Impacts of multivariate GARCH innovations on hypothesis testing for cointegrating vectors

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Abstract

This note investigates impacts of multivariate generalised autoregressive conditional heteroskedasticity (GARCH) errors on hypothesis testing for cointegrating vectors. The study reviews a cointegrated vector autoregressive model incorporating multivariate GARCH innovations and a regularity condition required for valid asymptotic inferences. Monte Carlo experiments are then conducted on a test statistic for a hypothesis on the cointegrating vectors. The experiments demonstrate that the regularity condition plays a crucial role in rendering the hypothesis testing operational. It is also shown that the Bartlett correction and wild bootstrapping are useful in improving the small-sample performance of the test statistic of interest.
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JEL Classification Codes: C32, C52, C63.

1 Introduction

The objective of this note is to investigate impacts of multivariate generalised autoregressive conditional heteroskedasticity (GARCH) errors on hypothesis testing for cointegrating vectors. This paper, built on asymptotic theory developed by Rahbek, Hansen and Dennis (2002), performs various computational analyses to pursue the objective. The introductory section briefly reviews the literature on cointegration and GARCH-type models, and then describes the most significant aspects of this note.

Economic and financial time series data often exhibit non-stationary behaviour and need to be treated as integrated processes rather than stationary. Cointegration introduced by Granger (1981) therefore plays an important role in time series econometrics. A cointegrated vector autoregressive (VAR) model explored by Johansen (1988, 1996) is based on Gaussian innovations, so that the maximum likelihood analysis of cointegration

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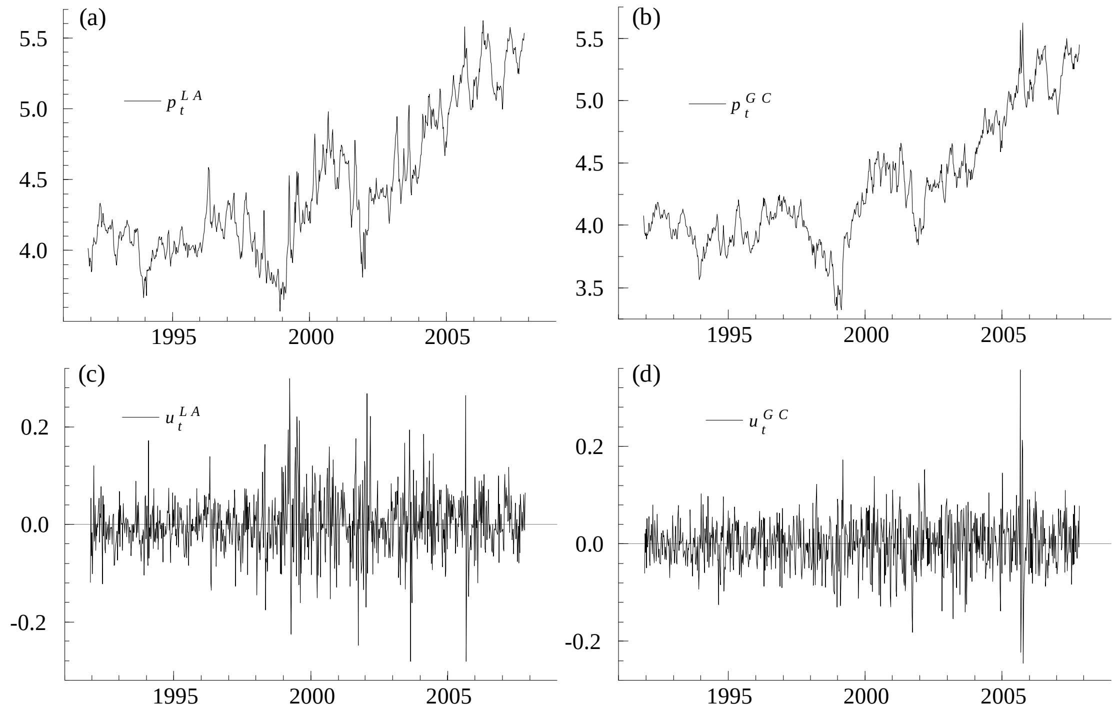

Figure 1: An overview of US gasoline prices and residuals from vector autoregression

can be conducted. Johansen's cointegrated VAR model has become a major econometric tool for applied economists. See Hendry and Mizon (1993), Juselius (2006), and Kurita (2007), *inter alia*, for empirical research using cointegrated VAR models.

