliefs and that together with assumptions ID-1 to ID-3 is sufficient to nonparametrically identify payoffs and beliefs in the model.

ASSUMPTION ID-4: Let $\mathcal{S}^{(R)} \subset \mathcal{S}$ be a subset of values in the set $\mathcal{S}$, with dimension $|\mathcal{S}^{(R)}| \equiv R$ that is strictly smaller than $|\mathcal{S}|$.

(a) For every state $\mathbf{X} = (\mathbf{S}, \mathbf{W})$ with $S_j \in \mathcal{S}^{(R)}$, the beliefs of player i on the behavior of player j are such that $B_{ijt}(y_j|\mathbf{X}) = P_{jt}(y_j|\mathbf{X})$, where here $P_{jt}(y_j|\mathbf{X})$ represents either the actual conditional choice probability of player j or consistent estimates of beliefs based on elicited beliefs data.

(b) Let $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ be the $R^{N-1} \times A^{N-1}$ matrix with elements $\{P_{-it}(\mathbf{y}_{-i}|S_i, \mathbf{S}_{-i}, \mathbf{W}) : \mathbf{y_{-i}} \in \mathcal{Y}^{\mathbf{N-1}}, \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$. For every period t and any value of $(S_i, \mathbf{W})$, this matrix has rank A^{N-1}.

Condition (a) establishes that there are some values of the opponents' stock variables $\mathbf{S}_{-i}$ for which strategic uncertainty disappears and beliefs about opponents' choice probabilities become unbiased. Alternatively, this assumption could be motivated by the availability of data on elicited beliefs for a limited number of states. Since $\mathcal{S}^{(R)}$ is a subset of the space $\mathcal{S}$, it is clear that Assumption ID-4(a) is weaker than the assumption of equilibrium beliefs, or alternatively, it is weaker than the condition of observing elicited beliefs for every possible value of the state variables. Note that the assumption does not necessarily mean that there is a subset of markets where beliefs are always in equilibrium. The assumption says that there is a subset of points in the state space such that a player's beliefs are unbiased every time that a point in that subset is reached, in any market. As such, in two markets m_1 and m_2, players may have beliefs out of equilibrium at some time period t, but the state in market m_1 may transit to a point where beliefs are unbiased at period $t+1$ while the state in market m_2 does not.

Condition (b) is needed for the rank condition of identification. A stronger but more intuitive condition than (b) is that $P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ is strictly monotonic with respect to $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, the actual choice probabilities of the other players depend monotonically on the state variables in $\mathbf{S}_{-i}$. Note that this intuition only applies to the case where $\mathbf{S}_{-i}$ is a scalar variable.

EXAMPLE 4: For the dynamic game in our example, we have that $S_{it} = (Z_i, Y_{i,t-1})$ such that the space $\mathcal{S}$ is equal to $\mathcal{Z} \times \mathcal{Y}$, with $\mathcal{Z}$ being the space of Z_i and $\mathcal{Y}$ is the binary set $\{0, 1\}$. Suppose that the set $\mathcal{S}^{(R)}$ consists of a pair of values $\{Z^*, 0\}$ and $\{Z^*, 1\}$, where Z^* is a particular point in the support $\mathcal{Z}$. Assumption ID-4 establishes that for every value of S_{it} we have that:

$$\begin{array}{rcl} B_{it}(1|S_{it}, S_{jt} = [Z^*, 0]) & = & P_{jt}(1|S_{it}, S_{jt} = [Z^*, 0]) \\ \\ B_{it}(1|S_{it}, S_{jt} = [Z^*, 1]) & = & P_{jt}(1|S_{it}, S_{jt} = [Z^*, 1]) \end{array}$$

That is, when the value of Z_j is Z^*, player i has unbiased beliefs about the behavior of player j whatever is the value of (S_{it}, Y_{jt-1}). In this example, $\mathbf{P}_{-it}^{(R)}(S_{it})$ is the 2×2 matrix:

$$\mathbf{P}_{jt}^{(R)}(S_{it}) = \left[\begin{array}{cc} P_{jt}(0|S_{it}, S_{jt} = [Z^*, 0]) \ , & P_{jt}(1|S_{it}, S_{jt} = [Z^*, 0]) \\ P_{jt}(0|S_{it}, S_{jt} = [Z^*, 1]) \ , & P_{jt}(1|S_{it}, S_{jt} = [Z^*, 1]) \end{array} \right]$$

Condition (b) on the rank of $\mathbf{P}_{jt}^{(R)}$ is satisfied if $P_{jt}(1|S_{it}, S_{jt} = [Z^*, 0]) \neq P_{jt}(1|S_{it}, S_{jt} = [Z^*, 1])$, i.e., if being an incumbent in the market at previous period has a non-zero effect on the probability of being in the market at current period. This is a very weak condition that we expect to be always satisfied in a dynamic game of market entry and exit. ■

The choice of the subset $\mathcal{S}^{(R)}$ of values where we impose the restriction of unbiased beliefs seems a potentially important modelling decision. In subsection 3.2.5 below, we discuss different approaches for the selection of subset $\mathcal{S}^{(R)}$.

Table 4 describes the order condition of identification under assumptions ID-1 to ID-4. This condition is satisfied if $1 - \frac{A^{N-1}}{|S|^{N-1}} - \left(1 - \frac{R^{N-1}}{|S|^{N-1}}\right)(N-1)$ is greater or equal than zero. When the number of players in the game is two, this condition becomes $R \geq A$, that can be satisfied for models where the number of states $|S|$ is strictly greater than the number of actions A. In games with more than two players, we have that $R \geq A$ is not sufficient to guarantee the order condition. However, if the support of the special state variable $|S|$ is large enough, then for any number of players N and any number of actions A, there is always a value of R between A and $|S|$ such that the order condition for identification is satisfied. For instance, with the number of players N is 3 and the number of actions A is 2, the order condition becomes $R \geq 1 + |S|/2$, that is compatible with $R < |S|$ when the number of states $|S|$ is greater than two. More generally, note that the order condition can be represented as:

$$\frac{1 - \left(\dfrac{R}{|\mathcal{S}|}\right)^{N-1}}{1 - \left(\dfrac{A}{|\mathcal{S}|}\right)^{N-1}} \leq \frac{1}{N-1} \tag{17}$$

If $|\mathcal{S}|$ is large enough such that $R/|\mathcal{S}|$ can be closed enough to 1, then it is clear that for any value of N and A, it is possible to find a value of R strictly smaller than $|\mathcal{S}|$ that satisfies this condition.

PROPOSITION 2: Suppose that assumptions MOD1 - MOD5 and ID-1 to ID-4 hold, and: (i) R is large enough such that the order condition $\left[1 - \left(\dfrac{R}{|\mathcal{S}|}\right)^{N-1}\right] / \left[1 - \left(\dfrac{A}{|\mathcal{S}|}\right)^{N-1}\right] \leq \dfrac{1}{N-1}$ *holds; and (ii) matrix* $\mathbf{Q}_{it}^{(R)}(S_i, \mathbf{W})$*, with dimension* $A \times R^{N-1}$ *and elements* $\{q_{it}(y_i, S_i, \mathbf{S}_{-i}, \mathbf{W}) : y_i \in \mathcal{Y}, \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$*, has rank equal to A. Then, the payoff functions* $\pi_{it}(y_i, \mathbf{y}_{-i}, S_i, \mathbf{W})$ *and the beliefs functions* $\{B_{ijt}(y_j|\mathbf{S}, \mathbf{W}) : j \neq i\}$ *are nonparametrically identified everywhere and at every period t.* ■

Proof. In the Appendix.

Remark 1. The condition that the rank of $\mathbf{Q}_{it}^{(R)}(S_i, \mathbf{W})$ is equal to A, in condition (ii), is satisfied if the conditional choice probability function of player i is strictly monotonic in $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, the actual choice probabilities of the other players depend monotonically on the state variables in $\mathbf{S}_{-i}$. Note that for the identification of the payoff function we need that beliefs (or the choice probabilities of players other than i) depend monotonically on $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. And for the identification of beliefs we also need that the choice probability of the own player i depends on $\mathbf{S}_{-i}$ over the subset $\mathcal{S}_{-i}^{(R)}$. That is, to identify beliefs we need that player i is playing a game such that the values of the state variables of the other players affect his decision through the effect of these variables in their beliefs. If the other players' actions do not have any effect on the payoff of player i, then his beliefs do not have any effect on his actions and therefore his actions cannot reveal any information about his beliefs.

Remark 2. In games with only two players, we can get identification of payoffs and beliefs by imposing the restriction of unbiased beliefs at only $R = A$ values of the special state variable. When the number of players increases, identification requires that we impose the restriction of unbiased beliefs in an increasing fraction of states. For instance, in a binary choice model ($A = 2$) with $|\mathcal{S}| = 10$ states, the minimum value of the ratio $R/|\mathcal{S}|$ to achieve identification is 20% in a model with two players, 72% with three players, 87% with four players, 93% with five players, and so on. In the limit, as the number of players goes to infinity, identification requires that the ratio $R/|\mathcal{S}|$ goes to one, i.e., in the limit we need to impose the restriction of unbiased beliefs at every possible state. This result is quite intuitive given that, as the number of players increases, the number of payoff parameters increases exponentially according to the function A^{N-1}. Nevertheless, when the number of players is not too large, such as $N \leq 5$, the model is fully identified even when we allow beliefs to be biased in a non-negligible fraction of states. Alternatively, a possible way to achieve identification when the number of players is large than two is to impose restrictions on the payoff function instead of beliefs. For instance, in some applications of interest, particularly in empirical IO, only the number of competitors taking an action, not the identity of the competitors, matters. The number of payoff parameters is substantially reduced in such applications and identification can be achieved even we allow beliefs to be biased in a large fraction of states.

For the rest of the paper we focus our attention on dynamic games with two players. We use subindexes i and j to represent the two players.

3.2.6 Where to assume unbiased beliefs?

As we mentioned above, the choice of the subset $\mathcal{S}^{(R)}$ where we impose the restriction of unbiased beliefs is a potentially important modelling decision. Here we describe three different approaches that may help the researcher when making this modelling decision.

(a) Minimization of the player's beliefs bias. Every choice of the set $\mathcal{S}^{(R)}$ implies a different estimate of payoffs and of the beliefs at states within and outside the set $\mathcal{S}^{(R)}$, and therefore a different distance between the vector of player beliefs $\mathbf{B}_{ijt}$ and the actual CCPs of player j, i.e., $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\|$. The researcher may want to be conservative and minimize the departure of his model

with respect to the paradigm of rational expectations or unbiased beliefs. If that is the case, the researcher can select the set $\mathcal{S}^{(R)}$ to minimize a bias criterion such as the distance $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\|$.

EXAMPLE 5. Consider a binary choice game with two players. Suppose that the set $\mathcal{S}$ has three values $\{s^{(1)}, s^{(2)}, s^{(3)}\}$. In this model, identification requires that we impose the restriction of unbiased beliefs in $R = A = 2$ values of S_j. There are three possible choices for the set $\mathcal{S}^{(R)}$: $\{s^{(1)}, s^{(2)}\}$, $\{s^{(1)}, s^{(3)}\}$, or $\{s^{(2)}, s^{(3)}\}$. These three choices of the set $\mathcal{S}^{(R)}$ have different implications on bias in the estimated beliefs. To show this, we can exploit the result from Proposition 1 in the previous subsection where we presented a test for the null hypothesis of unbiased beliefs. The restrictions of the model from player i's maximization of expected utility, and without any restriction from unbiased beliefs, implies the following restriction on beliefs:

$$\frac{q_{it}(1, S_i, S_j = s^{(3)}) - q_{it}(1, S_i, S_j = s^{(2)})}{q_{it}(1, S_i, S_j = s^{(2)}) - q_{it}(1, S_i, S_j = s^{(1)})} = \frac{B_{ijt}(1|S_i, S_j = s^{(3)}) - B_{ijt}(1|S_i, S_j = s^{(2)})}{B_{ijt}(1|S_i, S_j = s^{(2)}) - B_{ijt}(1|S_i, S_j = s^{(1)})} \tag{18}$$

This equation together with the restrictions of unbiased beliefs at two values in the state space identify the beliefs function $B_{jt}(1|S_i, S_j)$ at every value in $\mathcal{S}^2$. For notational simplicity, in this example we omit all the arguments in functions q_{it}, B_{ijt}, and P_{jt}, and use $q^{(k)}$, $B^{(k)}$, and $P^{(k)}$ to represent $q_{it}(1, S_i, S_j = s^{(k)})$, $B_{jit}(1|S_i, S_j = s^{(k)})$, and $P_{jt}(1|S_i, S_j = s^{(k)})$, respectively. Define $\delta_q \equiv \left[q^{(3)} - q^{(2)}\right] / \left[q^{(2)} - q^{(1)}\right]$, and $\delta_P \equiv \left[P^{(3)} - P^{(2)}\right] / \left[P^{(2)} - P^{(1)}\right]$. Then, we have that: (i) if $\mathcal{S}^{(R)} = \{s^{(1)}, s^{(2)}\}$, then $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\| = |B^{(3)} - P^{(3)}| = |\delta_q - \delta_P|\ |P^{(2)} - P^{(1)}|$; (ii) if $\mathcal{S}^{(R)} = \{s^{(1)}, s^{(3)}\}$, then $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\| = |B^{(2)} - P^{(2)}| = |\delta_q - \delta_P|\ |P^{(2)} - P^{(1)}|\ /\ |1 + \delta_q|$; and (iii) if $\mathcal{S}^{(R)} = \{s^{(2)}, s^{(3)}\}$, then $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\| = |B^{(1)} - P^{(1)}| = |\delta_q - \delta_P|\ |P^{(2)} - P^{(1)}|\ /\ |\delta_q|$. Note that when $|\delta_q - \delta_P| = 0$ the selection of the set $\mathcal{S}^{(R)}$ does not have any effect on our estimation of beliefs because the estimate of $\mathbf{B}_{ijt}$ will be equal to $\mathbf{P}_{jt}$ for any choice of the set $\mathcal{S}^{(R)}$. The value of $|\delta_q - \delta_P|$ corresponds to the statistic that we use to test for the null hypothesis of bias beliefs (see Example 3 above). Therefore, when there is not evidence of biased beliefs, the researcher's choice of the set $\mathcal{S}^{(R)}$ becomes irrelevant. However, $|\delta_q - \delta_P| > 0$, i.e., where there is evidence of biased beliefs, the researcher's choice of the set $\mathcal{S}^{(R)}$ matters. The set $\mathcal{S}^{(R)}$ that minimizes the bias criterion function $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\|$ is:

$$\begin{array}{lll} \mathcal{S}^{(R)} = \left\{s^{(1)}, s^{(2)}\right\} & \text{if} & -1 \leq \delta_q \leq 0 \\ \mathcal{S}^{(R)} = \left\{s^{(1)}, s^{(3)}\right\} & \text{if} & \delta_q \geq 0 \\ \mathcal{S}^{(R)} = \left\{s^{(2)}, s^{(3)}\right\} & \text{if} & \delta_q \leq -1 \end{array} \tag{19}$$

Taking into account the definition of δ_q, it is simple to verify that the set $\mathcal{S}^{(R)}$ that minimizes the bias $\|\mathbf{B}_{ijt} - \mathbf{P}_{jt}\|$ is the one that imposes unbiased beliefs at those states where $q^{(k)}$ takes its minimum and maximum value and leaves unrestricted the state with an intermediate value of $q^{(k)}$. ■

(b) Testing for the monotonicity of beliefs and using this restriction. Suppose that the CCP function $P_{jt}(y_j|S_i, S_j)$ is strictly monotonic in the state variable S_j. In subsection 3.2.4 above, we showed

that we can use our estimate of $\delta_q \equiv [q_{it}(1, S_i, S_j = s^{(3)}) - q_{it}(1, S_i, S_j = s^{(2)})] \ / \ [q_{it}(1, S_i, S_j = s^{(2)}) - q_{it}(1, S_i, S_j = s^{(1)})]$ to test for the monotonicity of the beliefs function B_{ijt} with respect to S_j, i.e., strict monotonicity implies that $\delta_q > 0$. Suppose that we cannot reject the strict monotonicity of the beliefs function. Then, if the data generating process is such that the special variable has a large support on the real line, the monotonicity of both CCPs and beliefs implies that the CCP function P_{jt} and the beliefs function B_{ijt} converge to each other at extreme points of the support, and it is natural to assume unbiased beliefs at these points.

(c) Most visited states. Suppose that the special state variable S_j has only variation across markets but it is constant over time. And suppose that the same players play the game in the different markets. Then, it is clear that players have played the game more times for the values of S_j that are more frequent, and as such may have learned more about other players' strategies and it is more likely that they have coordinated their beliefs. Therefore, in this context, a good candidate for values of S_j where beliefs are unbiased are those values which are more frequent in the data.

In our empirical application, in section 6, we apply these three arguments to justify our selection of the points in the state space where we impose the restriction of unbiased beliefs. We should note that the model selection methods proposed in this section can introduce a finite sample bias in our estimators of structural parameters and our inference using those estimators. This is the well-known problem of pre-testing (Leeb and Potscher, 2005) that is pervasive in many applications in econometrics. The sampling error at the model selection stage is not independent of the sampling error in the post-selection parameter estimates, and it can affect and distort the sampling distributions of these estimates. Different authors have advocated using bootstrap methods to construct correct post-selection inference methods. Recent work by Leeb and Potscher (2005, 2006) shows the limitations of some of these methods. In a recent paper, Berk et al. (2013) propose a new method to perform valid post model selection inference. Their method consists in doing simultaneous inference of the parameter estimates for all the possible models that can be selected. This method can be applied to our problem.

3.2.7 Closed-form expressions for payoffs and beliefs in terms of CCPs

Our proof of Proposition 2 is constructive and it provides closed-form expressions of the unknown parameters (payoffs and beliefs) in terms of the identified CCP functions. We use these formulas to construct two-step nonparametric estimators of payoffs and beliefs that we describe in section 4. Interestingly, the expressions describing the identification of payoffs and beliefs have an interpretation as a linear projection. This interpretation is useful not only for the actual implementation of the estimator and for the derivation of asymptotic properties but also to understand the conditions under which identification can be weaker or stronger.

Let $\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W})$ and $\mathbf{B}_{it}(\mathbf{X})$ be the $A \times 1$ vectors with payoffs $\pi_{it}(y_i, y_j, S_i, \mathbf{W})$ and beliefs $B_{it}(y_j|\mathbf{X})$, respectively, for every $y_j \in \mathcal{Y}$. At any period t and for any possible value $(y_i, S_i, \mathbf{W})$, the vector of payoffs $\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W})$ is identified as:

$$\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W}) = \left[\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \, \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})\right]^{-1} \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \, \tilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i, \mathbf{W}) \tag{20}$$

where the $R \times A$ matrix $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ has been defined in Assumption ID-4, and $\tilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i, \mathbf{W})$ is the $R \times 1$ vector with elements $\{\widetilde{q}_{it}(y_i, S_i, \mathbf{S}_{-i}, \mathbf{W}) : \mathbf{S}_{-i} \in \mathcal{S}_{-i}^{(R)}\}$ with

$$\widetilde{q}_{it}(y_i, \mathbf{X}) \equiv q_{it}(y_i, \mathbf{X}) - \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} P_{-it}(\mathbf{y}_{-i}|\mathbf{X}) \; c_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{X}) \tag{21}$$

$\tilde{\mathbf{q}}_{it}^{(R)}$ depends on the continuation values $c_{it}^{\mathbf{B}}$. At the last period T these continuation values are zero. For any period t before the last period, continuation values can be obtained using a simple recursive formula that we present in the Appendix. Note that the right-hand-side of equation (20), that describes the identification of the payoff function, is the estimator from a linear projection of $\tilde{\mathbf{q}}_{it}^{(R)}$ on $\mathbf{P}_{-iT}^{(R)}$. This interpretation has an important and useful implication for the choice of the subset $\mathcal{S}_{-i}^{(R)}$: the greater the variability of P_{it} over the set $\mathcal{S}_{-i}^{(R)}$, the more precise the estimation of the payoff function.

