Some Benefits of Cyclical Monetary Policy

Ricardo Cavalcanti* Ed Nosal†

Abstract

We show that modeling monetary circulation and cyclical activity offers insights about monetary policy that cannot be had in representative-agent models. Two fundamental ideas emerge: (i) the reflux of money back to the hands of those making current expenditures can be inefficient, and (ii) expansionary policy may accommodate more trade during high-demand seasons, at the expense of less trade in low-demand seasons and a less valuable currency. The paper provides a foundation for the optimality of a cyclical monetary policy.

1 Introduction

Should monetary policy be cyclical? Although this is an old question in monetary economics, there does not exist a consensus answer. One possible reason for this state of affairs is that in ongoing research in the pure theory of money—models where money arises endogenously as a solution to a trading problem—policy can only speak to affecting the rate of return of money. In this paper we consider an environment where policy can affect the *circulation* of money over a trade cycle. We find that seasonal movements in real activity has implications for the speed of circulation—or *reflux*—of money and that a cyclical monetary policy can improve matters.

There are two fundamental ideas in this paper. First, when the distribution of money holdings is determined by the exchange process, trade cycles may impair the flow of liquidity to the point where the future reflux of money back to the hands of those making current expenditures is compromised.[1] The second fundamental idea is that an expansionary policy, timed correctly, may accommodate more trade during high-demand seasons, at the expense of less trade in low-demand seasons and a less valuable currency.[2]

*Getulio Vargas Foundation, ricardoc@fgv.br

†Federal Reserve Bank of Cleveland, ed.nosal@clev.frb.org

[1] The *law of reflux* is a classical term related to the circulation of banknotes. Another body of work focuses on holdings of multiple assets by broad sectors in the economy using *flow-of-funds* data. A critical review of this literature is beyond the scope of this paper.

[2] The two fundamental ideas are obviously entwined to the extent that the existence of a role for interventions is itself a demonstration that a given distribution of money is inefficient. This inefficiency is, however, different from what has been identified in the literature to be discussed.

We formalize these ideas in a relatively simple random-matching model in the spirit of Kiyotaki and Wright (1989). Individuals alternate between being consumers and producers. At any date, half are consumers, half are producers. The trade cycle or *seasons* are generated by consumer preference shocks and alternating seasons of high and low demand affect the distributions of money. An inefficient distribution of money will be generated in the low demand season after trade concludes. In particular, since trade—and hence circulation of money—is low is the low demand season, a lot of money is kept in the hands of would-be producers, and not would-be consumers, precisely at the start of a season in which the desire to consume is high. Because of this, lump-sum transfer policies can redistribute purchasing power to the hands of consumers in seasons when demand is high.

On the one hand, the merits of cyclical and counter-cyclical monetary policy have been analyzed from myriad angles. But in most of these discussions, money's role as a medium of exchange is ignored. On the other hand, in discussions where this role is explicit, the issue of cyclical policy does not arise. This paper is an attempt to fill this void. Lucas (1972) was the first to present a pure theory of money with the short-run effects of monetary policy. However, an important ingredient in his analysis is an exogenous and random supply of money. Sargent and Wallace (1982) study allocations with alternating cycles in the demand for loans that are Pareto optimal and, therefore, there is no role for interventions in the money supply.[3]

Aggregate models, which are silent on the reflux of money, offer predictions about welfare losses associated with not following the optimal contractionary policy, the so-called Friedman rule.[4] In the context of Bewley-type models, Sheinkman and Weiss (1986) and Levine (1991)[5] argue that a monetary intervention that produces inflation provides insurance to individuals; inflation redistributes purchasing power to those who run out of money due to frequent consumption opportunities. Finally, the implications for circulation or reflux of money have been studied from the perspectives of banking stability (Cavalcanti et al., 1999), efficient allocation of capital (Cavalcanti, 2004), and banknote float (Wallace and Zhu, 2007). But these papers are also silent

[3] They also notice that central banking policies might be desired if a subset of savers is prevented from having access to borrowing because such policies equate the marginal rates of substitution between the various classes of consumers. In our model, by contrast, the main policy concern is not associated with improving the return of money but with its reflux, which incidentally has no meaning in the overlapping-generations models of Lucas (1972) and Sargent and Wallace (1982).

[4] Cash-in-advance models fall into this catogory. It is interesting to note that some applications of the Kiyotaki-Wright exchange environment bend over backwards to construct a representative-agent model in which the Friedman rule becomes the relevant benchmark. For example, Shi (1997) appeals to a large-househould construct and, more recently, Lagos and Wright (2005) appeal to quasilinearity and market trade.

[5] See also Molico (1999), and Deviatov and Wallace (2001).

on trade cycles and, therefore, cyclical interventions.

In order to provide a tractable mechanism-design analysis, our environment must be necessarily stylized: money is the only asset and cycles are deterministic. In addition, we restrict money holdings to be just 0 or 1. One novelty of our contribution is to provide analytical results on welfare improvements when it is known that lump-sum policies do not improve welfare in the "standard" 0-1 model without trade cycles. Let us elaborate on this point.

When money holdings are unbounded, lump-sum transfers produce a larger percent change in the holdings of the "poor" relative to the "rich" and, thus, there is a redistribution in purchasing power. One way to capture this effect in the standard model is to assume that an individual's money disappears with some probability and that people without money receive one unit according to another probability. But, in order to produce a stationary allocation, these probabilities must be chosen so that the quantity of money remains constant. As a result,"expansionary" policies in the standard model cause a reallocation of holdings that keeps constant the measure of people without money. This necessarily implies that such a policy is undesirable. In our model, however, we are able to promote a redistribution across *consumer-producer* status by using taxation in one season and transfers in the other. In both our model and the standard one, monetary policy reduces the return on money. Since the distribution of money holdings is unaffected by monetary policy in the standard model, policy is necessarily welfare reducing; this need not be the case in our model since monetary policy also alters the (consumer-producer) distribution of money.[6]

The rest of the paper is as follows: In section 2, we describe the environment formally. In section 3, we define symmetric and stationary allocations as well as the welfare criteria that guides the discussion of optimal monetary policies. In section 4, we define an implementable allocation. Section 5 analyzes extensive margin effects—the effect on the total number of trades—associated with a cycle monetary policy, and section 6 analyzes intensive margin effects—the amount produced in each trade meeting. Section 7 characterizes the optimal monetary policy. Section 8 concludes with a discussion of three issues: first, can our ideas be used to justify the creation of the Federal Reserve System in 1913?; second, are there any implications for modern policy?; and finally, can private banking substitute for policy interventions?

[6] The precise reason of why lump-sum transfers can work with 0-1 holdings depend on details of our simplifying assumptions, and future research may have to choose between replicating our results numerically with unbounded holdings, or making small changes to our model. One could ask, for instance, whether details of the matching function are important in the 0-1 case. For simplicity we use that in Cavalcanti (2004), having half the population consuming in one period and producing in the other. Adding periodicity to the standard random-matching function (where a fraction $\frac{1}{N}$ of people are consumers, $N > 2$) should shift the parameter region for which interventions are desired but, we conjecture, not the essence of our results.

2 The environment

Time is discrete and the horizon is infinite. There are two types of people, each defined on a $[0,1]$ continuum. Each type is specialized in consumption and production: A type e person consumes even-date goods and produces odd-date goods, whereas a type d person consumes odd-date goods and produces even-date goods. We find it convenient to refer to a type e individual in an even (odd) date, or a type d individual in an odd (even) date, as a *consumer* (*producer*). Each type maximizes expected discounted utility, with a common discount factor $\beta \in (0,1)$. Let $s \in \{e,d\}$ indicate the season and/or the type of person and $\delta \equiv \beta^2$.