Modelling volatility is also an important task in applied economics and finance. An ARCH model introduced by Engle (1982) has proved to be very useful for this purpose, and a vast class of ARCH-type models has been developed in the literature such as a GARCH model by Bollersev (1986) and an exponential GARCH model by Nelson (1991). See Engle (1995) for selected readings on a class of ARCH models. A multivariate GARCH model is also introduced and its theory is investigated by Bollerslev, Engle and Wooldridge (1988), Engle and Kroner (1995), Kroner and Ng (1998), Jeantheau (1998), Tse (2000), Comte and Lieberman (2003), *inter alia*. See Bauwens, Laurent and Rombouts (2006) for a survey of multivariate GARCH models.

A standard cointegrated VAR model assumes innovation errors to be independently and identically Gaussian distributed (i.i.d. Gaussian). However, this assumption is often violated in practice. Residuals from vector autoregression using financial and market data tend to show some temporal dependence, behaving like ARCH or GARCH processes. Figure 1 displays an example of such processes. Figures 1 (a) and (b) show time series plots of p_t^{LA} and p_t^{GC}, which are the logs of weekly spot prices for conventional regular gasoline (cents per gallon) in Los Angeles and Gulf Coast, respectively. The sample period runs from the second week in October 1991 to the final week in December 2007, and the data are taken from the webpage of energy information administration of the US. Both price series show non-stationary behaviour and wander around in a fairly similar way. The similarity in behaviour should be due to arbitrage trading in the gasoline market, suggesting the presence of a cointegrating relationship in the price data. Figures 1 (c) and (d) display residuals from a vector autoregression with two lags applied to p_t^{LA} and p_t^{GC}. Residuals for p_t^{LA} and p_t^{GC} are denoted by u_t^{LA} and u_t^{GC}, respectively. The former residuals, in particular, exhibit high volatility corresponding to GARCH features. The presence of cointegration in the data leads to the subsequent question of whether a test statistic for restrictions on the cointegrating vector is robust to influences of such GARCH innovations. It is of interest, in this empirical example, to see whether the cointegrating vector reflects the underlying arbitrage trading such that it can be expressed as a vector of homogeneous elements with opposite sign. This motivates a simulation setting in Section 4 of this note.

A cointegrated VAR model involving multivariate GARCH errors is employed for application by Bauwens, Deprins and Vandeuren (1997), and simulation studies are also carried out by Silvapulle and Podivinsky (2000) using univariate ARCH/GARCH error processes. Turning to econometric theory, Hansen (1992) explores asymptotic theory for cointegrated regressions in the presence of heteroskedasticity. Rahbek *et al.* (2002) investigate effects of multivariate ARCH innovations on inferences in a cointegrated VAR model, and provide a number of important theorems and simulation studies. Rahbek *et al.* (2002) are mainly concerned with a logged quasi-likelihood ratio ($\log QLR$) test for *the determination of cointegrating rank*, and demonstrate that the $\log QLR$ test is asymptotically robust to the ARCH errors under the fulfillment of a regularity condition. Seo (2007), allowing for multivariate GARCH innovations, explores the asymptotic properties of maximum likelihood estimators of cointegrating parameters.