At any period t and for any value of the vector of state variables $\mathbf{X}$, the $A \times 1$ vector of beliefs $\mathbf{B}_{it}(\mathbf{X})$ is identified as:

$$\mathbf{B}_{it}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X}) \tag{22}$$

$\mathbf{q}_{it}(\mathbf{X})$ is an $A \times 1$ vector with elements $\{q_{it}(1, \mathbf{X}), ..., q_{it}(A-1, \mathbf{X})\}$ at rows 1 to $A-1$, and a 1 at the last row. And $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is an $A \times A$ matrix where the element $(y_i, y_j + 1)$ is:

$$\pi_{it}(y_i, y_j, S_i, \mathbf{W}) + [c_{it}(y_i, y_j, \mathbf{X}) - c_{it}(0, y_j, \mathbf{X})] \tag{23}$$

and the last row of the matrix is a row of ones. By construction, the expression $\left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X})$ is equal to $\mathbf{P}_{-it}(\mathbf{X})$ for values of $\mathbf{X}$ such that $\mathbf{S}_{-i}$ belongs to $\mathcal{S}_{i}^{(R)}$. As in the case of the identification of payoffs, matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ depends on continuation values that in turn depend on future payoffs and beliefs. Using backwards induction we can obtain these continuation values recursively.[17]

3.2.8 Unobserved market-specific heterogeneity

In Assumption ID-1 we require that a player has the same beliefs at any two markets with the same observable (to the econometrician) characteristics. Formally, we require that, for every market m

[17] There is a somewhat subtle relationship between our idenfitication result here and the literature on nonparametric identification of finite mixture models (for example Hall and Zhou (2003) and Kasahara and Shimotsu (2009)). In particular, the result that payoffs are identified if beliefs are known and invertible at a sufficiently large subset of points in the state space has a parallel with the structure of finite mixture of distributions with an exclusion restriction. In that literature such an identification result is not particularly useful, as it requires knowledge of the mixture weights at different values of the excluded variable. In the present context it is motivated through elicited beliefs or theoretical assumptions.

with $\mathbf{X}_{mt} = \mathbf{X}$, $B_{ijmt}(y_j|\mathbf{X}) = B_{ijt}(y_j|\mathbf{X})$. Before proceeding to discuss how we may relax this assumption and still achieve identification, we would like to emphasize the consequences of allowing beliefs to vary freely across markets for unobserved (to the econometrician) reasons. Note that once we allow beliefs to vary freely in this way, we have as many free beliefs as we have observations in the data, because we also allow beliefs to vary over time. Clearly in such a case, we do not have identification of payoffs or beliefs. We thus require some type of homogeneity restrictions to make the identification problem feasible. An assumption typically maintained in the literature is that, in any two observationally equivalent markets the behavior of agents (choice probabilities) are the same, which of course implies that payoffs and beliefs are the same across markets too, as equilibrium beliefs is also a maintained assumption. So far, the assumption that we have made is strictly weaker than this, in that we require beliefs to be the same across markets that are observationally equivalent but do not require them to be correct.

Yet in many applications, the assumption that beliefs, either biased or unbiased, are the same across markets is quite restrictive and it is important to relax it. For instance, an scenario where it may be important to allow for beliefs being different across observationally equivalent (for the researcher) markets is when firms are competing in a new market for the first time. There are potentially multiple equilibria in the underlying game, and when the players are put into a new market they are perhaps uncertain about the equilibrium that will be played. It is possible that in such a context we observe players having different beliefs in different markets which have the same observable characteristics. We thus consider how we may allow for permanent unobserved differences across markets and still identify payoffs and beliefs.

To introduce unobserved market-specific heterogeneity, we follow the nonparametric finite-mixture approach in Kasahara and Shimotsu (2009) (hereafter KS).[18] They propose a method for identification of Conditional Choice Probabilities in Markov decision models with a finite number of unobserved "market types". In this type of model, players' choice probabilities are different across observationally equivalent markets. They show that, the conditional independence assumptions in Markov discrete decision models can provide identification of models with this type of time-invariant unobserved heterogeneity. Under some conditions that we describe below, the method of KS can be applied in our model to identify players' Conditional Choice Probabilities for every unobserved market type. Note that, in our model, allowing for time-invariant market-specific unobserved heterogeneity implies that we relax the assumption that players have the same beliefs across observationally equivalent markets.

Suppose that the vector of (common-knowledge) state variables in the game includes: (i) the vector $\mathbf{X}_{mt}$ that we have considered so far, that is observable to the researcher; and (ii) the variable

[18] Hu and Shum (2013) extend identification results in Kasahara and Shimotsu (2009) to a richer model with time-variant serially correlated unobserved heterogeneity with a Markov chain structure.

ω_m that is unobservable to the researcher and it is i.i.d. across markets with a distribution that has discrete and finite support $\{\omega^{(\ell)} : \ell = 1, 2, ..., L\}$ and probability density function $\{\lambda^{(\ell)} : \ell = 1, 2, ..., L\}$. Payoff, beliefs, and CCP functions include variable ω as an argument, i.e., $\pi_{it}(\mathbf{Y}, \mathbf{X}, \omega)$, $B_{ijt}(y_j|\mathbf{X}, \omega)$, and $P_{it}(y_i|\mathbf{X}, \omega)$. Suppose, for the moment, that the CCP function $P_{it}(y_i|\mathbf{X}, \omega)$ is identified everywhere and for every value in the support of ω. Then, we have identified the CCP function for each one of the L 'market-types' in the population. Under this condition, it is straightforward to show that all the identification results that we have presented in this section, and in particular Propositions 1 and 2, apply point-wise to each of the L market-types. Therefore, the only new identification problem associated to including unobserved market heterogeneity comes from the identification of the CCP function $P_{it}(y_i|\mathbf{X}, \omega)$.

Now, discuss the identification of the CCP function $P_{it}(y_i|\mathbf{X}, \omega)$ and how we can apply results from KS. The identification of the CCP function is based on the following restrictions of the model on the probability of a choice history for player i:

$$\Pr\left(Y_{i1}, \mathbf{X}_1, ..., Y_{iT}, \mathbf{X}_T\right) = \sum_{\ell=1}^{L} \lambda^{(\ell)} \Pr\left(\mathbf{X}_1|\omega^{(\ell)}\right) \left[\prod_{t=1}^{T} P_{it}(Y_{it}|\mathbf{X}_t, \omega^{(\ell)})\ f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)\right] \tag{24}$$

Given a large sample of M markets (infinite M), we can nonparametrically identify the probabilities in the right-hand-side of equation (24) for every possible history of Y_i and $\mathbf{X}$ between periods 1 and T. We can also identify nonparametrically the transition probabilities $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$. Then, the identification problem consists in obtaining conditions under which there is unique value of the probability functions $\{\lambda^{(\ell)}, \Pr\left(\mathbf{X}_1|\omega^{(\ell)}\right), P_{it}(Y_{it}|\mathbf{X}_t, \omega^{(\ell)}) : \ell = 1, 2, ..., L;\ t = 1, 2, .., T\}$ that solves the system of equations in (24).

Proposition 4 in KS provides conditions under which a researcher can identify nonparametrically this finite mixture model choice probabilities when the problem is non-stationary, i.e., time-dependant CCPs and transition probabilities. Given the inherent non-stationarity of finite horizon dynamic discrete choice models, this is the relevant identification result in KS to apply in our context. The requirements for identification of CCPs in KS Proposition 4 are the following (See KS page 142): (KS-1) $P_{it}(Y_{it}|\mathbf{X}_t, Y_{i,t-1}, \omega^{(\ell)}) = P_{it}(Y_{it}|\mathbf{X}_t, \omega^{(\ell)})$; (KS-2) $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$ does not depend on market type; and (KS-3) $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t) > 0$ for all t and any $(\mathbf{X}_{t+1}, \mathbf{Y}_t, \mathbf{X}_t)$. Assumptions (KS-1) and (KS-2) have no bite given our setup. Assumption (KS-1) simply says that, conditional on today's state, previous actions don't have an effect on the probability of taking any given action. We have implicitly assumed that choice probabilities are independent of previous actions except through their effect on the state vector. Assumption (KS-2) states that the transition probabilities are common across markets given the same observables. This is also implied in our framework. However, assumption (KS-3) imposes a relevant restriction in our model. It states that transition probability functions $f_t(.|\mathbf{Y}_t, \mathbf{X}_t)$ have full support over the whole state space $\mathcal{X}$. This condition rules out dynamic models where there is a deterministic relationship between the state today and actions in the previous period, e.g., games of market entry. However, models with transition rules of the form $\mathbf{X}_{t+1} = g_t(\mathbf{Y}_t, \mathbf{X}_t) + \eta_{t+1}$, where η_{t+1} is independent of $(\mathbf{Y}_t, \mathbf{X}_t)$, can satisfy condition (KS-3). An example from empirical IO is a game of dynamic investment, where capital depreciation is stochastic and it has enough variation. Models of this form satisfy the conditions of Proposition 4 in Kasahara and Shimotsu (2009), and thus we identify choice probabilities even in the presence of unobserved permanent differences across markets.

4 Estimation and Inference

Our constructive proofs of the identification results in Propositions 1 and 2 suggest methods for estimation and testing of the nonparametric model. Section 4.1 provides a description of a nonparametric estimation method. In most empirical applications, the payoff function is parametrically specified. For this reason, in section 4.2 we extend the estimation method to deal with parametric models. Section 4.3 presents our test for the null hypothesis of unbiased beliefs. In the Appendix, we derive the asymptotic properties of the estimators and tests.

4.1 Estimation with nonparametric payoff function

Nonparametric estimation proceeds in two steps.

Step 1: Nonparametric estimation of CCPs and transition probabilities. For every player, time period, and state $\mathbf{X}$, we estimate CCPs $P_{it}(y_i|\mathbf{X})$, and (if necessary) the transition probabilities $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{X}_t)$. We also construct estimates of $q_{it}(y_i, \mathbf{X})$ by inverting the mapping Λ, i.e., $q_{it}(y_i, \mathbf{X}) = \Lambda^{-1}(y_i, \mathbf{P}_{it}(\mathbf{X}))$.

Step 2: Recursive estimation of preferences and beliefs. We select the subset $\mathcal{S}^{(R)}$ with the values of S_j for which we assume that player i's beliefs are unbiased. Given this set and the estimates in step 1, we construct, for every period t and any value of $(y_i, S_i, \mathbf{W})$, the matrices $\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})$ and the vectors $\mathbf{q}_{it}(y_i, S_i, \mathbf{W})$. Then, we apply the following backwards induction procedure. We start at the last period T, where the continuation function is zero, and apply recursively the following three steps to estimate payoffs, beliefs, and continuation values functions at every period t: (i) the payoff function,

$$\boldsymbol{\pi}_{it}(y_i, S_i, \mathbf{W}) = \left[\mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \ \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})\right]^{-1} \mathbf{P}_{-it}^{(R)}(S_i, \mathbf{W})' \ \tilde{\mathbf{q}}_{it}^{(R)}(y_i, S_i, \mathbf{W}) \ ; \tag{25}$$

(ii) the beliefs function,

$$\mathbf{B}_{it}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X}) \ ; \tag{26}$$

and (iii) integrated value function,

$$c_{it}^{\mathbf{B}}(y_i, y_j, \mathbf{X}) \ = \ \beta \sum_{\mathbf{X}_{t+1}} \ln \left(\sum_{y_{it+1}} \exp\left\{ \mathbf{B}_{it}(\mathbf{X})' \left[\boldsymbol{\pi}_{it}(y_{it+1}, \mathbf{X}) + c_{it+1}^{\mathbf{B}}(y_{it+1}, \mathbf{X}) \right] \right\} \right) f_t(\mathbf{X}_{t+1}|y_i, y_j, \mathbf{X}) \tag{27}$$

This estimator is consistent and asymptotically normal (see Appendix). The derivation of the asymptotic variance is cumbersome. In our empirical application we use the bootstrap method to obtain standard errors and confidence intervals for the estimates.

4.2 Estimation with parametric payoff function

In most applications the researcher assumes a parametric specification of the payoff function. A class of parametric specifications that is common in empirical applications is the linear in parameters model:

$$\pi_{it}(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) = g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t)\ \boldsymbol{\theta}_{it} \tag{28}$$

where $g(Y_i, Y_j, S_i, \mathbf{W})$ is a $1 \times K$ vector of known functions, and $\boldsymbol{\theta}_{it}$ is a $K \times 1$ vector of unknown structural parameters in player i's payoff function. Let $\boldsymbol{\theta}_i$ be the vector with all the parameters in the payoff of player i: $\boldsymbol{\theta}_i \equiv \{\boldsymbol{\theta}_{it} : t = 1, 2, ..., T\}$.

EXAMPLE 6: Consider the dynamic game in Example 1. The profit function in equation (2) can be written as $g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t)\ \boldsymbol{\theta}_i$, where the vector of parameters $\boldsymbol{\theta}_i$ is $(\theta_i^M, \theta_i^D, \theta_{i0}^{FC}, \theta_{i1}^{FC}, \theta_i^{EC})'$ and

$$g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t) = Y_{it}\{\ H_t,\ -H_t Y_{jt},\ -1,\ -Z_i, -1\{Y_{it-1} = 0\}\ \} \tag{29}$$

■

To estimate $\boldsymbol{\theta}_i$ we propose a simple three steps method. The first two-steps are the same as for the nonparametric model.

Step 3: Given the estimates from step 2, we can apply a pseudo maximum likelihood method in the spirit of Aguirregabiria and Mira (2002) to estimate the structural parameters $\boldsymbol{\theta}_i$. Define the following pseudo likelihood function for the model with i.i.d. extreme value $\varepsilon's$:

$$Q(\boldsymbol{\theta}_i, \mathbf{B}_i, \mathbf{P}_i) \equiv \sum_{m=1}^{M}\sum_{t=1}^{T} \log\left(\frac{\exp\left\{\widetilde{g}_{it}^{\mathbf{B},\mathbf{P}}(Y_{imt}, \mathbf{X}_{mt})\ \boldsymbol{\theta}_i + \tilde{e}_{it}^{\mathbf{B},\mathbf{P}}(Y_{imt}, \mathbf{X}_{mt})\right\}}{\sum_{y_i=0}^{A-1}\exp\left\{\widetilde{g}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X}_{mt})\ \boldsymbol{\theta}_i + \tilde{e}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X}_{mt})\right\}}\right) \tag{30}$$

$\widetilde{g}_{it}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X})$ is the discounted sum of the expected values of $\{g(Y_{jt+s}Y_{jt+s}, \mathbf{X}_{t+s}) : s = 0,\ 1,\ ...,\ T-t\}$ given that the state at period t is $\mathbf{X}$, that player i chooses alternative y_i at period t and then behaves according to the choice probabilities in $\mathbf{P}$, and believes that player j behaves according to the probabilities in $\mathbf{B}$. And $\tilde{e}_{imt}^{\mathbf{B},\mathbf{P}}(y_i, \mathbf{X})$ is also a discounted sum, but of expected future values of $\sum_{y_i=0}^{A-1} P_{it+s}(y_i|\mathbf{X}_{mt+s})\ [\gamma - \ln P_{it+s}(y_i|\mathbf{X}_{mt+s})]$, that represents the expected value of $\varepsilon_{im,t+s}(Y_{imt+s})$ when $Y_{im,t+s}$ is optimally chosen, and γ is Euler's constant. From steps 1 and 2, we have consistent estimates of CCPs, $\hat{\mathbf{P}}_i$, and beliefs, $\hat{\mathbf{B}}_i$. Then, a consistent pseudo maximum likelihood estimator of $\boldsymbol{\theta}_i$ is defined as the value $\hat{\boldsymbol{\theta}}_i^{(1)}$ that maximizes $Q(\boldsymbol{\theta}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{P}}_i)$. Note that the sample criterion function $Q(\boldsymbol{\theta}_i, \hat{\mathbf{B}}_i, \hat{\mathbf{P}}_i)$ is just the log likelihood function of a Conditional Logit model with the restriction that the parameter multiplying the discounted sum $\tilde{e}_{it}^{\mathbf{B},\mathbf{P}}$ is equal to 1. The estimator is root-M consistent and asymptotically normal.

Steps 1 to 3 can be applied recursively to improve the statistical properties of our estimators. Let $\hat{\mathbf{P}}_i^{(1)} = \{\hat{P}_{it}^{(1)}(y_i|\mathbf{X})\}$ be the vector with the new estimates of CCPs implied by the parametric model:

$$\hat{P}_{it}^{(1)}(y_i|\mathbf{X}) = \frac{\exp\left\{\widetilde{g}_{it}^{\hat{\mathbf{B}}^0,\hat{\mathbf{P}}^0}(y_i,\mathbf{X})\ \widehat{\boldsymbol{\theta}}_i^{(1)} + \widetilde{e}_{it}^{\hat{\mathbf{B}}^0,\hat{\mathbf{P}}^0}(y_i,\mathbf{X})\right\}}{\sum_{y_i'=0}^{A-1}\exp\left\{\widetilde{g}_{it}^{\hat{\mathbf{B}}^0,\hat{\mathbf{P}}^0}(y_i',\mathbf{X})\ \widehat{\boldsymbol{\theta}}_i^{(1)} + \widetilde{e}_{it}^{\hat{\mathbf{B}}^0,\hat{\mathbf{P}}^0}(y_i',\mathbf{X})\right\}} \tag{31}$$

where $\hat{\mathbf{B}}^0$ and $\hat{\mathbf{P}}^0$ represent the initial nonparametric estimators of beliefs and CCPs, respectively, and $\hat{\boldsymbol{\theta}}_i^{(1)}$ is the pseudo maximum likelihood estimator of the parameters in the payoff function. We expect $\hat{\mathbf{P}}_i^{(1)}$ to have smaller asymptotic variance and finite sample bias than the initial $\hat{\mathbf{P}}_i^{(0)}$. Given $\hat{\mathbf{P}}_i^{(1)}$ and parametric estimates of payoffs, $g(Y_{it}, Y_{jt}, S_{it}, \mathbf{W}_t)\ \hat{\boldsymbol{\theta}}_{it}^{(1)}$, we can construct new estimates of beliefs using the formula in equation (26). That is:

$$\widehat{\mathbf{B}}_{it}^{(1)}(\mathbf{X}) = \left[\widehat{\mathbf{V}}_{it}^{(1)}(\mathbf{X})\right]^{-1}\widehat{\mathbf{q}}_{it}^{(1)}(\mathbf{X})\ , \tag{32}$$

where $\widehat{\mathbf{V}}_{it}^{(1)}$ and $\widehat{\mathbf{q}}_{it}^{(1)}$ are constructed using $\hat{\mathbf{P}}_i^{(1)}$ and $\hat{\boldsymbol{\theta}}_i^{(1)}$. We can apply this procedure recursively to update CCPs, beliefs, and structural parameters and to obtain a sequence of estimators $\{\hat{\boldsymbol{\theta}}^{(K)}$, $\hat{\mathbf{B}}^{(K)}$, $\hat{\mathbf{P}}^{(K)} : K \geq 1\}$.

4.3 Test of unbiased beliefs

In principle, we could use a standard Lagrange Multiplier (LM) or Score test of the null hypothesis of equilibrium beliefs. That test is based on the constrained maximum likelihood estimation (MLE) of structural parameters and beliefs.[19] In our context, the LM test has at least two important limitations. First, maximum likelihood estimation of dynamic games is computationally very demanding both because the high dimension of the state space and because of the existence of multiple equilibria. Second, this is a general specification test. The null hypothesis is not only that beliefs are in equilibrium but also that the parametric specification of preferences and the distribution of unobservables is correct. We would like to have a procedure that specifically tests for the equilibrium beliefs and not for other specification assumptions of the model.