The utility function for a consumer in season $s \in \{e,d\}$ is $\varepsilon_s u_s(y_s)$, where ε_s is the idiosyncratic shock affecting the consumer and $y_s \in R_+$ is the amount consumed. The shock $\varepsilon_s \in \{0,1\}$ and the probability that $\varepsilon_s = 1$, denoted $\pi_s \in (0,1)$, is indexed by the season s. A producer in season s can produce any choice of $y_s \geq 0$ at a utility cost normalized to be y_s itself. Utility in a period is thus $\varepsilon_s u_s(y_s)$ when consuming and $-y_s$ when producing. The function u_s is assumed to be increasing, twice differentiable, and satisfies $u_s(0) = 0$, $u_s'' < 0$, $u_s'(0) = \infty$ and $u_s'(\infty) < 1$. We assume that $\pi_e \geq \pi_d$ and $u_e' \geq u_d'$, so that even dates feature a higher desire for consumption—at both the individual and aggregate levels—than odd dates. It should be emphasized that a strict inequality for either of these gives rise to a cyclical demand for liquidity.

In every period, a type e person is matched randomly with a type d person. During meetings, the realization of preference shocks occurs and production may take place. All individuals are anonymous in the sense that they all have *private* histories. We also assume that people cannot commit to future actions, so that those who produce must get a tangible (future) reward for doing so. In this article, the reward takes the form of fiat money. To keep the model simple, we assume that each person can carry from one meeting to the next either 0 or 1 units of fiat money; this assumption makes the economy-wide distribution of money holdings tractable. Consequently, trade will take place in a match only when the consumer realizes $\varepsilon_s = 1$ and has money, and the producer has no money.

Monetary policy takes the simple form of a choice of the pair (σ, τ), where σ is the probability that a person without money gets one unit of money before meetings, and τ is the probability that a person with money loses the money before meetings. Let M_e denote the measure of individuals holding money in even periods and M_d the measure of individuals holding money in odd ones. We restrict attention to cases in which either $\tau = \sigma = 0$ in all dates, or $\sigma > 0$ in even dates and $\tau > 0$ in odd dates. This simple formulation is designed to limit our analysis to the specific question of whether periods of high desire for consumption should have an increase in the supply of money, which is offset by a reduction of economy-wide money balances in the subsequent period.

3 Stationarity and welfare criteria

Let the measure of consumers with money during meetings in season s be denoted by q_s and consumers without money denoted by $1-q_s$; and let the measure of producers without money during meetings in season s be denoted by p_s and producers with money denoted by $1-p_s$. In order to save on notation, let $y \equiv (y_e, y_d)$ denote the list of output levels, $x \equiv (p_e, q_e, p_d, q_d)$ denote an arbitrary *distribution of money holdings*, and use the superscript $+$, as in $x^+ \equiv (p_e^+, q_e^+, p_d^+, q_d^+)$, when the qualification that $\sigma > 0$ for that distribution becomes essential. A distribution $x \in [0,1]^4$ is considered *invariant* if and only if there exists $(\sigma, \tau) \in [0,1]^2$ such that

$$p_e = (1-\sigma)(1-q_d+\pi_d p_d q_d), \tag{1}$$

$$p_d = (1-\tau)(1-q_e+\pi_e p_e q_e)+\tau, \tag{2}$$

$$q_e = (1-\sigma)(1-p_d+\pi_d p_d q_d)+\sigma, \tag{3}$$

and

$$q_d = (1-\tau)(1-p_e+\pi_e p_e q_e), \tag{4}$$

where the distribution x is described *after* money is created or destroyed.

The stationarity requirement (1) can be explained as follows: During odd-date meetings, trade takes place after money is destroyed. The measure of consumers with money is q_d, and the measure of producers without money is p_d. Consumers without money, whose number is $1-q_d$, cannot buy goods; each of them faces a probability σ of finding money at the beginning of the *next* date. Hence, $1-\sigma$ times $1-q_d$ is the total flow of consumers who become producers without money in the next (even) date. Similarly, the measure of consumers with money in the odd date is q_d. Only a fraction π_d of these consumers will want to consume in the odd date, and only a fraction p_d of them will meet a producer without money. Therefore, $\pi_d p_d q_d$ represents the measure of consumers with money that will trade in date d, and $(1-\sigma)\pi_d p_d q_d$ represents the number of these consumers that become producers without money in the next (even) period, after money creation takes place.

Likewise, regarding requirement (2), we first notice that a measure $1-q_e+\pi_e p_e q_e$ producers arrive at the beginning of date d without money. Adding to that the mass of money destroyed from date e consumers with money who did not trade, $\tau q_e(1-\pi_e p_e)$, yields the right-hand side of (2). The same principle explains requirement (3). The measure of consumers with money at date e consists of the measure of producers who leave date d with money, $1-p_d+\pi_d p_d q_d$, plus the measure of producers who leave date d without money but obtain some when additional money is created at the beginning of date e, $\sigma p_d(1-\pi_d q_d)$. Finally, requirement (4) follows from imposing stationarity on the measure of consumers with money arriving at date d, $1-p_e+\pi_e p_e q_e$, after the destruction of money takes place with probability τ.

Our notion of stationarity amounts to restricting that output, y_s, as well as the measures p_s and q_s, to be constant functions of the season, s, only. These functions are used symmetrically in a measure of welfare as follows: We adopt an *ex ante* welfare criterion, with an expected discounted utility computed according to an invariant distribution and output function. Whenever trade takes place in a season, it is because money is changing hands from a fraction p_s of the mass of consumers $\pi_s q_s$, i.e., those consumers in position to trade with a producer. Since there is a producer for each consumer, the flow of total utility in season s is $\pi_s p_s q_s[u_s(y_s)-y_s]$. We call the term $\pi_s p_s q_s$ the *extensive margin* at s, and $u_s(y_s)-y_s$ the *intensive margin* at s. The extensive margin is a property of the distribution x, and the intensive margin is a property of outputs y. An *allocation* is a pair (x,y), where x and y are invariant and y has non-negative coordinates. The *welfare* U attained by an allocation is defined as the present discounted value[7]

$$U(x,y)=\sum_s \pi_s p_s q_s[u_s(y_s)-y_s].$$

The intensive margin at s is maximized at y_s^*, where $u_s'(y_s^*)=1$, which is uniquely defined by assumption. We refer to $y^*=(y_e^*,y_d^*)$ as the *first-best* output list.

4 Implementable allocations

The definition of the values of y consistent with incentive compatibility follows the notion of sequential individual rationality employed by Cavalcanti and Wallace (1999) and Cavalcanti (2004). Underlying their definitions of participation constraints is the idea that a social planner proposes an allocation but anonymous individuals may defect from that proposal by not trading in a given meeting. If individual(s) defect, then they do not lose any money holdings that were brought into the meeting. We adopt the same concept here, with the exception of the taxation of money holdings, which we assume cannot be avoided by individuals with money. The participation

[7] The welfare measure U assumes that the initial date can be even or odd with equal probability. That is, assume with probability .5 the first date is even and with complementary probability the first date is odd. The expected total utility—our measure of social welfare—is, therefore, proportional to U.

constraints are then defined by a set of allocations, according to the expected discounted utilities implied by the allocations. To be able to represent these constraints, we first need to describe the Bellman equations of the economy.

The value functions will be computed *before* the realization of the effects of creation and destruction of money for each individual in a given date. (Recall that money is created at the beginning of even dates and is destroyed at the beginning of odd dates.) The value function for consumers *with* money at s is denoted by v_s, and that for producers *without* money is denoted by w_s. Let $\bar{v}_s$ be the value function for consumers without money at s and $\bar{w}_s$ be that for producers with money. The Bellman equations, $(v, w) = (v_e, v_d, w_e, w_d)$, are defined by

$$\begin{aligned} v_e &= \pi_e p_e(u_e + \beta w_d) + (1 - \pi_e p_e)\beta \bar{w}_d \\ w_e &= \sigma\beta v_d + (1 - \sigma)[\pi_e q_e(-y_e + \beta v_d) + (1 - \pi_e q_e)\beta \bar{v}_d] \\ v_d &= \tau\beta w_e + (1 - \tau)[\pi_d p_d(u_d + \beta w_e) + (1 - \pi_d p_d)\beta \bar{w}_e] \\ w_d &= \pi_d q_d(-y_d + \beta v_e) + (1 - \pi_d q_d)\beta \bar{v}_e, \end{aligned} \tag{5}$$

where u_e and u_d, by an abuse of notation, stand for $u_e(y_e)$ and $u_d(y_d)$, respectively. The definition is completed by substituting for the values of $(\bar{v}, \bar{w})$ given by

$$\begin{aligned} \bar{v}_e &= \sigma v_e + (1 - \sigma)\beta w_d \\ \bar{w}_e &= \beta v_d \\ \bar{v}_d &= \beta w_e \\ \bar{w}_d &= \tau w_d + (1 - \tau)\beta v_e \end{aligned} \tag{6}$$

into the previous system.