The present note, given the cointegrating rank, conducts simulation studies of a $\log QLR$ test statistic for *restrictions on cointegrating vectors* in the presence of multivariate GARCH innovations. To the best of the author's knowledge, such simulation studies seem to be missing in the literature. Identifying long-run economic relationships in the data is often of primary interest in empirical modelling. Hence testing interpretable restrictions on cointegrating vectors plays a crucial role in a cointegrated VAR analysis. This note reviews that a $\log QLR$ test statistic for such restrictions is asymptotically χ^2-distributed, provided that the required regularity condition is satisfied. Monte Carlo experiments are then conducted on the $\log QLR$ test statistic. The experiments highlight a crucial role played by the regularity condition in drawing reliable statistical inferences for cointegrating vectors. Huge size distortions are observed when the regularity condition is violated; in contrast, if the condition is fulfilled, size distortions tend to be small as the number of observations increases. The Bartlett correction and wild bootstrapping are then adopted in the experiments in order to improve the small-sample performance of the $\log QLR$ test statistic. Both of the methods turn out to be useful in achieving the objective. These simulation-based findings convey important implications for applied research and complement asymptotic theory developed by Rahbek *et al.* (2002) and Seo (2007).

The organization of this note is as follows. Section 2 reviews a cointegrated VAR model with GARCH innovations, and introduces a regularity condition required for the subsequent asymptotic analysis. Section 3 then provides an asymptotic argument on inferences for cointegrating vectors. Section 4 conducts various Monte Carlo experiments involving the Bartlett correction and wild bootstrapping in order to see the performance of a $\log QLR$ test statistic for restrictions on cointegrating vectors. The overall summary and conclusion are provided in Section 5. All the numerical analyses and graphics in this note use *Ox* (Doornik, 2007) and *OxMetrics/PcGive* (Doornik and Hendry, 2007).

Some notational conventions are used throughout the note. For a certain matrix a with full column rank, $\overline{a} = a(a'a)^{-1}$ and so $a'\overline{a} = I$. An orthogonal complement $a_\perp$ is defined such that $a_\perp' a = 0$ with the matrix $(a, a_\perp)$ being of full rank. A spectral radius is defined as the maximum absolute value of eigenvalues of a certain square matrix A and is denoted by $\rho(A)$. Finally, a symbol $\overset{w}{\to}$ is used to signify weak convergence in a limiting theorem.

2 Cointegrated VAR model with GARCH errors

This section reviews a cointegrated VAR model incorporating multivariate GARCH innovations. The main reference is Rahbek *et al.* (2002). Let us consider an unrestricted VAR(k) model for a p-dimensional time series X_t conditional on the initial values $X_{-k+1}, ..., X_0$ as follows:

$$\Delta X_t = (\Pi, \Pi_c)\begin{pmatrix} X_{t-1} \\ 1 \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \xi_t, \quad \text{for} \quad t = 1, ..., T, \tag{1}$$

where $\Pi, \Gamma_i \in \mathbf{R}^{p\times p}$ and $\Pi_c \in \mathbf{R}^p$ all vary freely. The innovation sequence ξ_t is assumed to follow the following multivariate GARCH process introduced by Engle and Kroner (1995):

$$\begin{aligned} \xi_t &= \Omega_t^{1/2}\varepsilon_t, \\ \Omega_t &= \Omega + \sum_{i=1}^{l}\sum_{j=1}^{m}\left(\Psi_{ij}\xi_{t-i}\xi_{t-i}'\Psi_{ij}' + \Phi_{ij}\Omega_{t-i}\Phi_{ij}'\right), \end{aligned} \tag{2}$$

where ε_t has an independent and identical distribution with mean zero and variance I_p, with its density being positive continuous, Ω is a $p \times p$ positive definite and symmetric matrix, Ψ_{ij} and Φ_{ij} are $p \times p$ real matrices. The process (2) belongs to a class of BEKK-GARCH(l) processes (see Engle and Kroner, 1995). Note that the process (2) is assumed here, but it should be tested in practice using such methodology as suggested by Linton and Steigerwald (2000). In order to find the Granger-Johansen representation of (1), which provides a basis for $I(1)$ cointegration analysis, three conditions need to be introduced. These are given in Assumption 2.1.

Assumption 2.1 *(cf. Theorem 4.2 in Johansen, 1996)*

1. *The characteristic roots obey the equation* $|A(z)| = 0$, *where*

$$A(z) = (1-z)\,I_p - \Pi z - \sum_{i=1}^{k-1}\Gamma_j\,(1-z)\,z^i,$$

and the roots satisfy $|z| > 1$ *or* $z = 1$.