Our test is based on Proposition 1. Here we describe our test of unbiased beliefs at a single value of the state $(S_i, \mathbf{W})$. However, it is straightforward to generalize the test to multiple states or every possible state. Let $\delta_i(S_i, \mathbf{W})$ the be $(A-1)^2 \times 1$ vector with the elements of matrix $\Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W})$

[19] Define the log-likelihood function, $l(\boldsymbol{\theta}, \mathbf{P}) \equiv \sum_{m=1}^{M}\sum_{t=1}^{T}\sum_{i=1}^{N} \log\Lambda(Y_{imt}, v_{it}^{\mathbf{P}}(\mathbf{X}_{mt}, \boldsymbol{\theta}))$. The constrained MLE is defined as a vector $(\hat{\boldsymbol{\theta}}_{MLE}, \hat{\mathbf{P}}_{MLE})$ that maximizes the likelihood $l(\boldsymbol{\theta}, \mathbf{P})$ subject to the equilibrium constraints $\mathbf{P} = \Lambda(v^{\mathbf{P}}(\boldsymbol{\theta}))$. We want to test the null hypothesis $\mathbf{P} = \Lambda(v^{\mathbf{P}}(\boldsymbol{\theta}))$, that consists of $2(A-1)|\mathcal{X}|$ constrains on $(\boldsymbol{\theta}, \mathbf{P})$. We can use a standard LM test. Under the null hypothesis, the LM statistic is asymptotically distributed as a chi-square with $2(A-1)|\mathcal{X}|$ degrees of freedom.

$\left[\Delta\mathbf{Q}_{it}^{(b)}(S_i,\mathbf{W})\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(S_i,\mathbf{W}) \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i,\mathbf{W})\right]^{-1}$. And let $\widehat{\delta}_i(S_i,\mathbf{W})$ be a consistent estimator of this vector. Define the statistic:

$$\hat{D} = \widehat{\delta}_i(S_i,\mathbf{W}) \left[\widehat{Var}\left(\widehat{\delta}_i(S_i,\mathbf{W})\right)\right]^{-1} \widehat{\delta}_i(S_i,\mathbf{W}) \tag{33}$$

where $\widehat{Var}\left(\widehat{\delta}_i(S_i,\mathbf{W})\right)$ is a consistent estimator of the asymptotic variance of $\widehat{\delta}_i(S_i,\mathbf{W})$, that can be obtained using a nonparametric bootstrap method. Under the null hypothesis of unbiased beliefs, the statistic $\hat{D}$ is asymptotically distributed as a Chi-square with $(A-1)^2$ degrees of freedom. Under the alternative hypothesis, beliefs are biased, $\widehat{\delta}_i(S_i,\mathbf{W})$ does not converge in probability to zero, and the statistic $\hat{D}$ has a non-central chi-square asymptotic distribution.

5 Monte Carlo Experiments

In this section we use Monte Carlo methods to illustrate the identification and estimation framework presented in previous sections. Our purpose in implementing Monte Carlo experiments is threefold. First, we would like to assess the power of our identification assumptions in small samples. While preferences and beliefs are asymptotically identified given our identification assumptions, it might well be the case that when we replace the assumption of equilibrium beliefs by our identification conditions, estimates of preferences and beliefs become imprecise in the small samples that are common in actual applications. We want to evaluate the price of relaxing the assumption of unbiased beliefs in terms of precision of our estimates. The second purpose of our Monte Carlo experiments is to study the consequences of imposing the assumption of equilibrium beliefs when this assumption does not hold in the data generating process. In principle, incorrectly imposing the assumption of equilibrium beliefs can lead to significant bias in estimates of preferences and beliefs. We would like to understand the magnitude of this bias in the context of a simple application. Third, our identification results rest on an exclusion restriction: to identify a player's payoffs and beliefs, there must be a special variable which does not enter his own payoffs (directly), but does enter the other player's payoffs. Clearly, the exclusion itself is important, but in finite sample one must also be concerned with how much the excluded variable affects the other player's payoffs, or how much it "shifts" the other player's behavior. If the special variable does not shift the other players behavior very much, we will have problems identifying our objects of interest. It is interesting from a practical standpoint to know just how significant a role this plays in identification.

Together the results of the Monte Carlo experiments help to illustrate the trade-off a researcher faces when deciding whether or not to impose the assumption of equilibrium beliefs, and how this trade off depends on the underlying DGP. On the one hand, by imposing the assumption of equilibrium beliefs the researcher is able to rely on the identification power afforded by the equilibrium restrictions, which results in more precise estimates, though he must be concerned with the possibility of biased results (if beliefs are not actually in equilibrium in the underlying DGP). On the other hand by not imposing the assumption of equilibrium beliefs the researcher does not have to be concerned with the bias caused by making an incorrect assumption about players' beliefs in the underlying DGP, but must be concerned with decreased precision in the estimates.

The model we consider in our experiments is a particular case of the *dynamic game of market entry and exit* in Example 1. We consider a game with two players. The per period profit functions of the two players are given by:

$$\begin{aligned} \pi_{1mt}(1, Y_{2mt}, \mathbf{X}_{mt}) &= (1 - Y_{2mt})\ \theta_1^M + Y_{2mt}\ \theta_1^D - \theta_{01}^{FC} - (1 - Y_{1mt-1})\ \theta_1^{EC} \\ \pi_{2mt}(1, Y_{1mt}, \mathbf{X}_{mt}) &= (1 - Y_{1mt})\ \theta_2^M + Y_{1mt}\ \theta_2^D - \theta_{02}^{FC} - \theta_{12}^{FC}\ Z_{2m} - (1 - Y_{2mt-1})\ \theta_2^{EC} \end{aligned} \tag{34}$$

We normalize the profits to not being active to be zero for both players: $\pi_{1mt}(0, Y_{2mt}, \mathbf{X}_{mt}) = \pi_{2mt}(0, Y_{1mt}, \mathbf{X}_{mt}) = 0$. The players' payoffs to being active are symmetric except for the variable Z_{2m} which enters player 2's payoffs but not player 1's. Z_{2m} is an exogenous time-invariant characteristic which affects the fixed cost of player 2, but does not have a (direct) effect on the payoff of player 1. We assume that Z_{2m} is observable to the researcher. The vector of state variables is given by $\mathbf{X}_{mt} = (Z_{2m}, Y_{1mt-1}, Y_{2mt-1})$.

We focus on the estimation of the parameters in player 1's payoff and beliefs functions, i.e., $\{\theta_1^M, \theta_1^D, \theta_{01}^{FC}, \theta_1^{EC}\}$ and beliefs. Given the payoff structure in equation (34) above, only the payoffs and beliefs of player 1 are identified under our identification assumptions.[20] The variable Z_{2m} allows us to identify the payoffs and beliefs of player 1. As we discuss and illustrate with the Monte Carlo experiments below, the value of θ_{12}^{FC}, which determines the sensitivity of player 2's payoffs to the variable Z_{2m} plays an important role in our ability to identify our objects of interest.

As we are only concerned with identification and estimation of payoffs and beliefs of player 1, we focus our discussion on these for the remainder of the section. Note that given the payoff structure we have assumed, and more specifically, given that we do not include market size H_{mt} as a state variable, we cannot separately identify the market-invariant component of fixed cost, θ_{01}^{FC}, from the fixed component of the variable profit, θ_1^M, as the two parameters have the same impact on player 1's payoff. So define the parameters $\alpha_1 \equiv \theta_1^M - \theta_{01}^{FC} - \theta_1^{EC}$, and $\delta_1 \equiv \theta_1^M - \theta_1^D$. We can re-write player 1's per period profit function as:

$$\pi_{1mt}(1, Y_{2mt}, \mathbf{X}_{mt}) \;\; = \;\; \alpha_1 - \delta_1\ Y_{jmt} + Y_{imt-1}\ \theta_1^{EC} \tag{35}$$

The exogenous variable Z_{2m} is independently and identically distributed over markets, with a discrete uniform distribution with support $\{-2, -1, 0, 1, 2\}$. As we mentioned above, Z_{2m} is the

[20] Specifically, there is no variable with at least three points of support (i.e., $A+1 = 3$) that enters player 1's payoffs directly and does not enter player 2's payoffs directly. In principle Y_{1mt-1} could play the role of the "special variable" for identifying player 2's payoffs and beliefs, but since it can only take two values it is always at an "extreme point." In this case player 1's beliefs about player 2's behavior would always be correct. Relatedly, notice that in this set up we actually have over-identification, as the variable Y_{imt-1} is excluded from player j's payoffs for $i = 1, 2$. While these restrictions could be exploited in principle we do not do so here.

key variable which allows us to identify the payoffs and beliefs of player 1. Essentially Z_{2m} plays the role of an instrument for identifying the payoffs and beliefs of player 1 in the sense that it satisfies the usual exclusion restriction. It affects player 1's payoffs only through its effect on the behavior of player 1. As such, one must be concerned with the strength of the instrument. In our set-up here, the strength of the instrument, for a given support and distribution of Z_{2m} is completely determined by the value of θ_{12}^{FC}. By considering different values of θ_{12}^{FC} in the DGP, we use the Monte Carlo experiments to illustrate how a "weak instrument" may affect inference on payoffs and beliefs in a finite sample.

Table 5 presents the features of the DGPs in our experiments. We consider different values of θ_{12}^{FC} in the experiments, but keep α_i, δ_i, θ_i^{EC}, β_i constant. In order to provide an economic interpretation for the magnitude of these parameters, the table includes also some ratios implied by the payoff parameters.

In each experiment we consider, the sample is comprised by $M = 2,000$ markets and $T = 5$ time periods. We approximate the finite sample distribution of the estimators using $10,000$ Monte Carlo replications. The initial conditions for $\{Y_{1mt-1}, Y_{1mt-1}\}$ at $t = 1$ are drawn uniformly at random, as is the time invariant market specific variable Z_{2m}.

We implement four experiments: 1A, 1B, 2A, and 2B. The difference between experiments 1* and 2* is in the value of the parameter θ_{12}^{FC} associated with the power of the instrument Z_2: the value of this parameter is -0.5 in the 1A and 1B experiments, and -1.0 for the 2A and 2B experiments. The difference between experiments *A and *B is in the bias of players' beliefs. In experiments 1A and 2A beliefs are in equilibrium, while in experiments 1B and 2B players have biased beliefs. We now describe how players beliefs are determined in experiments 1B and 2B.

For for every player $i \in \{1, 2\}$ beliefs are $B_{imt}(\mathbf{X}_{mt}) = \lambda_{im}\ P_{jmt}(\mathbf{X}_{mt})$, where $\lambda_{im} \in [0, 1]$ is a parameter that captures player i's bias in beliefs in market m. Note that, given this specification of the DGP, beliefs are endogenous because they depend on the other player's choice probabilities that, of course, are endogenous. Therefore, to obtain these beliefs we need to solve for a particular equilibrium or fixed point problem. Given λ_{1m} and λ_{2m}, players' choice probabilities $P_{1mt}(\mathbf{X}_{mt})$ and $P_{2mt}(\mathbf{X}_{mt})$ solve a fixed point problem that we could describe as a *biased beliefs Markov Perfect Equilibrium* such that $P_{imt}(y|\mathbf{X}_{mt}) = \Lambda\left(y;\ \widetilde{\mathbf{v}}_{it}^{\mathbf{B}}(\mathbf{X}_{mt})\right)$ and $\mathbf{B}_{im} = \lambda_{im}\ \mathbf{P}_{jm}$. In Experiments 1A and 2A we fix $\lambda_{1m} = \lambda_{2m} = 1$ for every market m, such that beliefs are unbiased. In Experiments 1B and 2B we fix the following values for the bias parameters λ_{im}:

$$\lambda_{im} = \begin{cases} 1 & \text{if} \quad Z_{2m} \in \{-2, 2\} \\ \\ 0.5 & \text{if} \quad Z_{2m} \in \{-1, 0, 1\} \end{cases} \tag{36}$$

That is, if the exogenous characteristic Z_{2m} is at an 'extreme' value, i.e., $Z_m \in \{-2, 2\}$, then there is not any strategic uncertainty or bias beliefs: players' beliefs are in equilibrium. However, if Z_{2m}

lies in the interior of the support set, then beliefs are biased. More specifically, when beliefs are biased, both players are over-optimistic such that they underestimate (by 50%) the probability of the opponent will be active in the market. Note that given our choice of distribution of Z_{2m}, beliefs are (on average) out of equilibrium at 60% of the sample observations.[21]

Tables 6 and 7 summarize the results of our experiments. The tables report mean values and standard deviations from the Monte Carlo distribution of the estimators. As mentioned above, our interest in these experiments is threefold: to evaluate the loss of precision of our estimates when we relax the assumption of unbiased beliefs; to study the consequences of imposing the assumption of equilibrium beliefs when this assumption does not hold in the DGP; and to examine the role of the exclusion restriction in the precision of our estimates.

(a) Benchmark. Columns (1) and (2), in tables 6 and 7, present biases and standard errors of estimates when beliefs are unbiased in the DGP and we impose this restriction in the estimation. Relative to the true values of the parameters, biases and standard errors are always smaller than 5% and 10%, respectively. Therefore, we have a chosen a benchmark with quite precise estimates of payoffs and beliefs. Note that the parameter δ_1, that captures the competition effect, is more precisely estimated in the experiment with a high quality instrument: when we go from experiment 1A to 2A, the bias of the estimate declines from 3.3% to 2.6%, and the standard error from 7.8% to 6.1%. This result illustrates the importance of the exclusion restriction and the instrument to identify strategic interactions even in the model where we impose the restriction of equilibrium beliefs.

(b) Loss of precision when relaxing the assumption of unbiased beliefs. The main purpose of experiments 1A and 2A in each table is to evaluate the loss in identification power in finite sample when we do not impose the restrictions of equilibrium beliefs. With that purpose, we compare biases and standard errors in columns (3) and (4), where we relax the assumption of unbiased beliefs, with those in columns (1) and (2). The message in tables 6 and 7 is similar with respect to the identification power of equilibrium beliefs: biases and standard errors increase substantially when we do not enforce the assumption of equilibrium beliefs. For instance, for the payoff parameter δ_1 that measures the competition effect, in Table 6, the bias goes from 3.35% to 4.83%, and the standard error from 7.83% to 12.54%. The loss of precision in the estimation of the entry cost parameter is more substantial: the bias goes from 0.42% to 15.20% and the standard error from 13.30% to 22.35%. Nevertheless, when we do not impose equilibrium restrictions, the estimates are still quite informative about the true value of the parameters. Note that the loss of precision in the estimation of beliefs is substantially more severe than in the estimation of payoffs.

(c) Consequences of imposing the assumption of equilibrium beliefs when it is not true. In

[21] As a way of checking for possible coding errors in our program, we have also run all our experiments with $M = 100,000$ market observations instead of $2,000$. For the estimators of the correctly specified model, these experiments provide values of bias and standard deviations which are zero up to the fourth decimal place.

experiments 1B and 2B the DGP is such that beliefs are not in equilibrium. These experiments should help us to understand the bias induced by imposing the assumption of equilibrium beliefs incorrectly, as well as the trade-off between bias and variance in the estimation without imposing equilibrium restrictions. Thus these experiments are informative for a researcher in that the help to clearly establish the costs and benefits of imposing the assumption of equilibrium beliefs when beliefs may not in equilibrium. The bias induced by the incorrect assumption of equilibrium beliefs is substantial. For instance, in Table 6 column (5), the bias in the estimate of δ_1 is almost 25% of the true value, and for the entry cost parameter the bias is more than 62% of the true value. These biases reduce drastically when we relax the assumption of equilibrium beliefs: in column (7) of Table 6, we find that the biases of the parameters δ_1 and θ_1^{EC} become 2.1% and 2.4% of their true values, respectively. In terms of the non parametric estimates of beliefs, the message is similar: enforcing equilibrium beliefs significantly improves precision at the cost of significantly biased estimates. When we incorrectly impose equilibrium beliefs, the bias in the estimate of beliefs is roughly 100% of the true value, while the bias is usually much less than 10% if we do not impose the assumption. Though the precision of the estimates decreases significantly when we do not impose the assumption, the combination of bias and variance show that, in this example, there are very significant gains in the estimates of payoffs and beliefs when we allow for biased beliefs.

(d) Quality of the instrument. Comparing across tables 6 and 7, we see a general improvement in both the bias and precision of estimates in all cases when we go from the case of $\theta_{12}^{FC} = 0.5$ to $\theta_{12}^{FC} = 1.0$. This is sensible, particularly with respect to the improved precision. In the case of $\theta_{12}^{FC} = 1.0$ we simply have a stronger instrument for player 2's entry decision, because player 2's decision is more sensitive to the value of Z_{2m}. As such, our ability to identify the parameters of player 1's payoff function should improve. This is also illustrated in figure 1 below, where we plot the mean squared error of our estimates of the parameters of player 1's payoff function from Monte Carlo experiments with 10 different values of θ_{12}^{FC}, from $\theta_{12}^{FC} = -0.1$ to $\theta_{12}^{FC} = -1.0$. Clearly, as the instrument becomes stronger (θ_{12}^{FC} increases in absolute value), the mean squared error of the parameter estimates decreases.

6 Empirical Application

We illustrate our model and methods with an application of a dynamic game of store location. Recently there has been significant interest in the estimation of game theoretic models of market entry and store location by retail firms. Most studies have assumed static games: see Mazzeo (2002), Seim (2006), Jia (2008), Zhu and Singh (2009), and Nishida (2014), among others. Holmes (2011) estimates a single-agent dynamic model of store location by Wal-Mart. Beresteanu and Ellickson (2005), Suzuki (2013), and Walrath (2015) propose and estimate dynamic games of store location.

We study store location of McDonalds (MD) and Burger King (BK) using data for the United Kingdom during the period 1991-1995. The dataset was collected by Otto Toivanen and Michael Waterson, who use it in their paper Toivanen and Waterson (2005).[22] We divide the UK into local markets (districts) and study these companies' decision of how many stores, if any, to operate in each local market. The profits of a store in a market depends on local demand and cost conditions and on the degree of competition from other firms' stores and from stores of the same chain. There are sunk costs associated with opening a new store, and therefore this decision has implications for future profits. Firms are forward-looking and maximize the value of expected and discounted profits. Each firm has uncertainty about future demand and cost conditions in local markets. Firms also have uncertainty about the current and future behavior of the competitor. In this context, the standard assumption is that firms have rational expectations about other firms' strategies, and that these strategies constitute a Markov Perfect Equilibrium. Here we relax this assumption. The main question that we want to analyze in this empirical application is whether the beliefs of each of these companies about the store location strategy of the competitor are consistent with the actual behavior of the competitor. The interest of this question is motivated by Toivanen and Waterson (2005) empirical finding that these firms' entry decisions do not appear to be sensitive to whether the competitor is an incumbent in the market or not. As we have illustrated in our Monte Carlo experiments, imposing the restriction of equilibrium beliefs can generate an attenuation bias in the estimation of competition effects when this restriction is not true in the DGP. We investigate here this possible explanation.[23]

6.1 Data and descriptive evidence

Our working sample is a five year panel that tracks 422 *local authority districts* (local markets), including the information on the stock and flow of MD and BK stores into each district. It also contains socioeconomic variables at the district level such as population, density, age distribution, average rent, income per capita, local retail taxes, and distance to the UK headquarters of each of the firms. The *local authority district* is the smallest unit of local government in the UK, and generally consists of a city or a town sometimes with a surrounding rural area. There are almost

[22] We want to thank Otto Toivanen and Michael Waterson for generously sharing their data with us.