The participation constraint for producers at even dates is simply

$$-y_e + \beta v_d \geq \beta \bar{v}_d = \delta w_e, \tag{7}$$

since an even-date producer is bringing no money into a meeting and only has the option of leaving the meeting and becoming a producer two periods later. Producers at odd dates must take into account that if they disagree with producing the planned output y_d and walk away from a trade, then they have a chance of receiving money in the next period from the money-creation policy. Thus, the participation constraint for producers at odd dates can be stated as

$$-y_d + \beta v_e \geq \beta \bar{v}_e = \beta\sigma v_e + \delta(1 - \sigma)w_d. \tag{8}$$

For completeness, we state the participation constraint for consumers, which can be shown to be implied by the participation constraints of producers. They are

$$u_e + \beta w_d \geq \beta \bar{w}_d \tag{9}$$

and

$$u_d + \beta w_e \geq \beta \bar{w}_e. \tag{10}$$

An allocation (x, y) is said to be *implementable* if $x \equiv (p_e, q_e, p_d, q_d)$ is invariant for some policy (σ, τ) such that there exist (v, w) and $(\bar{v}, \bar{w})$, for which (5)-(10) hold. An allocation is said to be *optimal* if it maximizes $U(x, y)$ among the set of implementable allocations.

5 Extensive-margin effects

Monetary policy can be viewed as a choice of an invariant distribution x. Changes in x resulting from changes in (σ, τ) have direct effects on extensive margins, $\pi_s p_s q_s$, and indirect effects on intensive margins, $u_s(y_s) - y_s$, through the participation constraints; i.e., y depends on x. Note that the intensive margin effects can be "ignored" if, for both $s = e$ and $s = d$, the maximizer of $u_s(y_s) - y_s$, y_s^*, satisfies participation constraints. In this section, we ignore the intensive margin effects and investigate whether the maximizer of the sum $\sum_s \pi_s p_s q_s$—i.e., the maximizer of the extensive margin effect—among all invariant distributions of money holdings, x, is a *cyclical* policy x^+. A cyclical policy is one where $\sigma, \tau > 0$. We shall see that a cyclical monetary policy tends to increase the extensive margin in season e and to decrease that in season d. Since $u_e' \geq u_d'$, it will follow that *if* y^* satisfies participation constraints and the maximizer of the sum $\sum_s \pi_s p_s q_s$ is cyclical, then the optimal allocation is indeed characterized by a cyclical monetary policy. (If, however, y^* does not satisfy participation constraints, then the optimal allocation need not be a cyclical monetary policy.)

Acyclical distributions We start by pointing out an important property of the invariant distributions when the money supply is constant, i.e., when $\sigma = \tau = 0$. If x is invariant when $\sigma = \tau = 0$, we will say that x is *acyclical*, a label motivated by the following lemma:

Lemma 1 *Assume that x is acyclical. Then the extensive margin, $\pi_s p_s q_s$, is constant in s.*

Proof. Set $\sigma = \tau = 0$ in equations (1) and (4). It follows that $\pi_e p_e q_e = \pi_d p_d q_d$, i.e., the total number of trade matches is invariant to the season. ■

Interestingly, the property of constant extensive margins holds regardless of the relative values of π_s. We can offer an intuitive explanation for this property as follows: Let us consider the inflow and outflow of money for a set of individuals of the same type, say type e. Then, on one hand, the stationary measure of consumers of this type spending money is $(\pi_e p_e)q_e$, an event taking place at even dates. On the other hand, the stationary measure of producers of this type acquiring money is $(\pi_d q_d)p_d$, an event taking place at odd dates. Since the quantity of money in the hands of this group must be stationary, and all seasons have the same frequency, these two margins must be equalized, as stated in the lemma.

Some useful observations about acyclical distributions can be made with regard to the relative values of p_s and q_s.

Lemma 2 *Assume that x is acyclical. Then (i) $p_e - q_e = p_d - q_d$, and (ii) $p_e \leq p_d$ if and only if $\pi_d \leq \pi_e$.*

Proof. (i) Set $\sigma = \tau = 0$ in equations (1) and (2). Since, by lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, equations (1) and (2) imply that $p_e - q_e = p_d - q_d$. (ii) By lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, so $\pi_e \geq \pi_d$ if and only if $p_e q_e \leq p_d q_d$. Part (i) of this lemma implies that if $p_e q_e \leq p_d q_d$, then $p_e \leq p_d$ and $q_e \leq p_d$. ■

There is an alternative way to think about part (i) of lemma 1. The measure of individuals that hold money in period s, M_s, is the sum of consumers with money, q_s, and producers with money, $1 - p_s$. When $\sigma = \tau = 0$, the measures of individuals that hold money in odd and even periods are the same, i.e., $M_e = M_d$, which implies that $1 - p_e + q_e = 1 - p_d + q_d$, or that $p_e - q_e = p_d - q_d$.

An application of lemma 2 allows us to describe in rather simple terms the set of acyclical distributions when $\pi_e = \pi_d$.

Lemma 3 *Assume $\pi_e = \pi_d = \pi$. Then the set of acyclical distributions is fully described by $p_e = p_d = p$, $q_e = q_d = q$, and $p = 1 - q + \pi pq$ for $q \in [0, 1]$.*

Proof. Since $\pi_e = \pi_d$ then, by lemma 2, $p_e = p_d$, and consequently, by lemma 1, $q_e = q_d$. Equation (1) with $\sigma = 0$ thus proves the lemma. ■

The one-dimensional set described by lemma 3 is the symmetric set of distributions that appears in Cavalcanti (2004). The equation $p = 1 - q + \pi pq$ defines a strictly concave function for $q \in [0, 1]$, and the extensive margin πpq is maximized when $p = q = [1 - (1 - \pi)^{\frac{1}{2}}]/\pi$.

Properties similar to those described by lemma 3 also obtain when $\pi_e > \pi_d$; for example, every acyclical x can be indexed by a one-dimensional choice of q_d.

Lemma 4 *When $\pi_e > \pi_d$ there exists, for each q_s, a unique acyclical x. Moreover, x can be solved for analytically. The statement holds for any s in $\{e, d\}$.*

Proof. See appendix 1. ■

The extensive margin is maximized when the measure of consumers with money equals the measure of producers without money.

Lemma 5 *When $\pi_e > \pi_d$, the maximizer of $\pi_s p_s q_s$, among the set of acyclical distributions, is the unique x such that $p_s = q_s$ for $s \in \{e, d\}$, where*

$$p_d = \frac{1 + \sqrt{\frac{\pi_e}{\pi_d}} - \sqrt{(1 + \sqrt{\frac{\pi_e}{\pi_d}})^2 - 4\pi_d}}{2\pi_d},$$

and

$$p_e = 1 - p_d + \pi_d p_d^2.$$

Proof. See appendix 2. ■

Hence, when the money supply is constant, the distribution that maximizes the extensive margin is characterized by $p_d = q_d$ and $p_e = q_e$. This result echoes a standard result in many search models of money, namely, that it is optimal for half of the population to hold money. Such a distribution of money holdings maximizes the number of productive matches. To see that our model also has this feature, note that when $\sigma = \tau = 0$ and when $\pi_s p_s q_s$ is maximized, i.e., $p_s = q_s$ for $s \in \{e, d\}$, then the measure of individuals holding money at date s is $1 - p_s + q_s = 1$. Since the total measure of individuals in the economy is 2, having half the population holding money maximizes the extensive margin when $\sigma = \tau = 0$. Note that the value of x is easily computed when the extensive margin is maximized.

This completes our discussion of acyclical distributions, i.e., a constant money supply. We can now move on to cyclical money policy and cyclical distributions.