2. *Reduced rank condition:* $(\Pi, \Pi_c) = \alpha(\beta', \gamma')$ *for* $\alpha, \beta \in \mathbf{R}^{p\times r}$ *and* $\gamma' \in \mathbf{R}^r$ *with* $r < p$.

3. *Full rank condition: rank* $(\alpha'_\perp \Gamma \beta_\perp) = p - r$ *for* $\Gamma = I_p - \sum_{i=1}^{k-1} \Gamma_i$.

Let $\beta^{*\prime} = (\beta', \gamma')$, $X^*_{t-1} = \left(X'_{t-1}, 1\right)'$ and $p^* = p + 1$ for future reference. The first condition ensures that the process is neither explosive nor seasonally cointegrated. The second condition says that the matrix (Π, Π_c) is of reduced rank structure, in which α vectors are called adjustment vectors, while β^* vectors are referred to as cointegrating vectors or cointegrating parameters. This condition implies that there are at least $p - r$ common stochastic trends and cointegration arises when $r \geq 1$. The third condition prevents the process from being $I(2)$ or of higher order. The cointegrating rank r is, in practice, unknown to investigators, thus it needs to be determined based on the data analysis. A logged quasi-likelihood function for (1), which is introduced in the next subsection, is maximized by reduced rank regression, so that a $\log QLR$ test statistic for the determination of cointegrating rank is then constructed; the $\log QLR$ test statistic is given by the null hypothesis of r cointegration rank, denoted as $\mathsf{H}(r)$, against the alternative hypothesis $\mathsf{H}(p)$. See Rahbek *et al.* (2002) for details. This note is concerned with hypothesis testing on β^*, given the cointegrating rank.

If Assumption 2.1 is satisfied and the innovation sequence ξ_t is stationary and ergodic, it is then shown that (1) has the Granger-Johansen representation. Moreover, in order to fully validate the asymptotic analysis of cointegration using (1), it is necessary to introduce an additional regularity condition. Let $Y_t = \xi_t \xi_t'$ and $U_t = Y_t - \Omega_t$, and consider an autoregressive expression of Y_t based on (2):

$$\begin{aligned} Y_t &= \Omega_t + U_t \\ &= \Omega + \sum_{i=1}^{l} \sum_{j=1}^{m} \Psi_{ij} Y_{t-i} \Psi'_{ij} + \sum_{i=1}^{l} \sum_{j=1}^{m} \Phi_{ij} Y_{t-i} \Phi'_{ij} + V_t, \end{aligned}$$

where $V_t = U_t - \sum_{i=1}^{l} \sum_{j=1}^{m} \Phi_{ij} U_{t-i} \Phi'_{ij}$. Applying vectorisation operator vec to both sides leads to

$$vec\,(Y_t) = vec\,(\Omega) + \sum_{i=1}^{l} \sum_{j=1}^{m} \Upsilon_{ij} vec\,(Y_{t-i}) + vec\,(V_t)\,,$$

where $\Upsilon_{ij} = (\Psi_{ij} \otimes \Psi_{ij}) + (\Phi_{ij} \otimes \Phi_{ij})$. Defining $y_t = vec\,(Y_t)$, $\omega = vec\,(\Omega)$ and $v_t = vec\,(V_t)$, one finds the companion form of y_t to be

$$\begin{pmatrix} y_t \\ \vdots \\ y_{t-l+2} \\ y_{t-l+1} \end{pmatrix} = \begin{pmatrix} \omega \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \Upsilon \begin{pmatrix} y_{t-1} \\ \vdots \\ y_{t-l+1} \\ y_{t-l} \end{pmatrix} + \begin{pmatrix} v_t \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

where

$$\Upsilon = \begin{pmatrix} \sum_{j=1}^{m} \Upsilon_{1j} \cdots \sum_{j=1}^{m} \Upsilon_{l-1j} & \sum_{j=1}^{m} \Upsilon_{lj} \\ I_{p(l-1)} \otimes I_{p(l-1)} & 0_{p(l-1)\times p} \otimes 0_{p(l-1)\times p} \end{pmatrix}. \tag{3}$$

The regularity condition is then provided in Assumption 2.2.