[23] The nature of the econometric bias in the parameters that represent strategic interactions depends on the relationship between true and rational beliefs. It is useful to illustrate this issue using a simple model. Suppose that the relationship between q and the true beliefs B is $q(x) = \pi_0 + \pi_1\ B(x)$, where π_0 and π_1 are structural parameters from the payoff function. Suppose that the relationship between actual beliefs $B(x)$ and the rational beliefs $P(x)$ is $B(x) = f(P(x))$, where $f(.)$ is a continuous and differentiable function in the space of probabilities. And suppose that the researcher imposes the restriction of rational beliefs, such that he estimates the model $q(x) = \pi_0 + \pi_1\ P(x) + e$, where by construction the error term e is equal to $\pi_1[f(P(x)) - P(x)]$. It is straightforward to show that under mild regularity conditions the least square estimator of the parameter π_1 converges in probability to $f'(E_x[P(x)])$ π_1, where $f'(.)$ is the derivative of the function $f(.)$. In this case, we have an attenuation bias in the estimator of strategic interaction if $f'(E_x[P(x)]) < 1$. Note that this condition is different to a condition on the level of the bias of beliefs. That is, the condition $f'(E_x[P(x)]) < 1$ is compatible with $E_x[B(x)] > E_x[P(x)]$

500 *local authority districts* in Great Britain. Our working sample of 422 districts does not include those that belong to Greater London.[24] The median district in our sample has an area of 300 square kilometers and a population of 95,000 people.[25] Table 8 presents descriptive statistics for socioeconomic and geographic characteristics of our sample of local authority districts.

Table 9 presents descriptive statistics on the evolution of the number of stores for the two firms.[26] In 1990, MD had more than three times the number of stores of BK, and it was active in more than twice the number of local markets than BK. Conditional on being active in a local market, MD had also significantly more stores per market than BK. These differences between MD and BK have not declined significantly over the period 1991-1995. While BK has entered in more new local markets than MD (69 new markets for BK and 48 new markets for MD), MD has opened more stores (143 new stores for BK and 166 new stores for MD).

Table 10 presents the annual transition probabilities of market structure in local markets as described by the number of stores of the two firms. According to this transition matrix, opening a new store was an irreversible decision during this sample period, i.e., no store closings are observed during this sample period. In Britain during our sample period, the fast food hamburger industry was still young and expanding, as shown by the large proportion of observations/local markets without stores (41.6%). Although there is significant persistence in every state, the less persistent market structures are those where BK is the leader. For instance, if the state is "$BK = 1$ & $MD = 0$", there is a 20% probability that the next year MD opens at least one store in the market. Similarly, when the state is "$BK = 2$ & $MD = 1$", the chances that MD opens one more store the next year are 31%.

Table 11 presents estimates of reduced form Probit models for the decision to open a new store. We obtain separate estimates for MD and BK. Our main interest is in the estimation of the effect of the previous year's number of stores (own stores and competitor's stores) on the probability of opening new stores. We include as control variables population, GDP per capita, population density, proportion of population 5-14, proportion population 15-29, average rent, and proportion of claimants of unemployment benefits. To control for unobserved local market heterogeneity we also present two fixed effects estimations, one with county fixed effects and the other with local district fixed effects. We only report estimates of the marginal effects associated with the dummy variables that represent previous year number of stores. The main empirical result from table 11 is that, regardless of the set of control variables that we use, the own number of stores has a strong

[24] The reason we exclude the districts in Greater London from our sample is that they do not satisfy the standard criteria of isolated geographic markets.

[25] As a definition of geographic market for the fast food retail industry, the district is perhaps a bit wide. However, an advantage of using district as definition of local market is that most of the markets in our sample are geographically isolated. Most districts contain a single urban area. And, in contrast to North America where many fast food restaurants are in transit locations, in UK these restaurants are mainly located in the centers of urban areas.

[26] Toivanen and Waterson present a detailed discussion of why the retail chain fast food hamburger industry in the UK during this period can be assumed as a duopoly of BK and MD.

negative effect on the probability of opening a new store but the effect of the competitor's number of stores is either negligible or even positive. This finding is very robust to different specifications of the reduced form model and it is analogous to the result from the reduced form specifications in Toivanen and Waterson (2005). Controlling for unobserved heterogeneity using fixed effects reveals that the estimate of the marginal effect of the number of own stores without fixed effects suffers from significant upward bias. However, the estimated marginal effect of the number of competitor's stores barely changes. The estimates show also a certain asymmetry between the two firms: the absence of response to the competitor's number of stores is more clear for BK than for MD. In particular, when BK has three stores in the market there is a significant reduction in MD's probability of opening a new store. That negative effect does not appear in the reduced form probit for BK.

This empirical evidence cannot be explained by a standard static model of store location by firms that sell substitute products. Here we explore three, non-mutually exclusive, explanations: (a) spillover effects; (b) forward looking behavior (dynamic game); and (c) biased beliefs about the behavior of the competitor.

(a) Spillover effects. The competitor's presence may have a positive spillover effect on the profit of a firm. There are several possible sources of this spillover effect. For example one firm may infer from another's decision to open a store in a particular market that market conditions are favorable (informational spillover effects). Alternatively, one firm may benefit from another firm's entry through cost reductions, or from product expansion through advertising. As such, we allow for the possibility of spillover effects in our specification of demand, but since we do not have price and quantity data at the level of local markets, we do not try to identify the source of the spillover effect. While the natural interpretation of the spillover effect in the context of our model is a product expansion due to an advertising effect of retail stores, this should be interpreted as a 'reduced form' specification of different possible spillover effects.

(b) Forward looking behavior. Opening a store is a partly irreversible decision that involves significant sunk costs. Therefore, it is reasonable to assume that firms are forward looking when they make this decision. Moreover, dynamic strategic effects may help explain the apparent absence of competitive effects when we study behavior in the context of a static model of entry. Suppose that firms anticipate, with some uncertainty, the total number of hamburger stores that a local market can sustain in the long-run given the size and the socioeconomic characteristics of the market. For simplicity, suppose that this number of "available slots" does not depend on the ownership of the stores because the products sold by the two firms are very close substitutes. In this context, firms play a game where they 'race' to fill as many 'slots' as possible with their own stores. Diseconomies of scale and scope may generate a negative effect of the own number of stores on the decision of opening new stores. However, in this model, during most of the period of expansion the number

of slots of the competitor has zero effect on the decision of opening a new store. Only when the market is filled or close to being filled do the competitor's stores have a significant effect on entry decisions.

(c) Biased beliefs. Competition in actual oligopoly industries is often characterized by strategic uncertainty. Firms face significant uncertainty about the strategies of their competitors. Although MD and BK should know a lot about each others strategies from a long history of play, the UK represented a relatively new market. So while MD and BK likely know the possible strategies and thus the set of potential equilibria, the firms are competing for the first time in a new setting and may have not been sure, particularly during the initial stages of competition, which of the equilibria would be played by the opponent. While the possible equilibrium best responses are common knowledge, there is strategic uncertainty about which of these will be played. In the context of our application, it may be the case that MD's or/and BK's beliefs overestimate the negative effect of the competitor's stores on the competitor's entry decisions. For instance, if MD has one store in a local market, BK may believe that the probability that MD opens a second store is close to zero. These over-optimistic beliefs about the competitor's behavior may generate an apparent lack of response of BK's entry decisions to the number of MD's stores.

6.2 Model

Consider two retail chains competing in a local market. Each firm sells a differentiated product using its stores. Let $K_{imt} \in \{0, 1, ..., |\mathcal{K}|\}$ be the state variable that represents the number of stores of firm i in market m at period $t-1$. And let $Y_{imt} \in \{0, 1, ..., A-1\}$ be the number of *new* stores that firm i opens in the market during period t.[27] Following the empirical evidence during our sample period, we assume that opening a store is an irreversible decision. Also, for almost all the observations in the data we have that $Y_{imt} \in \{0, 1\}$, and therefore we consider a binary choice model for Y_{imt}, i.e., $A = 2$.

The total number of stores of firm i in market m at period t is $N_{imt} \equiv K_{imt} + Y_{imt}$. Firm i is active in the market at period t if N_{imt} is strictly positive. Every period, the two firms know the 'stocks' of stores in the market, K_{imt} and K_{jmt}, and simultaneously choose the new (additional) number of stores, Y_{imt} and Y_{jmt}. Firm i's total profit function is equal to variable profits minus entry costs and minus fixed operating costs: $\Pi_{imt} = VP_{imt} - EC_{imt} - FC_{imt}$.

The specification of the variable profit function is:

$$VP_{imt} = (\mathbf{W}_m \ \gamma) \ N_{imt} \ \left[\theta_{0i}^{VP} + \theta_{can,i}^{VP} N_{imt} + \theta_{com,i}^{VP} N_{jmt}\right] \tag{37}$$

$\mathbf{W}_m$ is a vector of exogenous market characteristics such as population, population density, percentage of population in age group 15-29, GDP per capita, and unemployment rate. γ is a vector

[27] We abstract from store location within a local market and assume that every store of the same firm has the same demand.

of parameters where the coefficient associated to the Population variable in $\mathbf{W}_{mt}$ is normalized to one. Therefore, the index $\mathbf{W}_m\gamma$ is measured in number of people and we interpret it as "market size". According to this specification, the term $\theta_{0i}^{VP} + \theta_{can,i}^{VP}\ N_{imt} + \theta_{com,i}^{VP}\ N_{jmt}$ represents variable profits per-capita and per-store. $\theta_{0i}^{VP} + \theta_{can,i}^{VP}$ is the variable profit (per capita) when firm i has a single store in the market. The term $\theta_{can,i}^{VP} N_{imt}$ captures cannibalization effects between stores of the same chain as well as possible economies of scale and scope in variable costs. The term $\theta_{com,i}^{VP} N_{jmt}$ captures the effect of competition from the other chain.

Entry cost have the following form:

$$EC_{imt} \quad = \quad 1\{Y_{imt} > 0\} \ \left[\theta_{0i}^{EC} + \theta_{K,i}^{EC}\ 1\{K_{imt} > 0\} + \theta_{Z,i}^{EC}\ Z_{imt} + \varepsilon_{it}\right] \tag{38}$$

$1\{.\}$ is the indicator function, and θ_{0i}^{EC}, $\theta_{K,i}^{EC}$, and $\theta_{Z,i}^{EC}$ are parameters. θ_{0i}^{EC} is an entry cost that is paid the first time that the firm opens a store in the local market. $\theta_{0i}^{EC} + \theta_{K,i}^{EC}$ is the cost of opening a new store when the firm already has stores in the market. If there are economies of scope in the operation of multiple stores in a market, we expect the parameter $\theta_{K,i}^{EC}$ to be negative such that the entry cost of the first store is greater than the entry cost of additional stores. Z_{imt} represents the geographic distance between market m and the closest market where firm i has stores at period $t-1$ (i.e., Z_{imt} is zero if $K_{imt} > 0$). The term $\theta_{Z,i}^{EC}\ Z_{imt}$ tries to capture economies of density as in Holmes (2011). The random variable ε_{it} is a private information shock in the cost of opening a new store, and it is i.i.d. normally distributed.[28]

The specification of fixed costs is:

$$FC_{imt} \quad = \quad 1\{(N_{imt}) > 0\} \ \left[\theta_{0i}^{FC} + \theta_{lin,i}^{FC}(N_{imt}) + \theta_{qua,i}^{FC}(N_{imt})^2\right] \tag{39}$$

θ_{0i}^{FC} is a lump-sum cost associated with having any positive number of stores in the market. The term $\theta_{lin,i}^{FC}\ (N_{imt})\ + \theta_{qua,i}^{FC}\ (N_{imt})^2$ takes into account that operating costs may increase (or decline) with the number of stores in a quadratic form.

Given this specification, the vector of state variables involved in the exclusion restriction of Assumption ID-3 is $S_{imt} = (K_{imt}, Z_{imt})$. A firm's variable profit depends on his own and his opponents current number of stores in the market, and also on his own stock of stores at previous period, K_{imt}, and on the distance from market m to the closest store of the chain at year $t-1$. These two variables, K_{imt} and Z_{imt}, affect the entry cost of firm i in market m. However, the competitors' number of stores in the previous year, and the distance from market m to the closest store of the competitor in the previous year, do not directly affect the current profit of the firm. This satisfies the exclusion restriction in assumption ID-3. Of course a firm's beliefs about the probability distribution of the opponents' choice, Y_{jmt}, depend on $S_{jmt} = (K_{jmt}, Z_{jmt})$.

[28] Here we assume that the entry decision is made and the entry cost is paid at the same year that the store opens and starts operating in the market. In other words, we assume there is no "time-to-build", or at least that it is substantially shorter than one year. This timing assumption is quite realistic for franchise stores of large retail chains.

The maximum value of K_{imt} in the sample is 13, but K_{imt} is less than or equal to three for 99% of the observations in the sample. We assume that the set of possible values of K_{imt} is $\{0, 1, 2, 3\}$, where $K_{imt} = 3$ represents a number of stores greater or equal than three. When $K_{imt} = 3$, we impose the restriction that firm i does not open additional stores in this market: $P_{imt}(1|\mathbf{X}_{mt}$ with $K_{imt} = 3) = 0$. The variable Z_{imt}, that represents the distance to the closest chain store, is discretized into 8 cells of 30 miles intervals: $Z_{imt} = 1$ represents a distance of less than 30 miles, $Z_{imt} = 2$ for a distance of between 30 and 60 miles, ..., $Z_{imt} = 7$ for a distance of between 180 and 210 miles, and $Z_{imt} = 8$ for a distance greater than 210 miles. Market characteristics in the vector $\mathbf{W}_m$ have very little time variability in our sample and we treat them as time invariant state variables in order to reduce the dimensionality of the state space.[29] Therefore, the set $\mathcal{S}$ is equal to $\{0, 1, 2, 3\} \times \{1, 2, ..., 8\}$ and it has 32 grid points, and the whole state space $\mathcal{X}$ is equal to $\mathcal{S} \times \mathcal{S}$ and it has $1,024$ points.

Assumption ID-4, which restricts beliefs over a subset of the state space, takes the following form in this application. We assume that the two firms have unbiased beliefs about the entry behavior of the opponent in markets which are relatively close to the opponents network, i.e., for small values of the distance Z_{jmt}. However, beliefs may be biased for markets that are farther away to the opponent's network. More formally, we assume that:

$$B_{imt}(y_j|\mathbf{X}_{mt}) = P_{jmt}(y_j|\mathbf{X}_{mt}) \qquad \text{if } Z_{jmt} \leq Z^* \tag{40}$$

We have estimated the model for different values of Z^*. The main intuition behind this assumption is that markets that are far away from a firm's network are unexplored markets for which there is more strategic uncertainty.

The selection of the points in the support of Z where we impose the restriction of unbiased beliefs is based on the three criteria that we have proposed in section 3.2.6 above. *Criterion of 'most visited states'*: the most common markets are those with a closer distance to the firms' network of stores, i.e., with smaller values of the special state variable Z. *Criterion 'testing for the monotonicity of beliefs and using this restriction'*: the probabilities of market entry for BK and MD are strictly decreasing in their own distance variable Z. Furthermore, we cannot reject the monotonicity of beliefs with respect to this special variable. According to this criterion, we could impose unbiased beliefs either at the smallest or at the largest values in the support of variable Z. *Criterion 'minimization of the player's beliefs bias'*: we have estimated the model under different selections for the points in the support of Z where we impose unbiased beliefs. In table 12, we present estimates under two different selections: $Z \in \{0, 1\}$ and $Z \in \{0, 1, 2\}$. The estimation results

[29] For those market characteristics with some time variation, we fix their values at their means over the sample period. We have also estimated the model using different values, such as the value at the first year in the sample, or at the last year (i.e., perfect forecast), and all the estimated parameters did not change up to the fourth significant digit.

are very similar under these two selections. We have also estimated the model imposing unbiased beliefs at the largest values of Z, i.e., $Z \in \{6, 7, 8\}$. The estimation results were quite different. In particular, we obtained substantially larger biases for beliefs. Therefore, a conservative criterion, based on minimizing the deviation with respect to the paradigm of unbiased beliefs, recommends imposing the restriction of unbiased beliefs at small values of the special state variable.

Our assumption on players' beliefs implies that the degree of bias in firms' beliefs declines over time with the geographic expansion of these retail chains. Eventually, when the retail chains have sufficiently expanded geographically, the beliefs of firms become unbiased for every market and state. More formally, with probability one, there is a period in the future, say t^* with $t^* < \infty$, such that for any $t \geq t^*$, any market m, and any firm j, we have that $Z_{jmt} \leq Z^*$. It is straightforward to check if condition $Z_{jmt} \leq Z^*$ is satisfied for every market and firm in the data after some year in the sample, such that we can say that the sample period includes year t^*. For our choices of Z^*, this condition is almost, but not exactly, satisfied in the last year of our sample, 1995.

6.3 Estimation of the structural model

Table 12 presents estimates of the dynamic game under three different assumptions on beliefs. Columns (1) and (2) present estimates under the assumption that beliefs are unbiased for every value of the state variables. In columns (3) and (4), we impose the restriction of unbiased beliefs only when the distance to the competitor's network is shorter than 60 miles, i.e., $Z^* = 2$. In columns (5) and (6), beliefs are unbiased when that distance is shorter than 30 miles , i.e., $Z^* = 1$. For each of these three scenarios, the proportion of observations at year 1995 for which we impose the restriction of unbiased beliefs is 100%, 38%, and 29%, respectively.

(a) Estimation with unbiased beliefs. The estimation shows substantial differences between estimated parameters in the variable profit function of the two firms. The parameter θ_{can}^{VP} is negative and significant for BK but positive and also statistically significant for MD. Cannibalization effects dominate in the case of BK. In contrast, economies of scope in variable profits seem important for MD. The estimates of the parameter that captures the competitive effect, θ_{com}^{VP}, are smaller in magnitude than the estimates of θ_{can}^{VP}, but they are statistically significant. According to these estimates the competitive effect of MD's market presence on BK's profits is smaller than the reverse effect.

The estimates of fixed cost parameters illustrates a similarity across firms in the structure of fixed costs of operation. The fixed operating cost increases linearly, not quadratically, with the number of stores, and the lump-sum component of the cost is relatively small. However, there a substantial economic differences between the firms in the magnitude of these costs. The fixed cost that BK pays per additional store is almost twice the fixed cost MD pays.

Entry costs are particularly important in this setting because they play a key role in the identification of the dynamic game, through the exclusion restrictions. The estimates of these costs are very significant, both statistically and economically. Entry costs depend significantly on the number of installed stores of the firm, K, and on the distance to the firm's network, Z. The signs of these effects, negative for θ_K^{EC} and positive for θ_Z^{EC}, are consistent with the existence of economies of scope and density between the stores of the same chain. McDonalds has smaller entry costs, and a larger absolute value of the parameter θ_K^{EC}, which indicates that there are stronger economies of scope in the network of McDonalds stores.

In summary, the estimated model with unbiased beliefs shows significant differences in the variable profits and entry costs of the firms. Cannibalization is stronger between BK stores, while MD exhibits substantial economies of scope both in variables profits and entry costs. Competition effects seem relatively weak but statistically significant.

(b) Tests of unbiased beliefs. Our test of unbiased beliefs clearly rejects the null hypothesis for BK, with a p-value of 0.00029, though we cannot reject the hypothesis of unbiased beliefs for MD.[30]

(c) Estimation with biased beliefs. As expected, (bootstrap) standard errors increase significantly when we estimate the model allowing for biased beliefs. Nevertheless, these standard errors are not large and the estimation provides informative and meaningful results. Comparing these parameter estimates with those in the model with equilibrium restrictions, the most important changes are in the parameters of variable profits of BK. In particular, the estimate of the parameter that measures the competitive effect of MD on BK is now more than twice the initial estimate with equilibrium beliefs. In contrast to the result with unbiased beliefs, we find that the competitive effect of MD on BK is stronger that the effect of BK on MD. This result is consistent with the findings in our Monte Carlo experiments: imposing the restriction of unbiased beliefs when it is incorrect introduces a "measurement error" in beliefs which in turn generates an attenuation bias in the estimate of the parameter associated with the strategic interactions. For the identification of this structural parameter the sample variation in beliefs plays an important role.