Cyclical distributions We now consider small perturbations in the quantity of money. We consider cyclical distributions x^+ in a neighborhood of a given acyclical x. Our ultimate goal is to describe and sign the derivative of the sum $\sum_s \pi_s p_s q_s$ with respect to σ, evaluated at $\sigma = 0$ and $p_s = q_s$. It follows, by force of lemma 5, that if this derivative is positive, then the maximizer of the sum $\sum_s \pi_s p_s q_s$ must be cyclical. Clearly, the system (1)-(4) that defines x^+ depends on σ and τ. The existence of x^+ follows from a simple fixed-point argument.

Lemma 6 *Let $(\tau, \sigma) \in (0, 1)^2$ be fixed. Then there exists an invariant distribution x^+.*

Proof. The right-hand side of (1)-(4) defines a continuous function of x^+, with domain on the compact and convex set $[0, 1]^4$. The result then follows from Brower's fixed-point theorem. ■

If x^+ is invariant, then the quantity of money destroyed in season d must equal the quantity created in e, i.e.,

$$\tau(1 - p_e^+ + q_e^+) = \sigma(1 - q_d^+ + p_d^+). \tag{11}$$

It can be shown that the equality (11) is implied by the system (1)-(4). The quantity of money during season e meetings, just before trade, is given by the mass $1 - p_e^+$ of producers plus the mass q_e^+ of consumers. Since trade itself does change this quantity of money, and each money holder at the beginning of next season faces a probability τ of losing his money, then the total amount of money destroyed is given by the left-hand side of (11). Likewise, the measure of individuals without money at the end of season d is $1 - q_d^+ + p_d^+$, and since each of those finds money at the beginning of season e with probability σ, then the quantity of money created is expressed in the right-hand side of (11).

When $\sigma = \tau = 0$, there is a unique x (lemma 4) for each choice of q_d satisfying stationarity. When σ and τ are strictly positive, there is an inflow of money that must be matched by an outflow of the same quantity. Our numerical experiments indicate that only one level of q_d^+ produces quantities of money equalizing inflows and outflows for a given pair (σ, τ). A more practical approach is to restrict attention to stationary distributions x^+ that result from small interventions. We can pin down the neighborhood in which q_d^+ lies as follows: Because we want to associate x^+ with a given x, we find it useful to define the constant ϕ with the property that, for $\tau = \phi\sigma$, x^+ converges to x as σ approaches zero. Since the pair (σ, τ) must be consistent with the stationary quantities of money in the economy, expressed above by equation (11), the desired ratio of τ to σ, for a given $x = (p_e, q_e, p_d, q_d)$, is

$$\phi = \frac{1 - q_d + p_d}{1 - p_e + q_e}. \tag{12}$$

By lemmas 1 and 5, the maximizer of the sum $\sum_s \pi_s p_s q_s$ among the set of *acyclical* distributions is the unique x for which $\phi = 1$. We assess the effects of perturbations by differentiating the system (1)-(4) with respect to σ for ϕ fixed.

The lemma next can be viewed as generalizing lemma 2; in other words, the difference between the measures of consumers with money and producers without money will be equalized between seasons only if the distribution is acyclical.

Lemma 7 *If x is invariant and $\tau = \phi\sigma$, then $p_s - q_s = f_s(\sigma)$, where $f_e(\sigma) = \frac{\phi - \phi\sigma - 1}{1 + \phi - \phi\sigma}$ and $f_d(\sigma) = \frac{\phi + \phi\sigma - 1}{1 + \phi - \phi\sigma}$.*

Proof. See appendix 3. ■

Note that f_s does not depend on the fraction of consumers who desire to consume in season s, π_s.

The next proposition, which is the main result of this section, characterizes the sign of the derivative of the sum $\sum_s \pi_s p_s q_s$, evaluated at $p_s = q_s$, and $\sigma = 0$ (the latter two equalities characterize the optimal constant-money-supply policy).

Proposition 1 *The maximizer of the sum* $\sum_s \pi_s p_s q_s$ *is cyclical if and only if* $\pi_d \in [0, \Pi(\pi_e)]$*, where* $\Pi(\pi_e) \in (0, \pi_e)$ *can be solved for analytically as a function of* π_e*.*

Proof. See appendix 4. ■

This proposition effectively says that if the number of consumers who want to consume in the even season, π_e, is sufficiently larger those who want to consume in the odd season, π_d, then a cyclical policy maximizes the extensive margin. Obviously, maximizer is acyclical if $\pi_d = \pi_e$. The intuition behind which policy—acyclical or cyclical—maximizes the average extensive margin can be understood by way of a simple example. First note if $\tau = \sigma = 0$, then the distribution that maximizes the average trade across seasons features $p_e = q_e$ and $p_d = q_d$. If, for example, $\pi_e = 1$ and $\pi_d = \frac{1}{10}$, then $q_e = p_e = \frac{1}{4}$ and $q_d = p_d = \frac{4}{5}$. Note that in the low-trade season, d, the distribution of money is relatively good in the sense that most consumers have money and most producers do not. However, in the high-trade season, e, the distribution of money is relative bad since most consumers do not have money while most producers do. Now consider the marginal effect of policy $(\tau, \sigma) \neq (0, 0)$. If the initial intervention is to withdraw in an odd season, the measure of producers without money will change to $p_d + \frac{1}{5}\tau$, while that of consumers with money changes to $q_d - \frac{4}{5}\tau$. Notice that the effect on producers is weighted by $\frac{1}{5}$, while that on consumers is weighted by $\frac{4}{5}$, because the population affected by withdrawals—the holders of money—are distributed unevenly between consumer and producer status. If the initial intervention is an injection at an even season, the measure of consumers with money changes to $q_e + \frac{3}{4}\sigma$, while that of producers without money changes to $p_e - \frac{1}{4}\sigma$. If $\sigma \approx \tau \approx 0$, then the initial effect is an increase in $p_e q_e$—by the amount $\frac{1}{2}\sigma$— and a decrease in $p_d q_d$—by the amount $\frac{3}{5}\tau$. There, however, no guarantee that the *average* amount of trade, $\pi_e p_e q_e + \pi_d p_d q_d$, will increase. The average amount of trade will increase only if π_e is sufficiently greater than π_d since the increase in trading opportunities in the even season, $\frac{1}{2}\sigma$, is less than the decrease in trading opportunities in the odd season, $\frac{3}{5}\tau$. But if the amount of trade in the low-trade season is "small" compared to the high-trade season, i.e., π_d is "small" compared to π_e, then average trade can increase since a relatively small weight, π_d, is attached to the low-trade season.

Hence, if the fraction of potential consumers in odd seasons is sufficiently smaller than the fraction of potential consumers in even seasons—or if demand in the "high" season is sufficiently greater than demand in the "low" season—then a cyclical monetary policy will deliver a higher average extensive margin than the optimal acyclical policy.

6 Intensive-margin effects

The only participation constraints that are relevant, given our notion of stationarity, are those of producers, i.e., if the producer participation constraints are satisfied, then so are those of the consumer. In this section, we derive the producer participation constraints as functions of preference parameters, policy parameters, and allocations, without reference to value functions.

Lemma 8 *The participation constraints are satisfied if and only if*

$$u_d(y_d) \geq \frac{y_e}{\beta}\left[\frac{1}{(1-\tau)\pi_d p_d} - (1-\sigma)\delta\frac{1-\pi_d p_d}{\pi_d p_d}\right] \tag{13}$$

and

$$u_e(y_e) \geq \frac{y_d}{\beta}\left[\frac{1}{(1-\sigma)\pi_e p_e} - (1-\tau)\delta\frac{1-\pi_e p_e}{\pi_e p_e}\right]. \tag{14}$$

Proof. See appendix 5. ■

Inequalities (13) and (14) indicate that cyclical policies have a potentially negative effect on intensive margins, since the right-hand side of both inequalities is increasing in σ and τ. The intuition behind these potential negative effects is straightforward: In either case—whether money is injected or withdrawn from the economy—the value of money in a trade will fall compared to the situation where $\sigma = \tau = 0$. In the case where the money supply is contracted after production and trade, the value of currency falls because there is a chance that the producer will be unable to use his unit of currency in a future trade because it will be taken away; in the case where the money supply is expanded after production and trade, the fact that a producer may receive a unit of currency if he does not produce reduces the value of a unit of currency for a producer who does. A fall in the value of money implies that the amount of output received per unit of currency is reduced. If, however, β is sufficiently high, then inequalities (13) and (14) will not bind at $y = y^*$—the output levels that maximize the intensive margins—in the neighborhood of $\sigma = \tau = 0$; hence, the potential effects on the intensive margins do not materialize for small monetary interventions in this case.