Assumption 2.2
(cf. Proposition 2.7 in Engle and Kroner, 1995; Theorems 5 and 6 in Rahbek et al., 2002) The spectral radius of Υ *is smaller than unity, that is,* $\rho(\Upsilon) < 1$.

Rahbek *et al.* (2002) use Markov chain theory developed by Meyn and Tweedie (1993) to derive a regularity condition equivalent to Assumption 2.2, and prove a class of limiting theorems relating to cointegration. As demonstrated by Rahbek *et al.* (2002), the satisfaction of Assumption 2.2 implies (i) the existence of the second order moment of the sequence ξ_t such that $E(\xi_t \xi_t') = \Sigma$, where Σ is a positive definite and symmetric $p \times p$ matrix; (ii) the law of large numbers and the functional central limit theorem hold for ξ_t; and (iii) appropriate sums of ξ_t converge to stochastic integrals.

Note that Assumption 2.1 provides the same conditions as those in the conventional Gaussian case (see Theorem 4.2 of Johansen, 1996), whereas Assumption 2.2 is the additional regularity condition allowing for the existence of multivariate GARCH innovations. In the next section, the impacts of such GARCH innovations on hypothesis testing for cointegrating vectors are reviewed in an asymptotic context.

3 Asymptotic inferences for β^*

This section reviews asymptotic inferences for cointegrating vectors when multivariate GARCH innovations exist in the model. The presence of such GARCH innovations leads to a quasi maximum likelihood analysis, rather than the exact maximum likelihood analysis based on i.i.d. Gaussian innovations in Johansen (1996). This section is therefore concerned with the asymptotic properties of a quasi-maximum likelihood estimator and test statistic.

Let us introduce a concentrated quasi-likelihood function for (1) as follows:

$$\log QL(\alpha, \beta, \Sigma) = -\frac{T}{2}\log(\det \Sigma) - \frac{1}{2}\sum_{t=1}^{T}(R_{0t} - \alpha\beta^{*\prime}R_{1t})'\Sigma^{-1}(R_{0t} - \alpha\beta^{*\prime}R_{1t}), \qquad (4)$$

where R_{0t} and R_{0t} denote residuals from the regression of ΔX_t and X_{t-1}^* on a set of the past values $\Delta X_{t-1}, ..., \Delta X_{t-k+1}$, respectively. In addition, the sample product moments of the residuals are defined as

$$\begin{pmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} R_{0t} \\ R_{1t} \end{pmatrix}\begin{pmatrix} R_{0t} \\ R_{1t} \end{pmatrix}'.$$

Maximising (4) then boils down to solving the following eigenvalue problem:

$$\det(\lambda S_{11} - S_{10}S_{00}S_{01}) = 0,$$

which yields the quasi-maximum likelihood estimator, $\widehat{\beta}^*$, as a set of r eigenvectors corresponding to the r largest eigenvalues $\widehat{\lambda}_i$ for $i = 1, ..., r$.

Investigations of long-run economic relationships in the data are often of primary interest in empirical research. Applied economists therefore attach importance to testing theory-consistent restrictions on cointegrating vectors. The original logged likelihood ratio ($\log LR$) test based on i.i.d. Gaussian innovations is explored by Johansen (1988). See also Johansen (1996, Chs.7 and 13) for details. Restrictions on cointegrating vectors are formulated in various ways according to research objectives, and the simplest formulation of them would be

$$\mathsf{H}_0 : \beta^* = H\varphi, \tag{5}$$

where H is a $p^* \times s$ dimensional known matrix for $s \geq r$ and φ represents $s \times r$ dimensional unknown parameters to be estimated. Determining the cointegrating rank r using Johansen's procedure under the assumption of i.i.d. Gaussian innovations, one can then construct the $\log LR$ test statistic for the null hypothesis H_0 given in (5) against the alternative $\mathsf{H}(r)$. It is known that the $\log LR$ test statistic is asymptotically χ^2-distributed (see Johansen, Ch.13).