Interestingly, BK's estimated profit function has a lower level when we allow for biased beliefs than when we enforce unbiased beliefs: variable profits are lower, and fixed costs and entry costs are larger. This is fully consistent with our finding that the bias in BK's beliefs are mostly in the direction of underestimating the true probability that MD will enter in unexplored markets. If we impose the assumption of unbiased beliefs, BK's profit must be relatively high in order to rationalize entry into markets where MD is also likely to enter or to expand its number of stores. Once we take into account the over-optimistic beliefs of BK about the behavior of MD, revealed preference shows that BK profits are not as high as before. In fact, in the estimates that allow for biased beliefs we find that the differences in the profit function of MD and BK are even larger.

[30] To implement this test we use a vector $\widehat{\delta}_i = \{\widehat{\delta}_i(S_i) : S_i \in \mathcal{S}\}$ of $|\mathcal{S}| = 32$ statistics.

7 Conclusion

This paper studies a class of dynamic games of incomplete information where players' beliefs about the other players' actions may not be in equilibrium. We present new results on identification, estimation, and inference of structural parameters and beliefs in this class of games when the researcher does not have data on elicited beliefs, or these data are limited to players' beliefs at only some values of the state variables. Specifically, we derive sufficient conditions under which payoffs and beliefs are point identified. These conditions then lead naturally to a sequential estimator of payoffs and beliefs. We also present a procedure for testing the null hypothesis that beliefs are in equilibrium. We illustrate our model and methods using both Monte Carlo experiments and an empirical application of a dynamic game of store location by McDonalds and Burger King. They key conditions for the identification of beliefs and payoffs in our application are the following. The first condition is an exclusion restriction in a firm's profit function that establishes that the previous year's network of stores of the competitor does not have a direct effect on the profit of a firm, but the firm's own network of stores at previous year does affect its profit through the existence of sunk entry costs and economies of density in these costs. The second condition restricts firms' beliefs to be unbiased in those markets that are close, in a geographic sense, to the opponent's network of stores. However, beliefs are unrestricted, and potentially biased, for unexplored markets which are farther away from the competitors' network. Our estimates show significant evidence of biased beliefs for Burger King. We find that Burger King underestimated the probability of entry of McDonalds in markets that were relatively far away from McDonalds' network of stores. Furthermore, we find that imposing the restriction of unbiased beliefs, when this restriction is rejected, generates a substantial attenuation bias in the estimation of the competition effects.

APPENDIX

[A.1] Aradillas-Lopez and Tamer's bounds approach in dynamic games

The purpose of this part of the appendix is to explain why Aradillas-Lopez and Tamer's bounds approach, while useful for identification and estimation of static binary choice games, has very limited applicability to dynamic games. Aradillas-Lopez and Tamer (2008) (we use the abbreviation ALT from now on) consider a static, two-player, binary-choice game of incomplete information. The model they consider can be seen as a specific case of our framework. To see this, consider the final period of the game T in our model. For the sake of notational simplicity, we omit here the vector of state variables $\mathbf{X}$ as an argument of payoff and belief functions. At the last period T, the decision problem facing the players is equivalent to that of a static game. At period T there is no future and the difference between the conditional choice value functions is simply the difference between the conditional choice current profits. For the binary choice game, there is only one difference between current profits: $\pi_{iT}^{\mathbf{B}}(1) - \pi_{iT}^{\mathbf{B}}(0)$. Taking into account that the game has only two players, we have $\pi_{iT}^{\mathbf{B}}(1) - \pi_{iT}^{\mathbf{B}}(0)$ is equal to $B_{iT}(0)\ [\pi_{iT}(1,0) - \pi_{iT}(0,0)] + B_{iT}(1)\ [\pi_{iT}(1,1) - \pi_{iT}(0,1)]$. Therefore, the Best Response Probability Function (BRP) function is:

$$P_{iT}(1) = \Lambda\left(\ B_{iT}(0)\ [\pi_{iT}(1,0) - \pi_{iT}(0,0)] + B_{iT}(1)\ [\pi_{iT}(1,1) - \pi_{iT}(0,1)]\ \right) \tag{A.1.1}$$

ALT assume that players' payoffs are submodular in players' decisions (Y_i,Y_j), i.e., for every value of the state variables $\mathbf{X}$, we have that $[\pi_{it}(1,0) - \pi_{it}(0,0)] > [\pi_{it}(1,1) - \pi_{it}(0,1)]$. Under this restriction, they derive informative bounds around players' conditional choice probabilities when players are *level-k rational*, and show that the bounds become tighter as k increases. For instance, without further restrictions on beliefs (i.e., rationality of level 1), player i's conditional choice probability $P_{iT}(1)$ takes its largest possible value when $B_{iT}(1) = 0$, and it takes its smallest possible value when beliefs are $B_{iT}(0) = 1$. This result yields informative bounds on the period T choice probabilities of player i:

$$\Lambda\left(\pi_{iT}(1,1) - \pi_{iT}(0,1)\right) \leq P_{iT}(1) \leq \Lambda\left(\pi_{iT}(1,0) - \pi_{iT}(0,0)\right) \tag{A.1.2}$$

These bounds on conditional choice probabilities can be used to "set-identify" the structural parameters in players' preferences.

In their setup, the monotonicity of players' payoffs in the decisions of other players implies monotonicity of players' BRP functions in the beliefs about other players actions. This type of monotonicity is very convenient in their approach, not only from the perspective of identification, but also because it yields a very simple approach to calculate upper and lower bounds on conditional choice probabilities. In particular, the maximum and minimum possible values of the CCPs are reached when the belief probability is equal to 0 or 1, respectively. Unfortunately, this property does not extend to dynamic games, even the simpler ones. We now discuss this issue.

Consider the two-players, binary-choice, dynamic game at some period t smaller than T. To obtain bounds on players' choice probabilities analogous to the ones obtained at the last period, we need to find, for every value of the state variables $\mathbf{X}$, the value of beliefs $\mathbf{B}$ that generate the smallest (and the largest) values of the best response probability $\Lambda(v_{it}^{\mathbf{B}}(1,\mathbf{X}) - v_{it}^{\mathbf{B}}(0,\mathbf{X}))$. That is, we need to minimize (or maximize) this best response probability with respect to the vector of beliefs $\{B_{it},\ B_{it+1},\ ...,B_{iT}\}$. Without making further assumptions, this best response function is not monotonic in beliefs at every possible state. In fact, this monotonicity is only achieved under very strong conditions not only on the payoff function but also on the transition probability of the state variables and on belief functions themselves.

Therefore, in a dynamic game, to find the largest and smallest value of a best response (and ultimately the bounds on choice probabilities) at periods $t < T$, one needs to explicitly solve a non-trivial optimization problem. In fact, the maximization (minimization) of the BRP function with respect to beliefs is a extremely complex task. The main reason is that the best response probability evaluated at a value of the state variables depends on beliefs at every period in the future and at every possible value of the state variables in the future. Therefore, to find bounds on best responses we must solve an optimization problem with a dimension equal to the number of values in the space of state variables times the number of future periods. This is because, in general, the maximization (or minimization) of a best response with respect to beliefs does not have a time-recursive structure except under very special assumptions (see Aguirregabiria, 2008). For instance, though $B_{iT}(1|\mathbf{X}_T) = 0$ maximizes the best response at the last period T, in general the maximization of a best response at period $T-1$ is not achieved setting $B_{iT}(1|\mathbf{X}_T) = 0$ for any value of $\mathbf{X}_T$. More generally, the beliefs from period t to T that provide the maximum (minimum) value of the best response at period t are not equal to the beliefs from period t to T that provide the maximum (minimum) value of the best response at $t-1$. So at each point in time we need to re-optimize with respect to beliefs about strategies at every period in the future. That is, while the optimization of expected and discounted payoffs has the well-known time-recursive structure, the maximization (or minimization) of the value of BRP functions does not.

[A.2] Integrated Value Function and Continuation Values

Our proofs of Propositions 1 and 2 apply the concepts of *integrated value function* and *continuation value function* as well as recursive formulas to calculate these functions. The *integrated value function* is defined as $\bar{V}^{\mathbf{B}}_{it}(\mathbf{X}_t) \equiv \int V^{\mathbf{B}}_{it}(\mathbf{X}_t, \varepsilon_{it})\ dG_{it}(\varepsilon_{it})$ (see Rust, 1994). Applying this definition to the Bellman equation, we obtained the *integrated Bellman equation:*

$$\begin{aligned}\bar{V}^{\mathbf{B}}_{it}(\mathbf{X}_t) &= \int \max_{y_i \in \mathcal{Y}} \left\{\ v^{\mathbf{B}}_{it}(y_i, \mathbf{X}_t) + \varepsilon_{it}(y_i)\ \right\}\ dG_{it}(\varepsilon_{it}) \\ &= \int \max_{y_i \in \mathcal{Y}} \left\{\ \pi^{\mathbf{B}}_{it}(y_i, \mathbf{X}_t) + \beta \sum_{\mathbf{X}_{t+1}} \bar{V}^{\mathbf{B}}_{it+1}(\mathbf{X}_{t+1})\ f^{\mathbf{B}}_{it}(\mathbf{X}_{t+1}|y_i, \mathbf{X}_t) + \varepsilon_{it}(y_i)\ \right\} dG_{it}(\varepsilon_{it})\end{aligned} \tag{A.2.1}$$

If $\{\varepsilon_{it}(0), \varepsilon_{it}(1), ..., \varepsilon_{it}(A-1)\}$ are i.i.d. extreme value type 1, the integrated Bellman equation has the following closed-form expression:

$$\bar{V}^{\mathbf{B}}_{it}(\mathbf{X}_t) = \ln\left(\sum_{y_i \in \mathcal{Y}} \exp\left\{v^{\mathbf{B}}_{it}(y_i, \mathbf{X}_t)\right\}\right) \tag{A.2.2}$$

If we knew payoffs and beliefs, we could use this formula to obtain the integrated value function by backwards induction, starting at the last period T where $\bar{V}^{\mathbf{B}}_{iT}(\mathbf{X}) = \ln\left(\sum_{y_i\in\mathcal{Y}} \exp\left\{\pi^{\mathbf{B}}_{it}(y_i,\mathbf{X}_t)\right\}\right)$.

The *continuation value function* provides the expected and discounted value of *future* payoffs given future beliefs of player i and current choices of all the players. It is defined as:

$$c^{\mathbf{B}}_{it}(\mathbf{Y}_t,\mathbf{X}_t) \equiv \beta \sum_{\mathbf{X}_{t+1}} \bar{V}^{\mathbf{B}}_{it+1}(\mathbf{X}_{t+1})\ f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t,\mathbf{X}_t) \tag{A.2.3}$$

Note that continuation values $c^{\mathbf{B}}_{it}$ depend on beliefs at periods $t+1$ and later, but not on beliefs at period t. By definition, the relationship between the conditional choice value function $v^{\mathbf{B}}_{it}$ and the continuation value function $c^{\mathbf{B}}_{it}$ is the following:

$$v^{\mathbf{B}}_{it}(y_i,\mathbf{X}_t) \quad = \sum_{\mathbf{y}_{-i}\in\mathcal{Y}^{N-1}} \left[\pi_{it}(y_i,\mathbf{y}_{-i},\mathbf{X}_t) + c^{\mathbf{B}}_{it}(y_i,\mathbf{y}_{-i},\mathbf{X}_t)\right]\ B_{it}(\mathbf{y}_{-i}|\mathbf{X}_t) \tag{A.2.4}$$

[A.3] Proof of Proposition 2

[Part (i): Identification of payoffs] The restrictions of the model that come from best response behavior of player i can be represented using the following equation. For any $(y_i,\mathbf{X})\in\mathcal{Y}\times\mathcal{X}$,

$$q_{it}(y_i,\mathbf{X}) \quad = \mathbf{B}_{it}(\mathbf{X})'\ \left[\boldsymbol{\pi}_{it}(y_i,\mathbf{X}) + \widetilde{\mathbf{c}}^{\mathbf{B}}_{it}(y_i,\mathbf{X})\right] \tag{A.3.1}$$

where $\mathbf{B}_{it}(\mathbf{X})$, $\boldsymbol{\pi}_{it}(y_i,\mathbf{X})$, and $\widetilde{\mathbf{c}}^{\mathbf{B}}_{it}(y_i,\mathbf{X})$ are vectors with dimension $A^{N-1}\times 1$ containing beliefs, payoffs, and continuation values, respectively, for every possible value of $\mathbf{y}_{-i}$ in the set $\mathcal{Y}^{N-1}$. Let $\mathcal{S}^{(R)}_{-i}$ be the set $[\mathcal{S}^{(R)}]^{N-1}$. By assumption ID-4, for any $\mathbf{X}$ such that $\mathbf{S}_{-i}\in\mathcal{S}^{(R)}_{-i}$ we have that $B_{it}(\mathbf{y}_{-i}|\mathbf{X}) = P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ and $P_{-it}(\mathbf{y}_{-i}|\mathbf{X})$ is known to the researcher. Consider the system of equations formed by equation (A.3.1) at a fixed value of $(y_i,S_i,\mathbf{W})$ and for every value of $\mathbf{S}_{-i}$ in $\mathcal{S}^{(R)}_{-i}$. This is a system of R^{N-1} equations, and we can represent this system in vector form using the following expression:

$$\tilde{\mathbf{q}}^{(R)}_{it}(y_i,S_i) \quad = \mathbf{P}^{(R)}_{-it}(S_i)\ \boldsymbol{\pi}_{it}(y_i,S_i) \tag{A.3.2}$$

where: $\boldsymbol{\pi}_{it}(y_i,S_i)$ is the $A^{N-1}\times 1$ vector $\{\pi_{it}(y_i,\mathbf{y}_{-i},\mathbf{S}_{-i}) : \mathbf{y_{-i}}\in\mathcal{Y}^{\mathbf{N-1}}\}$; $\mathbf{P}^{(R)}_{-it}(S_i)$ is the $R^{N-1}\times A^{N-1}$ matrix $\{P_{-it}(\mathbf{y}_{-i}|S_i,\mathbf{S}_{-i}) : \mathbf{y_{-i}}\in\mathcal{Y}^{\mathbf{N-1}},\ \mathbf{S}_{-i}\in\mathcal{S}^{(R)}_{-i}\}$; and $\tilde{\mathbf{q}}^{(R)}_{it}(y_i,S_i)$ is the $R^{N-1}\times 1$ vector with elements $\{\widetilde{q}_{it}(y_i,S_i,\mathbf{S}_{-i},\mathbf{W}) : \mathbf{S}_{-i}\in\mathcal{S}^{(R)}_{-i}\}$ where

$$\widetilde{q}_{it}(y_i,\mathbf{X}) \equiv q_{it}(y_i,\mathbf{X}) - \sum_{\mathbf{y}_{-i}\in\mathcal{Y}^{N-1}} P_{-it}(\mathbf{y}_{-i}|\mathbf{X})\ c^{\mathbf{B}}_{it}(y_i,\mathbf{y}_{-i},\mathbf{X})$$

Under condition (i) in Proposition 2, matrix $\mathbf{P}^{(R)}_{-it}(S_i)'\mathbf{P}^{(R)}_{-it}(S_i)$ is non-singular and therefore we can solve for vector $\boldsymbol{\pi}_{it}(y_i,S_i)$ in the previous system of equations:

$$\boldsymbol{\pi}_{it}(y_i,S_i) = \left[\mathbf{P}^{(R)}_{-it}(S_i)'\ \mathbf{P}^{(R)}_{-it}(S_i)\right]^{-1} \mathbf{P}^{(R)}_{-it}(S_i)'\ \tilde{\mathbf{q}}^{(R)}_{it}(y_i,S_i) \tag{(A.3.3)}$$

This expression shows that, given continuation values at period t, the vector of payoffs $\boldsymbol{\pi}_{it}(y_i,S_i)$ is identified, i.e., part (i) of Proposition 2.

[Part (ii): Identification of beliefs] Now, we show the identification of the beliefs function for states outside the subset $\mathcal{S}_{-i}^{(R)}$. Again, we start with the system equations implied by the best response restrictions, but now we take into account that the vector $\boldsymbol{\pi}_{it}(y_i, S_i)$ is identified and then look at the identification of beliefs at states $\mathbf{X}$ with $\mathbf{S}_{-i}$ outside the subset $\mathcal{S}_{-i}^{(R)}$. We stack equation (A.3.1) for every value of $y_i \in \mathcal{Y} - \{0\}$ to obtain a system of equations. Note that $\mathbf{B}_{it}(\mathbf{X})$ is a vector of A^{N-1} probabilities, one element for each value of $\mathbf{y}_{-i}$ in $\mathcal{Y}^{N-1}$. The probabilities in this vector should sum to one, and therefore, $\mathbf{B}_{it}(\mathbf{X})$ satisfies the restriction $\mathbf{1}'\mathbf{B}_{it}(\mathbf{X}) = 1$, where $\mathbf{1}$ is a vector of ones. Therefore, we have the following system of A equations:

$$\mathbf{q}_{it}(\mathbf{X}) \quad = \widetilde{\mathbf{V}}_{it}(\mathbf{X}) \ \mathbf{B}_{it}(\mathbf{X}) \tag{A.3.4}$$

$\mathbf{q}_{it}(\mathbf{X})$ is an $A \times 1$ vector with elements $\{q_{it}(1, \mathbf{X}), ..., q_{it}(A-1, \mathbf{X})\}$ at rows 1 to $A-1$, and a 1 at the last row. And $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is an $A \times A^{N-1}$ matrix with elements:

$$\pi_{it}(y_i, \mathbf{y}_{-i}, S_i, \mathbf{W}) + [c_{it}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{X}) - c_{it}^{\mathbf{B}}(0, \mathbf{y}_{-i}, \mathbf{X})] \tag{A.3.5}$$

and the last row of the matrix is a row of ones. When $N = 2$, matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is an $A \times A$ matrix, and condition (ii) in Proposition 2 implies that this matrix is non-singular. Therefore, if continuation values at period t are known to the researcher, we can identify beliefs in the vector $\mathbf{B}_{it}(\mathbf{X})$ as:

$$\mathbf{B}_{it}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{it}(\mathbf{X})\right]^{-1} \mathbf{q}_{it}(\mathbf{X}) \tag{A.3.6}$$

Now, we prove that condition (ii) implies that matrix $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ is non-singular. Our proof of part (i) implies that:

$$\widetilde{\mathbf{V}}_{it}(\mathbf{X}) = \mathbf{Q}_{it}^{(R)}(S_i) \ \mathbf{P}_{-it}^{(R)}(S_i) \ \left[\mathbf{P}_{-it}^{(R)}(S_i)' \ \mathbf{P}_{-it}^{(R)}(S_i)\right]^{-1} \tag{A.3.7}$$

and $\mathbf{Q}_{it}^{(R)}(S_i)$ is the $A \times R$ matrix with $\mathbf{q}_{it}^{(R)}(y_i, S_i)'$ at the first $A-1$ rows, and ones at the last row. By Assumption ID-4, $\mathbf{P}_{-it}^{(R)}(S_i)$ is full column rank, and then a sufficient condition for $\widetilde{\mathbf{V}}_{it}(\mathbf{X})$ to be non-singular matrix is that $\mathbf{Q}_{it}^{(R)}(S_i)$ has rank A, which is a condition in part (ii) of Proposition 2.