Suppose that neither participation constraint binds when $\sigma = \tau = 0$. Then, it turns out that if β is reduced, the first participation constraint to be violated is the participation constraint for date-e producers, (13). Hence,

Lemma 9 *If the participation constraint for date-e producers is satisfied for x acyclical and $y = y^*$, then (x, y) is implementable.*

Proof. Since $u'_e \geq u'_d$ and $u_e(0) = u_d(0)$, then $u'_e(y^*_d) \geq 1$, so that $y^*_e \geq y^*_d$ and $u^*_e(y^*_e) \geq u^*_d(y^*_d)$. Now, it has been established in the previous section that, if x is acyclical, then $\pi_d \leq \pi_e$ implies $q_d \leq q_e$. As a result, since the equality $\pi_e p_e q_e = \pi_d p_d q_d$ holds for all acyclical x, $\pi_d \leq \pi_e$ implies $\pi_d p_d \leq \pi_e p_e$. Since the right-hand side of (13) or (14) is increasing in $\pi_s p_s$, and since $u_e^*(y_e^*)/y_d^* \geq u_d^*(y_d^*)/y_e^*$, then the result follows. ■

Lemma 9 indicates that it suffices to look at the participation constraint for date-e producers in order to find a value of β such that small interventions have no negative effects on intensive margins. The next proposition characterizes the critical β for the optimal *acyclical* distribution such that the participation constraint for the date-e producer "just" binds.

Proposition 2 *Let x take the value of the acyclical distribution with $p_s = q_s$, and let $\beta > \bar{\beta}$, where*

$$\bar{\beta} = \frac{-\frac{u_d(y_d^*)}{y_e^*} + \sqrt{\left(\frac{u_d(y_d^*)}{y_e^*}\right)^2 + 4\frac{1-\pi_d p_d}{(\pi_d p_d)^2}}}{2\frac{1-\pi_d p_d}{\pi_d p_d}}.$$

Then, if σ is sufficiently small, the cyclical allocation (x^+, y^*) for x^+, in a neighborhood of x, is implementable.

Proof. The cutoff value $\bar{\beta}$ is constructed so that (x, y^*) is implementable at $\beta = \bar{\beta}$ when the policy is the optimal acyclical one. The cutoff $\bar{\beta}$ is constructed by setting $y = y^*$ and $\sigma = \tau = 0$ in (13) with a strict equality. Since the participation constraint sets vary continuously with (σ, τ), the result follows. ■

Looking ahead, proposition 1 tells us when a cyclical policy maximizes the extensive margin and proposition 2 gives us a scenario where a cyclical policy does not have an adverse affect on the intensive margin. These two propositions will be helpful when thinking about optimal policies.

7 Optimal policies

On one hand, our results regarding extensive-margin effects show that there exists a cutoff value for π_d, called $\Pi(\pi_e)$, such that the maximizer of the average extensive margin is cyclical if and only if $\pi_d < \Pi(\pi_e)$. On the other hand, our results on intensive margins show that there exists a cutoff value of β, called $\bar{\beta}$, such that for $\beta > \bar{\beta}$, small interventions around the allocation (x, y^*), where $p_s = q_s$, are implementable. It follows, therefore, that the optimum is cyclical for a large set of parameters, including π_s and β such that $\pi_d < \Pi(\pi_e)$ and $\beta > \bar{\beta}$.

Proposition 3 *If $\pi_d < \Pi(\pi_e)$ and $\beta \geq \bar{\beta}$, then the optimum monetary policy is cyclical.*

Proof. Welfare is proportional to $\sum_s E_s I_s$, where E_s is the extensive margin at s, $\pi_s p_s q_s$, and I_s is the intensive margin at s, $u_s(y_s) - y_s$. By lemma 1, $E_e = E_d$ for all acyclical policies, so that for fixed (I_e, I_d), the acyclical x that maximizes welfare features $p_s = q_s$. Since $\beta \geq \bar{\beta}$, y^* satisfies participation constraints evaluated at this maximizer, so that the allocation that attains the highest welfare among acyclical policies is (x, y^*). Since a small intervention increases E_e and $E_e + E_d$ when $\pi_d < \Pi(\pi_e)$, and $I_e \geq I_d$ for $y = y^*$, and such intervention is implementable according to our last proposition, then the optimal cannot be acyclical. ■

Proposition 3 provides only sufficient conditions for the optimality of a cyclical monetary policy. These conditions are not necessary. To see this, note that when $u_e = u_d$, $\pi_d = \Pi(\pi_e)$, and $\beta = \bar{\beta}$, proposition 1 and 2 implies that

$$\sum_s \pi_s p_s^+ q_s^+ \left[u_s\left(y_s^+\right) - y_s^+\right] = \sum_s \pi_s p_s q_s \left[u_s\left(y_s^*\right) - y_s^*\right],$$

where x^+ is a cyclical distribution, x is the (optimal) acyclical distribution and, by construction, $y_e^+ = y_d^+ = y_d^* = y_e^*$. If, however, $u_e' > u_d'$, $\pi_d = \Pi(\pi_e)$, and $\beta = \bar{\beta}$, then

$$\sum_s \pi_s p_s^+ q_s^+ \left[u_s\left(y_s^+\right) - y_s^+\right] > \sum_s \pi_s p_s q_s \left[u_s\left(y_s^*\right) - y_s^*\right], \tag{15}$$

since $u_e\left(y_e^+\right) - y_e^+ = u_e\left(y_e^*\right) - y_e^* > u_d\left(y_d^+\right) - y_d^+ = u_d\left(y_d^*\right) - y_d^*$, $\pi_e p_e^+ q_e^+ > \pi_e p_e^* q_e^*$, and $\sum_s \pi_s p_s^+ q_s^+ = \sum_s \pi_s p_s q_s$. Therefore, when $u_e' > u_d'$ there exist (non-unique) numbers $\hat{\beta} < \bar{\beta}$ and $\hat{\pi} < \Pi(\pi_e)$ such that for any $\beta \in \left(\hat{\beta}, \bar{\beta}\right)$ and $\pi \in (\hat{\pi}, \Pi(\pi_e))$, inequality (15) holds. Therefore, proposition 3 describes the conditions that are sufficient, but not necessary, for the optimal money policy to be cyclical. (We have documented these properties with numerical simulations, which are available upon request.) As a result, cyclical monetary policy may be optimal for some economies where the conditions of proposition 3 do not hold.

8 Conclusion

The fundamental idea behind this paper is quite simple. It is *not* about the rate of return to money nor about how policy can equate marginal rates of substitution, as in Sargent and Wallace (1982). It *is* about how the reflux of money can be inefficient and how cyclical policies may accommodate more trade during high-demand seasons at the expense of less trade in low-demand seasons and a less valuable currency. The paper provides a foundation for the optimality of a cyclical monetary policy.

We admit that our model environment is extreme. For example, there is an absence of borrowing and lending and other institutions that resemble modern economies. These omissions have to be addressed.

The issue of borrowing and lending is important because, potentially, they could substitute for a cyclical policy. However, very little is known about this issue. In Walrasian markets credit works well; so well, in fact, that it ends up substituting for money. The coexistence of money and credit is only meaningful when monitoring is imperfect and, as result, there is imperfect insurance. Hence, money holdings will be distributed unevenly due to imperfect insurance. But then, problems associated with the reflux and circulation of money arise and the need for cyclical policy may be warranted. In fact, we believe that our model provides a justification for the creation of the Fed.[8]

The preamble to the Federal Reserve Act states that the reserve banks were established to furnish an *elastic* currency. According to Meltzer (2003), the term elasticity has two meanings in the context of the Federal Reserve Act. One has to do with the central bank's ability to pool reserves and lend them out (in the event of a banking or financial crisis) and the other—which is the relevant one for this paper—has to do with the central bank's ability to expand and contract liquidity over the various seasons of the year.