A question of interest that follows is whether a $\log QLR$ test statistic for the null hypothesis H_0 against $\mathsf{H}(r)$ in the presence of multivariate GARCH innovations also has an asymptotic χ^2 distribution. The $\log QLR$ test statistic of interest can be expressed as

$$\log QLR(\mathsf{H}_0 \,|\, \mathsf{H}(r)) = T \sum_{i=1}^{r} \log \left\{ \left(1 - \widetilde{\lambda}_i\right) / \left(1 - \widehat{\lambda}_i\right) \right\}, \tag{6}$$

where $\widetilde{\lambda}_i$ for $i = 1, ..., r$ correspond to the r largest eigenvalues solved from

$$\det\left(\lambda H' S_{11} H - H' S_{10} S_{00} S_{01} H\right) = 0.$$

See Johansen (1997, Ch.7) for details. The limiting distribution of $\widehat{\beta}^*$, together with that of the $\log QLR$ test statistic, are provided in Theorem 3.1, in which $\widehat{\beta}^*$ is normalised on $c' = (\beta', 0)$ so that $\left(\widehat{\beta}_c', \widehat{\gamma}_c'\right)' = \widehat{\beta}^* \left(\overline{c}'\widehat{\beta}^*\right)^{-1} = \widehat{\beta}^* \left(\overline{\beta}'\widehat{\beta}\right)^{-1}$. Theorem 3.1 is regarded as a modified version of Corollary 3 in Rahbek *et al.* (2002) in that the restricted intercept is taken into account and multivariate GARCH innovations are assumed rather than multivariate ARCH.

Theorem 3.1 *(cf. Corollary 3 in Rahbek et al., 2002)*
Suppose that both Assumptions 2.1 and 2.2 are fulfilled. The asymptotic distribution of $\widehat{\beta}^*$ normalised on $c' = (\beta', 0)$ is then given by

$$\left\{ \begin{array}{c} T\overline{\beta}_{\perp}^{*\prime}\left(\widehat{\beta}_c - \beta\right) \\ T^{\frac{1}{2}}\left(\widehat{\gamma}_c - \gamma\right) \end{array} \right\} \xrightarrow{w} \left(\int_0^1 G_u G_u' du\right)^{-1} \int_0^1 G_u \left(dV_u\right)',$$

where

$$G_u = \begin{pmatrix} \beta_{\perp}' C W_u \\ 1 \end{pmatrix} \quad \textit{and} \quad V_u = \left(\alpha' \Sigma^{-1} \alpha\right)^{-1} \alpha' \Sigma^{-1} W_u,$$

for $C = \beta_{\perp}\left(\alpha_{\perp}' \Gamma \beta_{\perp}\right)^{-1} \alpha_{\perp}'$. The process W_u denotes a p-dimensional Brownian motion for $u \in [0, 1]$ with variance Σ, so that G_u and V_u are independent. Furthermore, the $\log QLR$

test statistic (6) is asymptotically χ^2-distributed with the degree of freedom $r\,(p^*-s)$, that is,

$$\log QLR(\mathsf{H}_0\,|\mathsf{H}\,(r)\,) \xrightarrow{w} \chi^2_{r(p^*-s)}.$$

Proof. Follow the proof of Corollary 3 in Rahbek *et al.* (2002) and the proof of Theorem 2 in Hansen and Johansen (1999) using Υ defined in (3) above, instead of using a matrix (13) in Rahbek *et al.* (2002). ■

Thus one is able to conduct a standard χ^2-based asymptotic inference for restrictions on the cointegrating vectors, even in a situation where multivariate GARCH innovations are involved in the model. This is an encouraging finding for applied economists willing to analyse financial data with GARCH features. The next section, using Monte Carlo experiments, checks the validity of Theorem 3.1 in the context of asymptotic inferences and also examines small-sample properties of the $\log QLR$ test statistic.