[Full identification] Given parts (i) and (ii) of Proposition 2, it is straightforward to show, using backwards induction, the identification of payoffs and beliefs at every period t. At the last period T, continuation values are zero, and therefore $\boldsymbol{\pi}_{iT}$ and $\mathbf{B}_{iT}$ are identified as:

$$\boldsymbol{\pi}_{iT}(y_i, S_i) = \left[\mathbf{P}_{-iT}^{(R)}(S_i)' \ \mathbf{P}_{-iT}^{(R)}(S_i)\right]^{-1} \mathbf{P}_{-iT}^{(R)}(S_i)' \ \mathbf{q}_{iT}^{(R)}(y_i, S_i) \tag{A.3.8}$$

and

$$\mathbf{B}_{iT}(\mathbf{X}) = \left[\widetilde{\mathbf{V}}_{iT}(\mathbf{X})\right]^{-1} \mathbf{q}_{iT}(\mathbf{X}) \tag{A.3.9}$$

For any period $t < T$, given payoffs, beliefs, and continuation values at period $t+1$, we can construct continuation values at period t. First, we obtain conditional choice value functions at period $t+1$:

$$v_{it+1}^{\mathbf{B}}(y_i, \mathbf{X}) \quad = \sum_{\mathbf{y}_{-i} \in \mathcal{Y}^{N-1}} \left[\pi_{it+1}(y_i, \mathbf{y}_{-i}, \mathbf{X}) + c_{it+1}^{\mathbf{B}}(y_i, \mathbf{y}_{-i}, \mathbf{X})\right] \ B_{it+1}(\mathbf{y}_{-i}|\mathbf{X}) \tag{A.3.10}$$

Second, we obtain the integrated value function at period $t+1$:

$$\bar{V}^{\mathbf{B}}_{it+1}(\mathbf{X}) \;\; = \;\; \ln \left(\sum_{y_i \in \mathcal{Y}} \exp\left\{ v^{\mathbf{B}}_{it+1}(y_i, \mathbf{X}) \right\} \right) \tag{A.3.11}$$

And finally, we calculate the continuation values at period t:

$$c^{\mathbf{B}}_{it}(\mathbf{Y}_t, \mathbf{X}_t) \;\; = \;\; \beta \sum\nolimits_{\mathbf{X}_{t+1} \in \mathcal{X}} \bar{V}^{\mathbf{B}}_{it+1}(\mathbf{X}_{t+1}) \; f_t(\mathbf{X}_{t+1} | \mathbf{Y}_t, \mathbf{X}_t) \tag{A.3.12}$$

Given these continuation values, we apply the formulas in ((A.3.3)) and (A.3.6) to obtain payoffs and beliefs at t. By using backwards induction we identify beliefs and payoff functions at every period t. ■

[A.4] Proof of Proposition 1

The proof has two parts. First, we show that given CCPs of player i only, it is possible to identify a function that depends on beliefs of players but not on payoffs. Second, under the assumption of equilibrium beliefs, the identified function of beliefs can be also identified using only CCPs of player j. Therefore, we have identified the same object using two different sources of data. If the hypothesis of equilibrium beliefs is correct, the two approaches should give us the same result, but if beliefs are biased the two approaches provide different results. This can be used to construct a test statistic.

There are $N = 2$ players, i and j, the vector of state variables $\mathbf{X}$ is $(S_i, S_j, \mathbf{W})$, and players' actions are y_i and y_j. Under the condition in Proposition 1 that the transition of the state variables has the form $f_t(\mathbf{X}_{t+1}|\mathbf{Y}_t, \mathbf{W}_t)$, we have that continuation values $c^{\mathbf{B}}_{it}(\mathbf{Y}_t, \mathbf{X}_t)$ do not depend on $\mathbf{S}_t$. Therefore, the restrictions of the model can be written as:

$$q_{it}(y_i, \mathbf{X}) \;\; = \mathbf{B}_{it}(\mathbf{X})' \; \widetilde{\mathbf{v}}^{\mathbf{B}}_{it}(y_i, S_i, \mathbf{W}) \tag{A.4.1}$$

where $\mathbf{B}_{it}(\mathbf{X})$ is the $A \times 1$ vector defined above, and $\widetilde{\mathbf{v}}^{\mathbf{B}}_{it}(y_i, S_i, \mathbf{W})$ is the $A \times 1$ vector with elements $\{\pi_{it}(y_i, y_j, S_i, \mathbf{W}) + \widetilde{c}^{\mathbf{B}}_{it}(y_i, y_j, \mathbf{W}) : y_j \in \mathcal{Y}\}$. For notational simplicity and without loss of generality, we omit $\mathbf{W}$ for the rest of this proof.

Let s^0_j be an arbitrary value of in the set $\mathcal{S}$. And let $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ be two different subsets included in the set $\mathcal{S} - \{s^0_j\}$ such that they satisfy two conditions: (1) each of these sets has $A-1$ elements; and (2) $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ have at least one element that is different. Since $|\mathcal{S}| \geq A+1$, it is always possible to construct two subsets that satisfy these conditions. Given one of these subsets, say $\mathcal{S}^{(a)}$, we can construct the following system of $A-1$ equations:

$$\Delta \mathbf{q}^{(a)}_{it}(y_i, S_i) \;\; = \Delta \mathbf{B}^{(a)}_{it}(S_i) \; \widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i) \tag{A.4.2}$$

where: $\Delta \mathbf{q}^{(a)}_{it}(y_i, S_i)$ is an $(A-1) \times 1$ vector with elements $\{q_{it}(y_i, S_i, S_j) - q_{it}(y_i, S_i, s^0_j) :$ for $S_j \in \mathcal{S}^{(a)}\}$; $\Delta \mathbf{B}^{(a)}_{it}(S_i)$ is a $(A-1) \times (A-1)$ matrix with elements $\{B_{it}(y_j, S_i, S_j) - B_{it}(y_j, S_i, s^0_j) :$ for $y_j \in \mathcal{Y} - \{0\}$ and $S_j \in \mathcal{S}^{(a)}\}$; and $\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i, S_i)$ is a $(A-1) \times 1$ vector with elements $\{\pi_{it}(y_i, y_j, S_i) + \widetilde{c}^{\mathbf{B}}_{it}(y_i, y_j) - \pi_{it}(y_i, 0, S_i) - \widetilde{c}^{\mathbf{B}}_{it}(y_i, 0) : y_j \in \mathcal{Y}\}$. Using the other subset, $\mathcal{S}^{(b)}$, we can construct a similar system of $A-1$ equations. Given that matrices $\Delta\mathbf{B}_{it}^{(a)}(S_i)$ and $\Delta\mathbf{B}_{it}^{(b)}(S_i)$ are non-singular, we can use these systems to obtain to different solutions for $\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i,S_i)$:

$$\begin{aligned}\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i,S_i) &= \left[\Delta\mathbf{B}_{it}^{(a)}(S_i)\right]^{-1}\Delta\mathbf{q}_{it}^{(a)}(y_i,S_i)\\ &= \left[\Delta\mathbf{B}_{it}^{(b)}(S_i)\right]^{-1}\Delta\mathbf{q}_{it}^{(b)}(y_i,S_i)\end{aligned} \tag{A.4.3}$$

For given S_i, we have these two solutions of $\widetilde{\widetilde{\mathbf{v}}}_{it}(y_i,S_i)$ for every value of y_i in the set $\mathcal{Y}-\{0\}$. Putting these $A-1$ solutions in matrix form, we have:

$$\left[\Delta\mathbf{B}_{it}^{(a)}(S_i)\right]^{-1}\Delta\mathbf{Q}_{it}^{(a)}(S_i) = \left[\Delta\mathbf{B}_{it}^{(b)}(S_i)\right]^{-1}\Delta\mathbf{Q}_{it}^{(b)}(S_i) \tag{A.4.4}$$

where $\Delta\mathbf{Q}_{it}^{(a)}(S_i)$ and $\Delta\mathbf{Q}_{it}^{(b)}(S_i)$ are $(A-1)\times(A-1)$ matrices with columns $\Delta\mathbf{q}_{it}^{(a)}(y_i,S_i)$ and $\Delta\mathbf{q}_{it}^{(b)}(y_i,S_i)$, respectively. Given that $\Delta\mathbf{Q}_{it}^{(a)}(S_i)$ is an invertible matrix, we can rearrange the previous system in the following way:

$$\Delta\mathbf{B}_{it}^{(a)}(S_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(S_i)\right]^{-1} = \Delta\mathbf{Q}_{it}^{(a)}(S_i)\ \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i)\right]^{-1} \tag{A.4.5}$$

This expression shows that we can identify the $(A-1)\times(A-1)$ matrix $\Delta\mathbf{B}_{it}^{(a)}(S_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(S_i)\right]^{-1}$ that depends only on beliefs, using only the CCPs of player i. That is, we can identify $(A-1)\times(A-1)$ objects or functions of beliefs.

Under the assumption of unbiased beliefs, we can use the CCPs of the other player, j, to identify matrix $\Delta\mathbf{B}_{it}^{(a)}(S_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(S_i)\right]^{-1}$:

$$\Delta\mathbf{B}_{it}^{(a)}(S_i)\ \left[\Delta\mathbf{B}_{it}^{(b)}(S_i)\right]^{-1} = \Delta\mathbf{P}_{jt}^{(a)}(S_i)\ \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i)\right]^{-1} \tag{A.4.6}$$

where $\Delta\mathbf{P}_{jt}^{(a)}(S_i)$ is $(A-1)\times(A-1)$ matrix with elements $\{P_{jt}(y_j,S_i,S_j)-P_{jt}(y_j,S_i,s_j^0) : \text{for } y_j \in \mathcal{Y}-\{0\} \text{ and } S_j \in \mathcal{S}^{(a)}\}$, and $\Delta\mathbf{P}_{jt}^{(b)}(S_i)$ has a similar definition. Therefore, under the assumption of unbiased beliefs by player i the CCPs of player i and player j should satisfy the following $(A-1)^2$ restrictions:

$$\Delta\mathbf{Q}_{it}^{(a)}(S_i)\ \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i)\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(S_i)\ \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i)\right]^{-1} = \mathbf{0} \tag{A.4.7}$$

These restrictions are testable. ■

[A.5] Asymptotic distribution of two-step estimators and test statistics

The derivation of the asymptotic distribution of our two-step estimators of payoffs and beliefs is a straightforward application of properties of two-step semiparametric estimators as shown in Newey (1994), Andrews (1994), and McFadden and Newey (1994). In fact, given our maintained assumption that the space of state variables $\mathcal{X}$ is discrete and finite, all the structural functions in our model live in a finite dimensional Euclidean space. Therefore, we do not need to apply stochastic equicontinuity results, as in Newey (1994) and Andrews (1994), to show root-M consistency and asymptotic normality of these estimators. Here we apply results in Newey (1984) who provides a methods of moments interpretation of sequential estimators.

We begin by establishing the consistency and asymptotic normality of our estimator of CCPs. The estimator of the CCP $P_{it}(y|\mathbf{x})$ is based on the moment condition:

$$\mathbb{E}\left[1\{\mathbf{X}_{mt}=\mathbf{x}\}\ \ (1\{Y_{imt}=y\}-P_{it}(y|\mathbf{x}))\right]=0$$

In vector, we have the system

$$\mathbb{E}\left[g_{\mathbf{x}}(\mathbf{X}_{mt},Y_{imt},\mathbf{P}_{it,\mathbf{x}})\right]\equiv\mathbb{E}\left[1\{\mathbf{X}_{mt}=\mathbf{x}\}\ \ (\mathbf{1}_{Y_{imt}}-\mathbf{P}_{it,\mathbf{x}})\right]=\mathbf{0}$$

where $\mathbf{1}_{Y_{imt}}$ and $\mathbf{P}_{it,\mathbf{x}}$ are the $(A-1)\times 1$ vectors $\mathbf{1}_{Y_{imt}}\equiv\{1\{Y_{imt}=y\}:y=1,2,...,A-1\}$ and $\mathbf{P}_{it,\mathbf{x}}\equiv\{P_{it}(y|\mathbf{x}):y=1,2,...,A-1\}$, respectively. The corresponding sample moment condition that defines the estimator $\widehat{\mathbf{P}}_{it,\mathbf{x}}$ is:

$$\sum_{m=1}^{M}g_{\mathbf{x}}(\mathbf{X}_{mt},Y_{imt},\widehat{\mathbf{P}}_{it,\mathbf{x}})\equiv\sum_{m=1}^{M}1\left\{\mathbf{X}_{mt}=\mathbf{x}\right\}\ \ \left[\mathbf{1}_{Y_{imt}}-\widehat{\mathbf{P}}_{it,\mathbf{x}}\right]=\mathbf{0}$$

For notational simplicity, for the rest of this Appendix we omit the player and time subindexes (i,t) from variables, parameters, and functions. As the observations are *i.i.d.* across markets, this estimator satisfies the standard regularity conditions for consistency and asymptotic normality of the Method of Moments estimator, such that as M goes to infinity, we have that $\widehat{\mathbf{P}}_{\mathbf{x}}\rightarrow_p\mathbf{P}_{\mathbf{x}}$, and

$$\sqrt{M}\left(\widehat{\mathbf{P}}_{\mathbf{x}}-\mathbf{P}_{\mathbf{x}}\right)\rightarrow_d N\left(0\ ,\ \mathbf{G}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{gg}\ \mathbf{G}_{\mathbf{x}}^{-1\prime}\right)$$

where $\mathbf{G}_{\mathbf{x}}\equiv\mathbb{E}\left[\partial g_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})/\partial\mathbf{P}_{\mathbf{x}}^{\prime}\right]$ and $\mathbf{\Omega}_{gg}\equiv\mathbb{E}\left[g_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})\ g_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})^{\prime}\right]$.

We now establish the consistency and asymptotic distribution of the estimator of payoffs. The population restrictions that our estimator of payoff must satisfy at a given value of $y_i,S_i,\mathbf{W}$ are given by:

$$\mathbf{q}_i^{(R)}(y_i,S_i,\mathbf{W})-\mathbf{P}^{(R)}(S_i,\mathbf{W})\ \boldsymbol{\pi}_i(y_i,S_i,\mathbf{W})=\mathbf{0}$$

Or in vector form, for any value of y_i (and omitting the player subindex i),

$$h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})\equiv\mathbf{q}^{(R)}(.,S,\mathbf{W})-\mathbf{P}^{(R)}(.,S,\mathbf{W})\ \boldsymbol{\pi}_{S,\mathbf{W}}(.,S,\mathbf{W})=0$$

In the just-identified nonparametric model, the estimator $\widehat{\boldsymbol{\pi}}_{S,\mathbf{W}}$ of the vector of payoffs $\boldsymbol{\pi}_{S,\mathbf{W}}$ is the value that solves the system of equations $h_{S,\mathbf{W}}(\widehat{\mathbf{P}}_{\mathbf{x}},\widehat{\boldsymbol{\pi}}_{S,\mathbf{W}})=0$. Under the conditions of Proposition 2, the mapping $h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem (or Mann-Wald Theorem) such that as M goes to infinity, we have that $\widehat{\boldsymbol{\pi}}_{S,\mathbf{W}}\rightarrow_p\boldsymbol{\pi}_{S,\mathbf{W}}$, and $\sqrt{M}\left(\widehat{\boldsymbol{\pi}}_{S,\mathbf{W}}-\boldsymbol{\pi}_{S,\mathbf{W}}\right)\rightarrow_d N\left(0\ ,\ \mathbf{V}_{\boldsymbol{\pi}_{S,\mathbf{W}}}\right)$, where applying Newey (1984)

$$\mathbf{V}_{\boldsymbol{\pi}_{S,\mathbf{W}}}=\mathbf{H}_{\boldsymbol{\pi}}^{-1}\left(\ \mathbf{\Omega}_{hh}+\mathbf{H}_{\mathbf{P}}[\mathbf{G}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{gg}\ \mathbf{G}_{\mathbf{x}}^{-1\prime}]\ \mathbf{H}_{\mathbf{P}}^{\prime}-\mathbf{H}_{\mathbf{P}}[\mathbf{G}_{\mathbf{x}}^{-1}\ \mathbf{\Omega}_{g,h}\ +\ \mathbf{\Omega}_{h,g}\mathbf{G}_{\mathbf{x}}^{-1\prime}]\mathbf{H}_{\mathbf{P}}^{\prime}\ \right)\ \mathbf{H}_{\boldsymbol{\pi}}^{-1\prime}$$

with $\mathbf{H}_{\boldsymbol{\pi}}\equiv\partial h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})/\partial\boldsymbol{\pi}_{S,\mathbf{W}}^{\prime}$, $\mathbf{H}_{\mathbf{P}}\equiv\partial h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})/\partial\mathbf{P}_{\mathbf{x}}^{\prime}$, $\mathbf{\Omega}_{hh}\equiv\mathbb{E}[h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})\ h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})^{\prime}]$, $\mathbf{\Omega}_{gh}\equiv\mathbb{E}[g_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})\ h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})^{\prime}]$, and $\mathbf{\Omega}_{hg}\equiv\mathbb{E}[h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}},\boldsymbol{\pi}_{S,\mathbf{W}})\ g_{\mathbf{x}}(\mathbf{X},Y,\mathbf{P}_{\mathbf{x}})^{\prime}]$.

Our estimator of beliefs takes as given the estimates of CCPs and payoffs. Specifically, for a given vector of the state variables $\mathbf{x}$, beliefs are given by the system

$$\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right) \equiv \widetilde{V}_i(\mathbf{X})\ \mathbf{B}_i(\mathbf{X}) - \mathbf{q}_i(\mathbf{X}) = \mathbf{0}$$

where $\mathbf{q}_i(\mathbf{X})$ is an $A \times 1$ vector with elements $\{q_i(1, \mathbf{X}), ..., q_i(A-1, \mathbf{X})\}$ at rows 1 to $A-1$, and a 1 at the last row, and $\widetilde{\mathbf{V}}_i(\mathbf{X})$ is an $A \times A$ matrix where the element $(y_i, y_j + 1)$ is $\pi_i(y_i, y_j, S_i, \mathbf{W})$, and the last row of the matrix is a row of ones. In the just identified case, we use the system of equations $h_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}, \boldsymbol{\pi}_{S,\mathbf{W}}) = 0$ to derive a closed form expression for $\boldsymbol{\pi}_{S,\mathbf{W}}$ in terms of $\mathbf{P}_{\mathbf{x}}$,. Therefore, in the system of equations $\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right) = 0$ we can omit $\boldsymbol{\pi}_{S,\mathbf{W}}$ as an argument because we have represented $\boldsymbol{\pi}_{S,\mathbf{W}}$ as a function of $\mathbf{P}_{\mathbf{x}}$. The estimator $\widehat{\mathbf{B}}_{\mathbf{x}}$ of the vector of beliefs $\mathbf{B}_{\mathbf{x}}$ is the value that solves the system of equations $\ell_{\mathbf{x}}\left(\widehat{\mathbf{P}}_{\mathbf{x}}, \widehat{\mathbf{B}}_{\mathbf{x}}\right) = 0$. Under the conditions of Proposition 2, the mapping $\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem such that as M goes to infinity, we have that $\widehat{\mathbf{B}}_{\mathbf{x}} \rightarrow_p \mathbf{B}_{\mathbf{x}}$, and $\sqrt{M}\left(\widehat{\mathbf{B}}_{\mathbf{x}} - \mathbf{B}_{\mathbf{x}}\right) \rightarrow_d N\left(0\ ,\ \mathbf{V}_{\mathbf{B}_{\mathbf{x}}}\right)$, where applying Newey (1984),

$$\mathbf{V}_{\mathbf{B}_{\mathbf{x}}} = \mathbf{L}_{\mathbf{B}}^{-1}\left(\ \boldsymbol{\Omega}_{\ell\ell} + \mathbf{L}_{\mathbf{P}}[\mathbf{G}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{gg}\ \mathbf{G}_{\mathbf{x}}^{-1\prime}]\ \mathbf{L}_{\mathbf{P}}' - \mathbf{L}_{\mathbf{P}}[\mathbf{G}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{g,\ell}\ +\ \boldsymbol{\Omega}_{\ell,g}\mathbf{G}_{\mathbf{x}}^{-1\prime}]\mathbf{L}_{\mathbf{P}}'\ \right)\ \mathbf{L}_{\mathbf{B}}^{-1\prime}$$

with $\mathbf{L}_{\mathbf{B}} \equiv \partial\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)/\partial\mathbf{B}_{\mathbf{x}}'$, $\mathbf{L}_{\mathbf{P}} \equiv \partial\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)/\partial\mathbf{P}_{\mathbf{x}}'$, $\boldsymbol{\Omega}_{\ell\ell} \equiv \mathbb{E}[\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)\ \ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)']$, $\boldsymbol{\Omega}_{g,\ell} \equiv \mathbb{E}[g_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})\ \ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)']$, and $\boldsymbol{\Omega}_{\ell,g} \equiv \mathbb{E}[\ell_{\mathbf{x}}\left(\mathbf{P}_{\mathbf{x}}, \mathbf{B}_{\mathbf{x}}\right)\ g_{\mathbf{x}}(\mathbf{X}, Y, \mathbf{P}_{\mathbf{x}})']$.