At the time of the founding of the Fed, seasonal demand for currency posed a real problem for the agricultural sector and, given the size of the agricultural sector at that time, for the economy as a whole. Farmers needed large amounts of cash (or credit backed by cash) in both the spring and autumn seasons in order to plant and harvest, respectively, their crops. However, prior to the founding of the Fed, it was not unusual to have this needed liquidity sitting at major banking centers instead of in the hands of farmers during these seasons of the year.

It appears that US economy lacked a mechanism that would permit farmers to "get their hands" on cash when they needed it. One proposed solution to this problem was the creation of a national wide banking system—the Federal Reserve System—that had the ability to furnish an elastic currency, where the notion of an elastic currency has to do with increasing currency in seasons of high economic activity and then decreasing it in seasons of low economic activity. The Federal Reserve "solution" to this season problem is close in spirit to our cyclical policy (although our model does not have banks to implement monetary policy and instead relies on lump-sum taxes and transfers).

Although our model is useful for understanding central bank policies from a century ago, is it relevant for understanding modern policies? The modern counterpart of money holdings in our model is some measure of liquidity, which is composed of

[8] Any other mechanism-design formulation deriving welfare benefits from interventions, like that in Levine (1991), could motivate the creation of Fed. Because our model does not predict ongoing inflation (or deflation), we think that our motivation is more compelling. Should banking stability, which we ignore, be the front-runner motivation? We leave this decision for the reader.

many assets whose returns are affected by Fed-controlled interest rates. To the extent that changes in Fed policies affect the return of different assets and the distribution and circulation of "liquidity" across businesses and individuals, the fundamental idea behind this paper still applies. But there are many issues to be resolved. For example, although the distribution and circulation of money have precise meaning in our model, when we move to modern economies, what exactly do we mean by the distribution and circulation of liquidity? This is a challenge for future research.

References

Bewley, T. 1980 "The Optimum Quantity of Money," in J. Kareken and N. Wallace, eds., *Models of Monetary Economics* (Federal Reserve Bank of Minneapolis, Minneapolis, MN).

Cavalcanti, R. de O. 2004 "A Monetary Mechanism for Sharing Capital: Diamond and Dybvig Meet Kiyotaki and Wright," *Economic Theory* 24 769–88.

Cavalcanti, R., A. Erosa and T. Temzelides 1999 "Private Money and Reserve Management in a Random-Matching Model," *Journal of Political Economy* 107 929-945.

Cavalcanti, R. de O. and N. Wallace 1999 "A Model of Private Banknote Issue," *Review of Economic Dynamics* 2 104-36.

Deviatov, A. and N. Wallace 2001 "Another Example in which Money Creation is Beneficial," *Advances in Macroeconomics* 1, article 1.

Friedman, M. 1969 "The Optimum Quantity of Money," in *The Optimum Quantity of Money and Other Essays* (Aldine, Chicago IL).

Kiyotaki, N. and R. Wright 1989 "On Money as a Medium of Exchange," *Journal of Political Economy* 91 927–54.

Lagos, R. and R. Wright 2005 "A Unified Framework for Monetary Theory and Policy Analysis," *Journal of Political Economy* 113 463-484.

Levine, D. K. 1991 "Asset Trading Mechanisms and Expansionary Policy," *Journal of Economic Theory* 54 148–64.

Lucas, R. E. 1972 "Expectations and the Neutrality of Money," *Journal of Economic Theory* 4 103–24.

Meltzer, A. 2003 *A History of the Federal Reserve*, Volume 1: 1913-1951 (University of Chicago Press, Chicago IL).

Miron J. 1986 "Financial Panics, the Seasonality of the Nominal Interest Rate and the Founding of the Fed," *American Economic Review* 76 125–40.

Molico, M. 2006 "The Distribution of Money and Prices in Search Equilibrium," *International Economic Review* 47 701-722.

Sargent, T and N. Wallace 1982 "The Real-Bills Doctrine versus the Quantity Theory: A Reconsideration," *Journal of Political Economy* 90 1212-36.

Sheinkman, J. A. and L. Weiss 1986 "Borrowing Constraints and Aggregate Economic Activity," *Econometrica* 54 23–45.

Shi, S. 1997 "A Divisible Search Model of Fiat Money" Econometrica 65 75–102.

Wallace, N. and T. Zhu "Float on a Note," *Journal of Monetary Economics* forthcoming.

Appendix 1

Lemma 4 *When $\pi_e > \pi_d$ there exists, for each q_s, a unique acyclical x. Moreover, x can be solved for analytically. The statement holds for any s in $\{e, d\}$.*

Proof. We shall make repeated use of the system (1-4) with $\sigma = \tau = 0$. According to lemma 2, $p_s = q_s + a$ for some a that does not depend on s. We shall first solve for a analytically. For this purpose, let $A \equiv 1 + \pi_s p_s q_s$, which, by force of lemma 1, does not depend on s as well. Equation (1) now reads (i) $p_e = A - q_d$. Using (ii) $p_d = q_d + a$, we can write (2) as (iii) $q_e = A - (a + q_d)$. The equality $p_e q_e = \theta p_d q_d$ for $\theta = \pi_d/\pi_e$, can be written, using (i), (ii), and (iii), as $(A - q_d)^2 - a(A - q_d) - \theta q_d(a + q_d) = 0$. The only relevant solution of this quadratic equation is given by (iv) $2(A - q_d) = a + \sqrt{a^2 + 4\theta b}$, where $b = q_d(a + q_d)$. Since $A = 1 + \pi_d q_d p_d = 1 + \pi_d b$, we can rewrite (iv) as (v) $a^2 + 4\theta b = [2\pi_d b + 2(1 - q_d) - a]^2$. Expanding (v) as a quadratic equation in b, we find that the only relevant solution is given by (vi) $2\pi_d^2 b = \theta + a\pi_d - 2\pi_d(1 - q_d) + \sqrt{\theta^2 - 4\pi_d(1 - q_d)\theta + \pi_d^2 a^2 + 2\theta\pi_d a}$. Substituting in (vi) the expression $b = q_d(a + q_d)$, produces a quadratic equation in a as a function of q_d. The only relevant solution of the latter is (vii) $a = [-k_2 - \sqrt{k_2^2 - 4k_1k_3}]/(2k_1)$, where $k_1 = \pi_d^2[(2\pi_d q_d - 1)^2 - 1]$, $k_2 = 2\pi_d\{(2\pi_d q_d - 1)[2(\pi_d q_d)^2 + 2\pi_d(1 - q_d) - \theta] - \theta\}$, and $k_3 = [2(\pi_d q_d)^2 + 2\pi_d(1 - q_d) - \theta]^2 - \theta^2 + 4\pi_d(1 - q_d)\theta$. If q_d is fixed, then $p_d = q_d + a$ determines p_d. Using (1) and $p_e = q_e + a$, the values of p_e and q_e are also determined. Since the system (1-4) is symmetric in e and d, when $\sigma = \tau = 0$, similar conclusions follow when q_e is given, instead of q_d. ■

Appendix 2

Lemma 5 *When $\pi_e > \pi_d$, the maximizer of $\pi_s p_s q_s$, among the set of acyclical distributions, is the unique x such that $p_s = q_s$ for $s \in \{e, d\}$, where*

$$p_d = \frac{1 + \sqrt{\frac{\pi_e}{\pi_d}} - \sqrt{(1 + \sqrt{\frac{\pi_e}{\pi_d}})^2 - 4\pi_d}}{2\pi_d},$$

and

$$p_e = 1 - p_d + \pi_d p_d^2.$$

Proof. The set of acyclical distributions is closed, and $\pi_s p_s q_s$ is continuous in x for each s, so that a maximizer exists. Let us fix $x = x^1$, with $p_s^1 \neq q_s^1$ for some s, and show that x^1 cannot be the maximizer. Note that, by lemma 2, $p_s^1 \neq q_s^1$ if and only if $p_{s'}^1 \neq q_{s'}^1$. We start by constructing x^2, the "mirror image" of x^1, with the equalities $p_s^2 = q_s^1$ and $q_s^2 = p_s^1$ for $s \in \{e, d\}$. Also, for $\alpha \in (0, 1)$, let $x^\alpha \equiv \alpha x^1 + (1 - \alpha)x^2$. It is clear that, for all s, $\alpha p_s^1 q_s^1 + (1 - \alpha)p_s^2 q_s^2 < p_s^\alpha q_s^a$. Thus the distribution of x^α

attains a higher extensive margin than that of x^1, although x^α is not invariant if it does not satisfy (1-4) with equality. However, using now lemma 1, one can rewrite each equation in the system (1-4), when $\sigma = \tau = 0$, as $p_s + q_{s'} = 1 + \pi_s p_s q_s$ or $p_{s'} + q_s = 1 + \pi_s p_s q_s$, where $s' \neq s$, so that each right-hand side is increasing in the extensive margin. Since $p_s^\alpha + q_{s'}^\alpha < 1 + \pi_s p_s^\alpha q_s^\alpha$ and $p_{s'}^\alpha + q_s^\alpha < 1 + \pi_s p_s^\alpha q_s^\alpha$, then there exists an acyclical $\bar{x}$, with $\bar{x} \geq x^\alpha$, that attains a higher extensive margin than that of x.

Since by lemma 1, $\pi_e p_e q_e = \pi_d p_d q_d$, equation (2) with $\tau = 0$ yields $p_d = 1 - q_e + \pi_e p_e q_e$. Because $q_s = p_s$, then $p_e = \sqrt{\frac{\pi_d}{\pi_e}} p_d$ and $p_d = 1 - p_e + \pi_d p_d^2$. The last two expressions yield a quadratic equation in p_d whose only relevant solution is as stated. The value for p_e can be computed from the last expression once p_d is determined. The proof is now complete. ■

Appendix 3

Lemma 7 *If x is invariant and $\tau = \phi\sigma$, then $p_s - q_s = f_s(\sigma)$, where $f_e(\alpha) = \frac{\phi - \phi\alpha - 1}{1 + \phi - \phi\alpha}$ and $f_d(\alpha) = \frac{\phi + \phi\alpha - 1}{1 + \phi - \phi\alpha}$.*

Proof. The system (1-4) can be rewritten as

$$\hat{p}_e = 1 - (1 - \tau)\hat{q}_d + \pi_d p_d q_d, \tag{16}$$

$$\hat{p}_d = 1 - (1 - \sigma)\hat{q}_e + \pi_d p_e q_e + \frac{\tau}{1 - \tau}, \tag{17}$$

$$\hat{q}_e = 1 - (1 - \tau)\hat{p}_d + \pi_d p_d q_d + \frac{\sigma}{1 - \sigma}, \tag{18}$$

and

$$\hat{q}_d = 1 - (1 - \sigma)\hat{p}_e + \pi_e p_e q_e, \tag{19}$$

where $\hat{p}_e = p_e/(1 - \sigma)$, $\hat{p}_d = p_d/(1 - \tau)$, $\hat{q}_e = q_e/(1 - \sigma)$, and $\hat{q}_d = q_d/(1 - \tau)$. Eliminating $\pi_d p_d q_d$ between equations (16) and (18), and $\pi_e p_e q_e$ between (17) and (19), yields

$$\hat{p}_e - \hat{q}_e = (1 - \tau)(\hat{p}_d - \hat{q}_d) - \frac{\sigma}{1 - \sigma}$$

and

$$\hat{p}_d - \hat{q}_d = (1 - \sigma)(\hat{p}_e - \hat{q}_e) + \frac{\tau}{1 - \tau},$$

which can now be solved as

$$\hat{p}_e - \hat{q}_e = \frac{(1 - \sigma)\tau - \sigma}{(1 - \sigma)[1 - (1 - \tau)(1 - \sigma)]} \tag{20}$$

and

$$\hat{p}_d - \hat{q}_d = \frac{\tau - (1 - \tau)\sigma}{(1 - \tau)[1 - (1 - \tau)(1 - \sigma)]}. \tag{21}$$

One can now multiply both sides of (20) by $1-\sigma$ to obtain the expression $p_e - q_e = f_e(\sigma)$, and multiply both sides of (21) by $1-\tau$ to obtain the expression $p_d - q_d = f_d(\sigma)$. ■

Appendix 4

Before we can provide a proof for proposition 1, the following two lemmas are needed. From lemma 7 we can use the expression $q_s = p_s - f_s$ to reduce (1)-(4) to a system in (p_e, p_d), which allows us to write the derivatives of p_s with respect to σ as follows.

Lemma 10 *If x is invariant and $\tau = \phi\sigma$, then the derivatives of p_s with respect to σ, evaluated at $\sigma = 0$, satisfy*

$$\begin{bmatrix} 1 & 1-\pi_d(2p_d - f_d) \\ 1-\pi_e(2p_e - f_e) & 1 \end{bmatrix} \begin{bmatrix} p_e' \\ p_d' \end{bmatrix} = \begin{bmatrix} (1-\pi_d p_d)f_d' - p_e \\ (1-\pi_e p_e)f_e' - \phi p_d + \phi \end{bmatrix}.$$

Proof. Equations (1) and (2) can be written as

$$\frac{p_e^+}{1-\sigma} = 1 - p_d^+ + f_d + E_d \tag{22}$$

and

$$\frac{p_d^+}{1-\phi\sigma} - \frac{\phi\alpha}{1-\phi\sigma} = 1 - p_e^+ + f_e + E_e, \tag{23}$$

where $E_d = \pi_d p_d^+ (p_d^+ - f_d)$ and $E_e = \pi_e p_e^+ (p_e^+ - f_e)$. Taking derivatives on both sides of (22) and (23), with respect to σ, yields, for $\sigma = 0$,

$$p_e + p_e' = -p_d' + f_d' + E_d' \tag{24}$$

and

$$\phi p_d + p_d' - \phi = -p_e' + f_e' + E_e', \tag{25}$$

where $E_d' = \pi_d p_d'(2p_d - f_d) - \pi_d p_d f_d'$ and $E_e' = \pi_e p_e'(2p_e - f_e) - \pi_e p_e f_e'$. Substituting the expressions for E_d' and E_e' into equations (24) and (25) yields the result. ■

The total effect of changes in σ on extensive margins can also be expressed in a compact form.

Lemma 11 *If x is invariant and $\tau = \phi\sigma$, then the derivative of the sum $\sum_s \pi_s p_s q_s$, with respect to σ, evaluated at $\sigma = 0$, is equal to $p_e + \phi p_d - \phi - f_e' - f_d' + 2(p_e' + p_d')$.*

Proof. Using equations (24) and (25), derived in the proof of the previous lemma, yields the results because the derivative of the sum $\sum_s \pi_s p_s q_s$ is precisely $E_d' + E_e'$. ■

Using lemmas 6, 10 ,and 11, we can characterize the sign of the derivative of the sum $\sum_s \pi_s p_s q_s$, for $p_s = q_s$, as follows:

Proposition 1 *The maximizer of sum* $\sum_s \pi_s p_s q_s$ *is cyclical if and only if* $\pi_d \in [0, \Pi(\pi_e)]$, *where* $\Pi(\pi_e) \in (0, \pi_e)$ *can be solved for analytically as a function of* π_e.