Define the vector $d_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}})$ as:

$$d_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}}) \equiv \Delta\mathbf{Q}_{it}^{(a)}(S_i, \mathbf{W})\ \left[\Delta\mathbf{Q}_{it}^{(b)}(S_i, \mathbf{W})\right]^{-1} - \Delta\mathbf{P}_{jt}^{(a)}(S_i, \mathbf{W})\ \left[\Delta\mathbf{P}_{jt}^{(b)}(S_i, \mathbf{W})\right]^{-1}$$

And given our consistent estimator of the vector of CCPs $\mathbf{P}_{\mathbf{x}}$, define the vector of statistics $\widehat{\mathbf{d}}_{S,\mathbf{W}} \equiv d_{S,\mathbf{W}}(\widehat{\mathbf{P}}_{\mathbf{x}})$. Under the conditions of Proposition 1, the mapping $d_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}})$ satisfies the regularity conditions to apply Slutsky's Theorem and the Continuous Mapping Theorem such that as M goes to infinity, we have that $\widehat{\mathbf{d}}_{S,\mathbf{W}} \rightarrow_p \mathbf{d}_{S,\mathbf{W}}$, and $\sqrt{M}\left(\widehat{\mathbf{d}}_{S,\mathbf{W}} - \mathbf{d}_{S,\mathbf{W}}\right) \rightarrow_d N\left(0\ ,\ \mathbf{V}_{\mathbf{d}}\right)$, where by the Delta Method,

$$\mathbf{V}_{\mathbf{d}} = \mathbf{D}_{\mathbf{P}}\ [\mathbf{G}_{\mathbf{x}}^{-1}\ \boldsymbol{\Omega}_{gg}\ \mathbf{G}_{\mathbf{x}}^{-1\prime}]\ \mathbf{D}_{\mathbf{P}}'$$

with $\mathbf{D}_{\mathbf{P}} \equiv \partial d_{S,\mathbf{W}}(\mathbf{P}_{\mathbf{x}})/\partial\mathbf{P}_{\mathbf{x}}'$. Now, consider the quadratic form. Under the null hypothesis of unbiased beliefs, the true value of the vector $\mathbf{d}_{S,\mathbf{W}}$ is equal to zero such that $\sqrt{M}\widehat{\mathbf{d}}_{S,\mathbf{W}} \rightarrow_d N\left(0\ ,\ \mathbf{V}_{\mathbf{d}}\right)$. Consider the quadratic form statistic:

$$\widehat{\mathbf{d}}_{S,\mathbf{W}}'\ \widehat{\mathbf{V}}_{\mathbf{d}}^{-1}\ \widehat{\mathbf{d}}_{S,\mathbf{W}}$$

where $\widehat{\mathbf{V}}_{\mathbf{d}}$ is a consistent estimator of $\mathbf{V}_{\mathbf{d}}$. By the Continuous Mapping Theorem, we have that under the null hypothesis of unbiased beliefs the quadratic-form statistic $\widehat{\mathbf{d}}_{S,\mathbf{W}}'\ \widehat{\mathbf{V}}_{\mathbf{d}}^{-1}\ \widehat{\mathbf{d}}_{S,\mathbf{W}}$ is asymptotically distributed as a Chi-squared with degrees of freedom equal to the dimension of the vector $\mathbf{d}_{S,\mathbf{W}}$.

References

[1] Aguirregabiria, V. (2008): "Comment on The Identification Power of Equilibrium in Games," *Journal of Business and Economic Statistics*, 26, 283-289.

[2] Aguirregabiria, V. (2010): "Another Look at the Identification of Dynamic Discrete Decision Processes: An Application to Retirement Behavior," *Journal of Business and Economic Statistics,* 28(2), 201–218.

[3] Aguirregabiria, V. and P. Mira (2002) "Identification and Estimation of Dynamic Discrete Games," manuscript presented at the *Conference on Industrial Organization and the Food Processing Industry*, Institut d'Economie Industrielle (IDEI). Toulouse, France, June, 2002: http://idei.fr/doc/conf/fpi/papers_2002/aguirrega.pdf

[4] Aguirregabiria, V. and P. Mira (2007) "Sequential Estimation of Dynamic Discrete Games," *Econometrica*, 75, 1–53.

[5] Andrews, W. (1994): "Asymptotics for Semiparametric Econometric Models via Stochastic Equicontinuity," *Econometrica*, 62, 43-72.

[6] Aradillas-Lopez, A. (2012): "Pairwise-Difference Estimation of Incomplete Information Games," *Journal of Econometrics*, 168, 120-140.

[7] An, Y. (2010): "Nonparametric Identification and Estimation of Level-k Auctions," manuxcript. Johns Hopkins University.

[8] Aradillas-Lopez, A. and E. Tamer (2008): "The Identification Power of Equilibrium in Games," *Journal of Business and Economic Statistics*, 26 (3), 261-283.

[9] Bajari, P., L. Benkard and J. Levin (2006): "Estimating Dynamic Models of Imperfect Competition," *Econometrica*, 75, 1331-1370.

[10] Bajari, P. and H. Hong (2005): "Semiparametric Estimation of a Dynamic Game of Incomplete Information. Manuscript," Duke University.

[11] Bajari, P., H. Hong, and S. Ryan (2010): "Identification and Estimation of a Discrete Game of Complete Information," *Econometrica*, 78, 1529-68.

[12] Bajari, P., H. Hong, J. Kreiner, and D. Nekipelov (2010): "Estimating Static Models of Strategic Interaction," *Journal of Business and Economic Statistics*, 28(4), 469-482.

[13] Battigalli, P. (1997): "On rationalizability in extensive games," *Journal of Economic Theory*, 74(1), 40-61.

[14] Beresteanu, A. and P. Ellickson (2005): "The Dynamics of Retail Oligopolies," manuscript. Duke University.

[15] Berk, R., L. Brown, A. Buja, K. Zhang, and L. Zhao(2013): "Valid Post-Selection Inference," The Annals of Statistics, 41(2), 802-837.

[16] Bernheim, B. (1984): "Rationalizable Strategic Behavior," *Econometrica*, 52, 1007-1028.

[17] Besanko, D., and U. Doraszelski (2004): "Capacity Dynamics and Endogenous Asymmetries in Firm Size," *RAND Journal of Economics,* 35, 23-49.

[18] Besanko, D., U. Doraszelski, L. Lu, and M. Satterthwaite (2010): "On the role of demand and strategic uncertainty in capacity investment and disinvestment dynamics," *International Journal of Industrial Organization,* 28(4), 383-389.

[19] Camerer, C. (2003): "Behavioral Game Theory: Experiments in Strategic Interaction," Princeton University Press: Princeton NJ.

[20] Camerer, C. F., T-H Ho, and J-K Chong (2004): "A Cognitive Hierarchy Model of Games," *Quarterly Journal of Economics*, 119, 861-898.

[21] Costa-Gomes, M., and G. Weizsäcker (2008): "Stated Beliefs and Play in Normal-Form Games," *Review of Economic Studies*, 75, 729-762.

[22] Crawford, V., and H. Haller (1990): "Learning How to Cooperate: Optimal Play in Repeated Coordination Games," *Econometrica*, 58(3), 571-595.

[23] Delavande, A. (2008): "Pill, Patch or Shot? Subjective Expectations and Birth Control Choice," *International Economic Review*, 49, 999-1042.

[24] Delavande, A. (2008): "Pill, Patch or Shot? Subjective Expectations and Birth Control Choice," *International Economic Review*, 49, 999-1042.

[25] Fershtman, C., and A. Pakes (2012). "Dynamic games with asymmetric information: A framework for empirical work." *The Quarterly Journal of Economics*: qjs025.

[26] Gillen, B (2010): "Identification and Estimation of Level-k Auctions," manuscript. California Institute of Technology.

[27] Goldfarb, A., and M. Xiao (2011): "Who thinks about the competition? Managerial ability and strategic entry in US local telephone markets," *American Economic Review*, 101(7), 3130-3161.

[28] Hall, P., and X. Zhou 2003. "Nonparametric estimation of component distributions in a multivariate mixture." *Annals of Statistics*, 201-224.

[29] Heinemann, F., R. Nagel, and P. Ockenfels (2009): "Measuring strategic uncertainty in coordination games," *The Review of Economic Studies*, 76(1), 181-221.

[30] Holmes, T. (2011): "The Diffusion of Wal-Mart and Economies of Density," *Econometrica*, 79(1), 253-302.

[31] Hotz, J., and R. Miller (1993): “Conditional choice probabilities and the estimation of dynamic models,” *Review of Economic Studies*, 60, 497-529.

[32] Hu, Y., and M. Shum (2013): “Identifying dynamic games with serially-correlated unobservables,” *Advances in Econometrics*, 31, 97-113.

[33] Jia, P. (2008): "What Happens When Wal-Mart Comes to Town: An empirical analysis of the Discount Industry," *Econometrica*, 76, 1263-1316.

[34] Kasahara, H. and K. Shimotsu (2009): "Nonparametric Identification of Finite Mixture Models of Dynamic Discrete Choices," *Econometrica*, 77, 135-175.

[35] Khan, S., and E. Tamer (2009): "Irregular Identification, Support Conditions, and Inverse Weight Estimation," manuscript. Duke University and Northwestern University.

[36] Kline, B., and E. Tamer (2012): "Bounds for best response functions in binary games," Journal of Econometrics, 166(1), 92-105.

[37] Leeb, H., and B. Potscher (2005): "Model Selection and Inference: Facts and Fiction," Econometric Theory, 21, 21-59.

[38] Leeb, H. and B. Potscher (2006): "Can One Estimate the Conditional Distribution of Post-Model-Selection Estimators?" The Annals of Statistics, 34, 2554-2591.

[39] Magnac, T. and Thesmar, D. (2002): "Identifying Dynamic Discrete Decision Processes," *Econometrica*, 70, 801-816.

[40] Manski, C. (2004): "Measuring Expectations," *Econometrica*, 72, 1329-1376.

[41] Matzkin, R. (1992): "Nonparametric and Distribution-Free Estimation of the Binary Choice and the Threshold Crossing Models”, *Econometrica*, 60, 239-270.

[42] Matzkin, R. (1993) "Nonparametric identification and estimation of polychotomous choice models" *Journal of Econometrics*, 58(1), 137-168.

[43] Matzkin, R. (1994): “Restrictions of Economic Theory in Nonparametric Methods,” *Handbook of Econometrics*, Vol. 4, edited by C.F. Engel and D.L. McFadden, Elsevier.

[44] Mazzeo, M. (2002): "Product choice and oligopoly market structure," *Rand Journal of Economics*, 33(2), 221–242.

[45] McFadden, D., and W. Newey (1994): "Large Sample Estimation and Hypothesis Testing," in *Handbook of Econometrics*, Vol. 4, ed. by R. E. Engle and D. McFadden, Amsterdam: North Holland.

[46] Morris, S., and H-S. Shin (2002): “Measuring Strategic Uncertainty,” manuscript. Department of Economics. Princeton University.

[47] Morris, S., and H-S. Shin (2004): “Coordination risk and the price of debt,” *European Economic Review*, 48, 133-153.

[48] Newey, W. (1984): “A method of moments interpretation of sequential estimators,” *Economics Letters*, 14(2), 201-206.

[49] Newey, W. (1994): “The Asymptotic Variance of Semiparametric Estimators,” *Econometrica*, 62, 1349-1382.

[50] Norets, A., and X. Tang (2014): “Semiparametric Inference in Dynamic Binary Choice Models,” *Review of Economic Studies*, 81, 1229-1262.

[51] Nishida, M. (2014): “Estimating a model of strategic network choice: The convenience-store industry in Okinawa,” *Marketing Science*, 34(1), 20-38.

[52] Pakes, A. (1994): "Dynamic structural models, problems and prospects," in C. Sims (ed.) Advances in Econometrics. Sixth World Congress, Cambridge University Press.

[53] Pakes, A. and P. McGuire (1994): “Computing Markov-Perfect Nash Equilibria: Numerical Implications of a Dynamic Differentiated Product Model," *RAND Journal of Economics*, 25(4), 555-589.

[54] Pakes, A., M. Ostrovsky, and S. Berry (2007): "Simple estimators for the parameters of discrete dynamic games (with entry/exit examples)," Rand Journal of Economics 38, 373-399.

[55] Palfrey, T., and S. Wang (2009): "On eliciting beliefs in strategic games," *Journal of Economic Behavior & Organization*, 71, 98–109.

[56] Pearce, D. (1984): "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica*, 52, 1029-1050.

[57] Pesendorfer, M., and P. Schmidt-Dengler (2003): "Identification and Estimation of Dynamic Games," NBER working paper 9726.

[58] Pesendorfer, M. and Schmidt-Dengler (2008): "Asymptotic Least Squares Estimators for Dynamic Games," *The Review of Economic Studies*, 75, 901-928.

[59] Rust, J. (1994): "Structural estimation of Markov decision processes," in R. E. Engle and McFadden (eds.) Handbook of Econometrics Volume 4, North-Holland. Amsterdam.

[60] Rutström, E., and N. Wilcox (2009): "Stated beliefs versus inferred beliefs: A methodological inquiry and experimental test," *Games and Economic Behavior*, 67, 616-632.

[61] Samuelson, P. (1938): "A Note on the Pure Theory of Consumers’ Behaviour," *Economica,* 5, 61-71.

[62] Seim, K. (2006): "An Empirical Model of Firm Entry with Endogenous Product-Type Choices," *The RAND Journal of Economics*, 37(3), 619–640.

[63] Suzuki, J. (2013): “Land Use Regulation as a Barrier to Entry: Evidence from the Texas Lodging Industry,” *International Economic Review*, 54(2), 495-523.

[64] Tamer, E. (2003): "Incomplete Simultaneous Discrete Response Model with Multiple Equilibria," *Review of Economic Studies*, 70, 147-165.

[65] Toivanen, O., and M. Waterson (2005): "Market Structure and Entry: Where's the Beef?," *RAND Journal of Economics*, 36(3), 680-699.

[66] Uetake, K., and Y. Watanabe (2013): "Estimating Supermodular Games Using Rationalizable Strategies," in E. Choo and M. Shum (eds.) Structural Econometric Models (Advances in Econometrics, Volume 31) Emerald Group Publishing Limited, 233-247.

[67] Van Huyck, J., R. Battalio, and R. Beil (1990): “Tacit Coordination Games, Strategic Uncertainty, and Coordination Failure,” *The American Economic Review*, 80(1), 234-248.

[68] Van der Klaauw, W. and K. Wolpin (2008): "Social Security and the Retirement and Savings Behavior of Low Income Households," *Journal of Econometrics*, 145, 21-42.

[69] Walker, J. (2003): "Pregnancy and Fertility Expectations: Estimates of Bounded Rationality and Unintended Births," Manuscript, University of Wisconsin.

[70] Walrath, M. (2015): "Entry Models Applied to Churches: Could Protestants Use a Catholic Bishop to Solve Excess Entry?," *Journal of Industrial Economics*, forthcoming.

[71] Zhu, T. and V. Singh (2009): "Spatial competition with endogenous location choices: An application to discount retailing," *Quantitative Marketing and Economics*, 7, 1–35.

Table 1

Sequence of Beliefs $B_{it}^{(t_0)}$

Period when beliefs are formed (t_0)	Period of the opponents' behavior (t)					
	$t=1$	$t=2$	$t=3$	...	$t=T-1$	$t=T$
$t_0 = 1$	$B_{i1}^{(1)}$	$B_{i2}^{(1)}$	$B_{i3}^{(1)}$	...	$B_{i,T-1}^{(1)}$	$B_{iT}^{(1)}$
$t_0 = 2$	-	$B_{i2}^{(2)}$	$B_{i3}^{(2)}$	...	$B_{i,T-1}^{(2)}$	$B_{iT}^{(2)}$
$t_0 = 3$	-	-	$B_{i3}^{(3)}$	...	$B_{i,T-1}^{(3)}$	$B_{iT}^{(3)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t_0 = T-1$	-	-	-	...	$B_{i,T-1}^{(T-1)}$	$B_{iT}^{(T-1)}$
$t_0 = T$	-	-	-	...	-	$B_{iT}^{(T)}$

Table 2. Order Condition for Identification
Models without Exclusion Restrictions in Payoffs

Number of parameters & restrictions for each player-period

Number of parameters & restrictions	**Models without exclusion restrictions (A) Unrestricted Beliefs**	**(B) Unbiased (Equil) Beliefs**
(1) Restrictions from observed behavior	$(A-1)\ \lvert\mathcal{X}\rvert$	$(A-1)\ \lvert\mathcal{X}\rvert$
(2) Restrictions from unbiased beliefs	0	$(N-1)\ (A-1)\ \lvert\mathcal{X}\rvert$
(3) Free parameters in payoffs	$(A-1)\ \lvert\mathcal{X}\rvert\ A^{N-1}$	$(A-1)\ \lvert\mathcal{X}\rvert\ A^{N-1}$
(4) Free parameters in beliefs	$(N-1)\ (A-1)\ \lvert\mathcal{X}\rvert$	$(N-1)\ (A-1)\ \lvert\mathcal{X}\rvert$
(1)+(2)-(3)-(4) Over-under identifying rest.	$(A-1)\lvert\mathcal{X}\rvert\left[1-A^{N-1}-(N-1)\right]$	$(A-1)\ \lvert\mathcal{X}\rvert\ \left[1-A^{N-1}\right]$
Is the Model identified?	**NO (For any N$\geq$ 2)**	**NO (For any N$\geq$ 2)**

Table 3. Order Condition for Identification
Models WITH Exclusion Restrictions in Payoffs
Number of parameters & restrictions for each player-period

Number of parameters & restrictions	Models with exclusion restrictions (A) Unrestricted Beliefs	(B) Unbiased (Equil) Beliefs
(1) Restrictions from observed behavior	$(A-1)\ \lvert\mathcal{S}\rvert^{N}$	$(A-1)\ \lvert\mathcal{S}\rvert^{N}$
(2) Restrictions from unbiased beliefs	0	$(N-1)\ (A-1)\ \lvert\mathcal{S}\rvert^{N}$
(3) Free parameters in payoffs	$(A-1)\ \lvert\mathcal{S}\rvert\ A^{N-1}$	$(A-1)\ \lvert\mathcal{S}\rvert\ A^{N-1}$
(4) Free parameters in beliefs	$(N-1)\ (A-1)\ \lvert\mathcal{S}\rvert^{N}$	$(N-1)\ (A-1)\ \lvert\mathcal{S}\rvert^{N}$
(1)+(2)-(3)-(4) Over-under identifying rest.	$(A-1)\lvert\mathcal{S}\rvert^{N}\left[1-\frac{A^{N-1}}{\lvert\mathcal{S}\rvert^{N-1}}-(N-1)\right]$	$(A-1)\lvert\mathcal{S}\rvert^{N}\left[1-\frac{A^{N-1}}{\lvert\mathcal{S}\rvert^{N-1}}\right]$
Is the Model identified?	**NO (For any N$\geq$ 2)**	**YES (For any $\lvert\mathcal{S}\rvert \geq A$)**

Table 4. Order Condition for Identification
Model WITH Exclusion Restrictions in Payoffs
and Unbiased Beliefs in a Subset of States