Proof. Lemmas 6, 10, and 11 allow the substitution of expressions for $p'_e + p'_d$ and $f'_e + f'_d$ into the expression of the derivative of $\sum_s \pi_s p_s q_s$, evaluated at $\sigma = 0$, $p_s = q_s$, and $\phi = 1$. Substituting also the analytical solution for p_e and p_d, when $p_s = q_s$ and $\sigma = 0$ from lemma 5, yields an expression for the derivative involving only parameters. After some tedious but straightforward algebra, the condition according to which this derivative is positive can be written as

$$2\pi_d \leq (1-\theta)\sqrt{2} - (1-\sqrt{\theta})^2,$$

where $\theta = \pi_d/\pi_e$. The inequality is not satisfied for $\theta = 1$ and $\pi_d > 0$. Hence, the cutoff value of π_d for which the derivative is positive must be below π_e. Imposing equality in this expression and substituting for the value of θ yields, after solving for the unique relevant solution of the implied quadratic equation in π_d^2,

$$\Pi(\pi_e) = \frac{1}{4}\left[\frac{2/\sqrt{\pi_e} + \sqrt{4/\pi_e - 4(2 + (1+\sqrt{2})/\pi_e)(1-\sqrt{2})}}{2 + (1+\sqrt{2})/\pi_e}\right]^2,$$

which has the properties stated in the proposition. ■

Appendix 5

Before providing a proof of lemma 8, we will first rewrite (v, w) in a convenient form and will then introduce two lemmas that will be needed in the proof. Substituting the values of $(\bar{v}, \bar{w})$ from equation (6) into equation (5) allows us to work with two independent systems of Bellman equations in (v, w), represented in matrix format as

$$\begin{bmatrix} v_s \\ w_{s'} \end{bmatrix} = \frac{1}{\det(M_{ss'})} M_{ss'} \begin{bmatrix} \mu_{us}\pi_s p_s u_s \\ -\mu_{ys'}\pi_{s'} q_{s'} y_{s'} \end{bmatrix}, \tag{26}$$

where $s, s' \in \{e, d\}$, $s' \neq s$, $\mu_{ue} = \mu_{yd} = 1$, $\mu_{ud} = 1 - \tau$, $\mu_{ye} = 1 - \sigma$, and

$$M_{ss'} = \begin{bmatrix} 1 - (1 - \pi_{s'} q_{s'})\delta(1-\sigma) & \tau\beta + (1-\tau)\pi_s p_s \beta \\ \sigma\beta + (1-\sigma)\pi_{s'} q_{s'} \beta & 1 - (1 - \pi_s p_s)\delta(1-\tau) \end{bmatrix}.$$

We start with the following lemma, which allows us to ignore $\det(M_{ss'})$ in the algebra that follows.

Lemma 12 *The determinant of* $M_{ss'}$ *is positive.*

Proof. For $a_d \equiv 1 - \pi_d q_d$ and $a_e \equiv 1 - \pi_e p_e$, the determinant of M_{ed} equals

$$(1 - \delta a_d + \sigma\delta a_d)(1 - \delta a_e + \tau\delta a_e) - \delta(\pi_d q_d + \sigma a_d)(\pi_e p_e + \tau a_e),$$

which can be written as the sum of two terms, k_0 and k_1, where k_0 contains all the terms without σ or τ, and k_1 contains the other terms. The expression for k_0 is

$$k_0 = [1 - \delta(1 - \pi_d q_d)][1 - \delta(1 - \pi_e q_e)] - \delta\pi_d q_d \pi_e q_e.$$

After some simple algebra, that expression becomes

$$k_0 = (1 - \delta)(1 - \delta + \delta\pi_d q_d + \delta\pi_e q_e - \delta\pi_d q_d \pi_e q_e),$$

which is positive if x is invariant. Likewise, since for $a_d \equiv 1 - \pi_d q_d$ and $a_e \equiv 1 - \pi_e p_e$, one can write k_1 as

$$\begin{aligned}
&\tau\delta a_e(1 - \delta a_d - \pi_d q_d) + \sigma\delta a_d(1 - \delta a_e - \pi_e p_e) + \delta\sigma a_d \tau a_e(\delta - 1), \text{ or} \\
&\tau\delta a_e(1 - \delta)(1 - \pi_d q_d) + \sigma\delta a_d(1 - \delta)(1 - \pi_e p_e) - \sigma\delta a_d(1 - \delta)\tau a_e, \text{ or} \\
&\tau\delta a_e(1 - \delta)(1 - \pi_d q_d) + \sigma\delta a_d(1 - \delta)(1 - \pi_e p_e)(1 - \tau),
\end{aligned}$$

which is nonnegative. A similar argument shows that $\det(M_{de})$ is also positive. ■

Next, we use the Bellman equation for $w_e$to write (7) in an equivalent format that does not depend on y_e explicitly.

Lemma 13 *The participation constraint for date-s producers is equivalent to* $[1 - (1 - \sigma)\delta]w_s \geq \sigma\beta v_{s'}$.

Proof. Let $s = e$. The Bellman equation for w_e can be written as

$$[1 - (1 - \sigma)\delta]w_e - \sigma\beta v_d = (1 - \sigma)\pi_e q_e(-y_e + \beta v_d - \beta\bar{v}_d),$$

then the result follows directly from (7). The argument for $s = d$ follows from the same steps. ■

We now use the previous two lemmas to write the slack in the producer constraint in matrix algebra as

$$\begin{bmatrix} -\sigma\beta & 1 - (1 - \sigma)\delta \end{bmatrix} \begin{bmatrix} v_s \\ w_{s'} \end{bmatrix} = \frac{1}{\det(M_{ss'})} \begin{bmatrix} m_{us} & m_{ys'} \end{bmatrix} \begin{bmatrix} \mu_{us}\pi_s p_s u_s \\ -\mu_{ys'}\pi_{s'} q_{s'} y_{s'} \end{bmatrix} \quad (27)$$

where the scalars m_{us} and $m_{ys'}$ are to be computed so that the sign of the participation constraint does not depend on the magnitude of $\det(M_{ss'})$. After some straightforward algebra is used to produce a simple expression for m_{us} and $m_{ys'}$, the desired inequalities are derived as follows.

Lemma 8 *The participation constraints are satisfied if and only if*

$$u_d(y_d) \geq \frac{y_e}{\beta}\left[\frac{1}{(1 - \tau)\pi_d p_d} - (1 - \sigma)\delta\frac{1 - \pi_d p_d}{\pi_d p_d}\right] \quad (28)$$

and

$$u_e(y_e) \geq \frac{y_d}{\beta}\left[\frac{1}{(1 - \sigma)\pi_e p_e} - (1 - \tau)\delta\frac{1 - \pi_e p_e}{\pi_e p_e}\right]. \quad (29)$$

Proof. The steps for deriving inequality (28) are simple; we omit the proof for inequality (29) because it is identical to the proof of inequality (28). Regarding participation constraint for date-e producers, we find it useful to set $\rho = \pi_d p_d$ and $\xi = \pi_e q_e$, so that the expression for m_{ud} can be written as

$$\begin{aligned}
-m_{ud} &= \sigma\beta - \sigma\beta(1-\sigma)(1-\xi)\delta - \sigma\beta - (1-\sigma)\xi\beta + \sigma\beta(1-\sigma)\delta + \\
&\quad (1-\sigma)\xi\beta(1-\sigma)\delta \\
&= -(1-\sigma)\xi\beta + (1-\sigma)\xi\beta\delta \\
&= -(1-\delta)(1-\sigma)\xi\beta.
\end{aligned}$$

The expression for m_{ye} is

$$\begin{aligned}
-m_{ye} &= \sigma\beta\tau\beta + \sigma\beta(1-\tau)\rho\beta - 1 + (1-\tau)(1-\rho)\delta + (1-\sigma)\delta + \\
&\quad -(1-\sigma)\delta(1-\tau)(1-\rho)\delta \\
&= \sigma\beta\tau\beta + \sigma(1-\tau)\rho\delta - 1 + (1-\sigma)\delta + \\
&\quad -(1-\tau)(1-\rho)\delta[(1-\sigma)\delta - 1] \\
&= -1 + \delta - \sigma\delta[1-\tau-(1-\tau)\rho] + \sigma\delta(1-\tau)(1-\rho)\delta + \\
&\quad (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -1 + \delta - \sigma\delta(1-\tau)(1-\rho) + \sigma\delta(1-\tau)(1-\rho)\delta + \\
&\quad (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -1 + \delta - (1-\delta)\sigma\delta(1-\tau)(1-\rho) + (1-\delta)(1-\tau)(1-\rho)\delta \\
&= -(1-\delta)[1-(1-\sigma)\delta(1-\tau)(1-\rho)].
\end{aligned}$$

Thus, the right-hand side of (27) equals

$$\frac{(1-\delta)(1-\sigma)\pi_e q_e}{\det(M_{de})} \begin{bmatrix} \beta & 1-\delta(1-\sigma)(1-\tau)(1-\pi_d p_d) \end{bmatrix} \begin{bmatrix} (1-\tau)\pi_d p_d u_d \\ -y_e \end{bmatrix},$$

so that (28) follows. ■