Number of parameters & restrictions for each player-period

Number of parameters & restrictions	**Model with exclusion restrictions and Unbiased Beliefs in a Subset of States**
(1) Restrictions from observed behavior	$(A-1)\ \lvert\mathcal{S}\rvert^{N}$
(2) Restrictions from unbiased beliefs	$(N-1)\ (A-1)\ \lvert\mathcal{S}\rvert^{N-1}R$
(3) Free parameters in payoffs	$(A-1)\ \lvert\mathcal{S}\rvert\ A^{N-1}$
(4) Free parameters in beliefs	$(N-1)\ (A-1)\ \lvert\mathcal{S}\rvert^{N}$
(1)+(2)-(3)-(4) Over-under identifying rest.	$(A-1)\lvert\mathcal{S}\rvert^{N}\left[1-\frac{A^{N-1}}{\lvert\mathcal{S}\rvert^{N-1}} - \left(1-\frac{R}{\lvert\mathcal{S}\rvert}\right)(N-1)\right]$
Is the Model identified?	**YES**

Table 5
Summary of DGPs in the Monte Carlo Experiments

For all the experiments:	$\alpha_1 = \alpha_2 = 2.4$; $\delta_1 = \delta_2 = 3.0$; $\theta_1^{EC} = \theta_2^{EC} = 0.5$; $\beta_1 = \beta_2 = 0.95$; $Z_{2m} \sim Uniform\ \{-2, -1, 0, +1, +2\}$		
	$M = 2,000$; $T = 5$; $MC\ replications = 10,000$		
Experiment 1A:	$\theta_{12}^{FC} = -0.5$	and	Unbiased beliefs
Experiment 1B:	$\theta_{12}^{FC} = -0.5$	and	Biased beliefs
Experiment 2A:	$\theta_{12}^{FC} = -1.0$	and	Unbiased beliefs
Experiment 2B:	$\theta_{12}^{FC} = -1.0$	and	Biased beliefs

Some Ratios Implied by these Parameter Values

Entry cost over average profit of a monopolist: $\theta_i^{EC}/(\theta_i^M - \theta_{0i}^{FC})$	17.1%
Profit reduction from monopoly to duopoly: $(\theta_i^M - \theta_i^D)/(\theta_i^M - \theta_{0i}^{FC})$	103.4%
Profit reduction for player 2 as monopolist if Z_2 *goes from* -2 *to* 2*:* $\theta_{12}^{FC}(2-(-2))\ /\ (\theta_2^M - \theta_{02}^{FC} - (-2)\theta_{12}^{FC})$	
with $\theta_{12}^{FC} = -0.5$	51.3%
with $\theta_{12}^{FC} = -1.0$	81.6%

Table 6
Monte Carlo Experiments 1A and 1B

	Experiment 1A DGP: $\theta_{12}^{FC} = -0.5$; Unbiased beliefs					Experiment 1B DGP: $\theta_{12}^{FC} = -0.5$; Biased beliefs			
	(1)	**(2)**	**(3)**	**(4)**		**(5)**	**(6)**	**(7)**	**(8)**
	Estimation with equilibrium restrictions		Estimation no equilibrium restrictions			Estimation with equilibrium restrictions		Estimation no equilibrium restrictions	
Parameter (True value)	Bias (%)	Std (%)	Bias (%)	Std (%)		Bias (%)	Std (%)	Bias (%)	Std (%)
Payoffs					**Payoffs**				
α_1 (2.4)	-0.0992	0.2208	0.1412	0.3702	α_1 (2.4)	0.0157	0.2999	-0.0657	0.3312
	(4.13)	(9.20)	(5.88)	(15.42)		(0.65)	(12.50)	(2.74)	(13.80)
δ_1 (3.0)	-0.1004	0.2349	0.1448	0.3763	δ_1 (3.0)	0.7481	0.3250	0.0617	0.3488
	(3.35)	(7.83)	(4.83)	(12.54)		(24.94)	(10.83)	(2.06)	(11.63)
θ_1^{EC} (0.5)	-0.0021	0.0665	-0.0760	0.1118	θ_1^{EC} (0.5)	-0.3114	0.0798	-0.0120	0.1478
	(0.42)	(13.30)	(15.20)	(22.35)		(62.28)	(15.96)	(2.41)	(29.55)
Beliefs: $t = 1$; $Z_2 = 0 : B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6993)	-0.0005	0.0460	0.0109	0.1563	$B_{1t}(0,0)$ (0.4145)	-0.4144	0.0378	0.0108	0.2401
	(0.07)	(6.57)	(1.57)	(22.36)		(99.99)	(9.12)	(2.61)	(57.92)
$B_{1t}(0,1)$ (0.8390)	0.0004	0.0369	0.0204	0.1259	$B_{1t}(0,1)$ (0.4454)	-0.4449	0.0313	0.0178	0.2464
	(0.05)	(4.40)	(2.43)	(15.01)		(99.88)	(7.03)	(3.99)	(55.32)
$B_{1t}(1,0)$ (0.6009)	-0.0001	0.0488	0.0169	0.1850	$B_{1t}(1,0)$ (0.4078)	-0.4076	0.0388	0.0083	0.2466
	(0.02)	(8.13)	(2.82)	(30.79)		(99.99)	(9.52)	(2.05)	(60.48)
$B_{1t}(1,1)$ (0.7603)	0.0002	0.0423	0.0159	0.1544	$B_{1t}(1,1)$ (0.4405)	0.4403	0.0323	0.0136	0.2500
	(0.02)	(5.56)	(2.09)	(20.31)		(99.99)	(7.33)	(3.08)	(56.75)
Beliefs: $t = 5$; $Z_2 = 0$; $B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6269)	-0.0018	0.0966	0.0099	0.2268	$B_{1t}(0,0)$ (0.3876)	-0.3765	0.1672	-0.0025	0.3006
	(0.29)	(15.41)	(1.58)	(36.17)		(97.14)	(43.14)	(0.63)	(77.56)
$B_{1t}(0,1)$ (0.8034)	-0.0002	0.0448	0.0330	0.1937	$B_{1t}(0,1)$ (0.4272)	-0.4271	0.0530	-0.0108	0.2822
	(0.02)	(5.58)	(4.11)	(24.11)		(99.99)	(12.42)	(2.52)	(66.07)
$B_{1t}(1,0)$ (0.4975)	0.0014	0.0568	-0.0266	0.1855	$B_{1t}(1,0)$ (0.3768)	-0.3768	0.0655	-0.0066	0.2047
	(0.28)	(11.41)	(5.35)	(37.30)		(99.99)	(17.39)	(1.75)	(54.32)
$B_{1t}(1,1)$ (0.6939)	0.0003	0.0315	0.0012	0.0756	$B_{1t}(1,1)$ (0.4190)	0.4187	0.0211	0.0073	0.1139
	(0.05)	(4.54)	(0.17)	(10.90)		(99.94)	(5.04)	(1.75)	(27.19)

Table 7
Monte Carlo Experiments 2A and 2B

	Experiment 2A DGP: $\theta_{12}^{FC}= -1.0$; Unbiased beliefs					Experiment 2B DGP: $\theta_{12}^{FC}= -1.0$; Biased beliefs			
	(1)	**(2)**	**(3)**	**(4)**		**(5)**	**(6)**	**(7)**	**(8)**
	Estimation with equilibrium restrictions		Estimation no equilibrium restrictions			Estimation with equilibrium restrictions		Estimation no equilibrium restrictions	
Parameter (True value)	Bias (%)	Std (%)	Bias (%)	Std (%)		Bias (%)	Std (%)	Bias (%)	Std (%)
Payoffs					**Payoffs**				
α_1 (2.4)	-0.0726	0.1832	0.2060	0.3045	α_1 (2.4)	-0.3332	0.2666	-0.0081	0.2829
	(3.03)	(7.63)	(8.58)	(12.69)		(13.88)	(11.11)	(0.34)	(11.79)
δ_1 (3.0)	-0.0793	0.1852	0.1802	0.3048	δ_1 (3.0)	0.2979	0.2746	0.1543	0.3071
	(2.64)	(6.17)	(6.01)	(10.16)		(9.93)	(9.15)	(5.14)	(10.24)
θ_1^{EC} (0.5)	-0.0042	0.0629	-0.0861	0.1229	θ_1^{EC} (0.5)	-0.3277	0.0778	0.0134	0.1482
	(0.84)	(12.57)	(17.21)	(24.59)		(65.55)	(15.56)	(2.68)	(29.63)
Beliefs: $t = 1$; $Z_2= 0 : B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6993)	-0.0005	0.0460	0.0257	0.1683	$B_{1t}(0,0)$ (0.4145)	-0.4144	0.0378	0.0283	0.2449
	(0.07)	(6.57)	(3.68)	(24.07)		(99.99)	(9.12)	(6.83)	(59.08)
$B_{1t}(0,1)$ (0.8390)	0.0004	0.0369	0.0308	0.1318	$B_{1t}(0,1)$ (0.4454)	-0.4449	0.0313	0.0389	0.2457
	(0.05)	(4.40)	(3.67)	(15.70)		(99.88)	(7.03)	(8.74)	(55.15)
$B_{1t}(1,0)$ (0.6009)	-0.0001	0.0488	0.0341	0.1982	$B_{1t}(1,0)$ (0.4078)	-0.4076	0.0388	0.0270	0.2506
	(0.02)	(8.13)	(5.68)	(32.99)		(99.99)	(9.52)	(6.61)	(61.45)
$B_{1t}(1,1)$ (0.7603)	0.0002	0.0423	0.0286	0.1638	$B_{1t}(1,1)$ (0.4405)	0.4403	0.0323	0.0359	0.2525
	(0.02)	(5.56)	(3.76)	(21.55)		(99.99)	(7.33)	(8.16)	(57.33)
Beliefs: $t = 5$; $Z_2= 0$; $B_{1t}(Z_2, Y_{1t-1}, Y_{2t-1})$									
$B_{1t}(0,0)$ (0.6269)	-0.0018	0.0966	0.0505	0.2644	$B_{1t}(0,0)$ (0.3876)	-0.3765	0.1672	0.0104	0.3080
	(0.29)	(15.41)	(8.06)	(42.17)		(97.14)	(43.14)	(2.68)	(79.47)
$B_{1t}(0,1)$ (0.8034)	-0.0002	0.0448	0.0066	0.1776	$B_{1t}(0,1)$ (0.4272)	-0.4271	0.0530	-0.1357	0.2419
	(0.00)	(5.58)	(0.83)	(22.11)		(99.99)	(12.42)	(31.77)	(56.63)
$B_{1t}(1,0)$ (0.4975)	0.0014	0.0568	-0.0387	0.2538	$B_{1t}(1,0)$ (0.3768)	-0.3768	0.0655	-0.0498	0.2789
	(0.28)	(11.41)	(7.77)	(51.03)		(99.99)	(17.39)	(13.22)	(74.02)
$B_{1t}(1,1)$ (0.6939)	0.0003	0.0315	0.0060	0.0969	$B_{1t}(1,1)$ (0.4190)	0.4187	0.0211	0.0110	0.1469
	(0.05)	(4.54)	(0.86)	(13.97)		(99.94)	(5.04)	(2.63)	(35.06)

Table 8
Descriptive Statistics on Local Markets (Year 1991)
422 local authority districts (excluding Greater London districts)

Variable	Median	Std. Dev.	Pctile 5%	Pctile 95%
Area (thousand square km)	0.30	0.73	0.03	1.67
Population (thousands)	94.85	93.04	37.10	280.50
Share of children: Age 5-14 (%)	12.43	1.00	10.74	14.07
Share of Young: 15-29 (%)	21.24	2.46	17.80	25.17
Share of Pensioners: 65-74 (%)	9.01	1.50	6.89	11.82
GDP per capita (thousand £)	92.00	12.14	74.40	112.70
Claimants of UB / Population ratio (%)	2.75	1.27	1.24	5.11
Avg. Weekly Rent per dwelling (£)	25.31	10.61	19.11	35.07
Council tax (thousand £)	0.24	0.05	0.11	0.31
Number of BK stores	0.00	0.62	0.00	1.00
Number of MD stores	1.00	1.16	0.00	3.00

Table 9
Evolution of the Number of Stores
422 local authority districts (excluding Greater London districts)

	Burger King					
	1990	**1991**	**1992**	**1993**	**1994**	**1995**
#Markets with Stores	71	98	104	118	131	150
Change in #Markets with Stores	-	17	6	14	13	19
# of Stores	79	115	128	153	181	222
Change in # of Stores	-	36	13	25	28	41
Mean #Stores per Market (Conditional on #Stores>0)	1.11	1.17	1.23	1.30	1.38	1.48
	McDonalds					
	1990	**1991**	**1992**	**1993**	**1994**	**1995**
#Markets with Stores	206	213	220	237	248	254
Change in #Markets with Stores		7	7	17	11	6
# of Stores	281	316	344	382	421	447
Change in # of Stores		35	28	38	39	26
Mean #Stores per Market (Conditional on #Stores>0)	1.36	1.49	1.56	1.61	1.70	1.76

Table 10
Transition Probability Matrix for Market Structure

Annual Transitions. Market structure: BK=x & MD=y, where x and y are number of stores

Market Structure at t	**% Market Structure at t+1** BK=0 MD=0	BK=0 MD=1	BK=0 MD≥ 2	BK=1 MD=0	BK=1 MD=1	BK=1 MD≥ 2	BK≥ 2 MD=0	BK≥ 2 MD=1	BK≥ 2 MD≥ 2
BK=0 &MD=0	**95.1**	3.6	0.2	1.0	-	-	-	0.1	-
BK=0 &MD=1	-	**87.2**	4.2	-	7.4	1.0	-	-	1.4
BK=0 &MD≥ 2	-	-	**82.7**	-	-	15.8	-	-	1.4
BK=1 &MD=0	-	-	-	**76.0**	18.0	2.0	4.0	-	-
BK=1 &MD=1	-	-	-	-	**87.1**	8.1	-	3.3	1.4
BK=1 &MD≥ 2	-	-	-	-	-	**86.5**	-	-	13.5
BK≥ 2 &MD=0	-	-	-	-	-	-	**84.6**	15.4	-
BK≥ 2 &MD=1	-	-	-	-	-	-	-	**69.0**	31.0
BK≥ 2 &MD≥ 2	-	-	-	-	-	-	-	-	**100.0**
Frequency	41.6	23.3	6.6	2.2	10.9	8.8	0.6	1.4	4.5

Table 11
Reduced Form Probits for the Decision to Open a Store

	Estimated Marginal Effects[1] ($\Delta P(x)$ when dummy from 0 to 1)					
	Burger King			**McDonalds**		
Explanatory Variable	**No FE**	**County FE**	**District FE**	**No FE**	**County FE**	**District FE**
Own number of stores at t-1						
Dummy: Own #stores = 1	-0.021**	-0.036**	-0.885**	-0.035**	-0.045**	-0.550**
	(0.005)	(0.007)	(0.063)	(0.010)	(0.012)	(0.056)
Dummy: Own #stores = 2	-0.023**	-0.030**	-0.210*	-0.047**	-0.060*	-0.757**
	(0.004)	(0.005)	(0.085)	(0.006)	(0.008)	(0.041)
Dummy: Own #stores $\geq$ 3	-0.019**	-0.027**	-0.056	-0.043**	-0.053**	-0.816**
	(0.005)	(0.005)	(0.036)	(0.006)	(0.008)	(0.038)
Competitor's number of stores at t-1						
Dummy: Comp.'s #stores = 1	0.032**	0.037*	-0.025	0.020	0.032*	0.052**
	(0.011)	(0.014)	(0.055)	(0.013)	(0.018)	(0.073)
Dummy: Comp.'s #stores = 2	0.045*	0.052*	-0.017	0.041	0.076	-0.007**
	(0.023)	(0.029)	(0.031)	(0.029)	(0.046)	(0.093)
Dummy: Comp.'s #stores $\geq$ 3	0.089*	0.101*	0.011	-0.041**	-0.050**	-0.104**
	(0.048)	(0.059)	(0.084)	(0.007)	(0.009)	(0.020)
Pred. Prob. Y=1 at mean X	0.024	0.027	0.014	0.045	0.054	0.085
Time dummies	YES	YES	YES	YES	YES	YES
Control variables[2]	YES	YES	YES	YES	YES	YES
County Fixed Effects	NO	YES	NO	NO	YES	NO
District Fixed Effects	NO	NO	YES	NO	NO	YES
Number of Observations[3]	2110	1715	535	2110	1855	640
Number of Local Districts[3]	422	343	107	422	371	128
log likelihood	-371.89	-340.26	-110.54	-467.46	-449.02	-198.50
Pseudo R-square	0.229	0.252	0.624	0.159	0.161	0.441

Note 1: Estimated Marginal Effects are evaluated at the mean value of the rest of the explanatory variables.
Note 2: Every estimation includes as control variables log-population, log-GDP per capita, log-population density, share population 5-14, share population 15-29, average rent, and proportion of claimants of unemployment benefits.
Note 3: FE estimations do not include districts where the dependent variable does not have enough time variation.

Table 12
Estimation of Dynamic Game for McDonalds and Burger King
Models with Unbiased and Biased Beliefs[1]

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)					
	(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
	Unbiased Beliefs		Biased Beliefs: $Z^*= 2$		Biased Beliefs: $Z^*= 1$	
	Burger King	McDonalds	Burger King	McDonalds	Burger King	McDonalds
Variable Profits:						
θ_0^{VP}	0.5413	0.8632	0.4017	0.8271	0.4342	0.8582
	(0.1265)*	(0.2284)*	(0.2515)*	(0.4278)*	(0.2820)	(0.4375)
θ_{can}^{VP} cannibalization	-0.2246	0.0705	-0.2062	0.0646	-0.1926	0.0640
	(0.0576)*	(0.0304)*	(0.1014)*	(0.0710)	(0.1140)*	(0.0972)
θ_{com}^{VP} competition	-0.0541	-0.0876	-0.1133	-0.0856	-0.1381	-0.0887
	(0.0226)*	(0.0272)	(0.0540)*	(0.0570)	(0.0689)*	(0.0622)
Fixed Costs:						
θ_0^{FC} fixed	0.0350	0.0374	0.0423	0.0307	0.0490	0.0339
	(0.0220)	(0.0265)	(0.0478)	(0.0489)	(0.0585)	(0.0658)
θ_{lin}^{FC} linear	0.0687	0.0377	0.0829	0.0467	0.0878	0.0473
	(0.0259)*	(0.0181)*	(0.0526)*	(0.0291)	(0.0665)	(0.0344)
θ_{qua}^{FC} quadratic	-0.0057	0.0001	-0.0007	0.0002	-0.0004	0.0004
	(0.0061)	(0.0163)	(0.0186)	(0.0198)	(0.0253)	(0.0246)
Entry Cost:						
θ_0^{EC} fixed	0.2378	0.1887	0.2586	0.1739	0.2422	0.1764
	(0.0709)*	(0.0679)*	(0.1282)*	(0.0989)*	(0.1504)	(0.1031)
θ_K^{EC} (K)	-0.0609	-0.107	-0.0415	-0.1190	-0.0419	-0.1271
	(0.043)	(0.0395)*	(0.096)	(0.0628)*	(0.109)*	(0.0762)*
θ_Z^{EC} (Z)	0.0881	0.0952	0.1030	0.1180	0.0902	0.1212
	(0.0368)*	(0.0340)*	(0.0541)*	(0.0654)*	(0.0628)	(0.0759)*
Log-Likelihood	-848.4		-840.4		-838.7	
Test of unbiased beliefs:						
For BK: $\widehat{D}$ (d.o.f) (p-value)			66.841 (32) (0.00029)		66.841 (32) (0.00029)	
For MD: $\widehat{D}$ (d.o.f) (p-value)			42.838 (32) (0.09549)		42.838 (32) (0.09549)	

Note 1: Bootstrap standard errors in parentheses.

Note 2: * and ** denote significance at the 5% and 1% level respectively

Figure 1. RMSE of Estimates as functions of the Instrument Quality