4 Monte Carlo experiments

This section conducts Monte Carlo experiments on the $\log QLR$ test statistic discussed in the previous section. The design of an artificial data generation process (DGP) is based on an empirical bivariate model estimated from the gasoline price data in Figure 1. A reduced rank regression using the price data finds a single cointegrating vector with its coefficients being almost homogenous with opposite sign. Hence the DGP is defined in such a way as it incorporates (i) a single cointegrating vector of homogeneity with opposite sign, and (ii) multivariate GARCH innovations:

$$\begin{aligned}
\begin{pmatrix} \Delta X_{1,t} \\ \Delta X_{2,t} \end{pmatrix} &= \begin{pmatrix} -0.2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0.1 \end{pmatrix} \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \\ 1 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0.3 & 0.0 \\ 0.1 & 0.1 \end{pmatrix} \begin{pmatrix} \Delta X_{1,t-1} \\ \Delta X_{2,t-1} \end{pmatrix} + \begin{pmatrix} \xi_{1,t} \\ \xi_{2,t} \end{pmatrix}, \\
\begin{pmatrix} \xi_{1,t} \\ \xi_{2,t} \end{pmatrix} &= \Omega_t^{1/2} \begin{pmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{pmatrix},
\end{aligned}$$

where $(\varepsilon_{1,t}, \varepsilon_{2,t})' \sim IN\,(0, I_2)$ and the parameters are given by

$$\Omega_t = \Omega + \Psi_{11}\left(\xi_{1,t-1}, \xi_{2,t-1}\right)'\left(\xi_{1,t-1}, \xi_{2,t-1}\right)\Psi_{11}' + \Phi_{11}\Omega_{t-1}\Phi_{11}',$$

$$\Omega = 0.06^2 \times \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}, \quad \Psi_{11} = \begin{pmatrix} \psi_{11} & \psi_{12} \\ 0 & \psi_{22} \end{pmatrix}, \quad \text{and} \quad \Phi_{11} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ 0 & \phi_{22} \end{pmatrix}.$$

Note that $X_{1,t}$ and $X_{2,t}$ correspond to p_t^{LA} and p_t^{GC} respectively, so that all the parameters apart from those in the GARCH innovations are close to round-off values of the estimates from the price data. The initial values are taken from the gasoline price data, and the number of replications is $10,000$ apart from wild bootstrap experiments in Section

4.3. The null hypothesis H_0 needs to reflect homogeneity with opposite sign and is thus formulated by

$$\mathsf{H}_0 : \beta^* = H\varphi,$$

where

$$H = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}',$$

and φ is a 2×1 unknown parameter vector to be estimated.

Let $\Upsilon = (\Psi_{11} \otimes \Psi_{11}) + (\Phi_{11} \otimes \Phi_{11})$, and the spectral radius of Υ is

$$\rho(\Upsilon) = \max\{(\psi_{11}^2 + \phi_{11}^2), (\psi_{22}^2 + \phi_{22}^2)\}.$$

The baseline setting for the experiments is as follows:

$$\begin{aligned} \Psi_{11} &: \psi_{11} = \psi_{22} = \psi_{12} = 0.5, \\ \Phi_{11} &: \phi_{12} = 0.5, \quad \phi_{11} = \phi_{22}. \end{aligned}$$

The spectral radius is thus given by $\rho(\Upsilon) = 0.5^2 + \phi_{11}^2$. There is empirical evidence that off-diagonal elements in the coefficient matrices such as ψ_{12} can be non-zero; see Bauwens *et al.* (1997), Rahbek *et al.* (2002), Pojarliev and Polasek (2003), *inter alia*. In order to allow for such evidence, both ψ_{12} and ϕ_{12} are different from zero in the above DGP.

The experiments conduced below permit ϕ_{11} to take four different values: 0.7, 0.8, 0.85, and 0.9. The corresponding values of the spectral radius are 0.74, 0.89, 0.9725, and 1.06, respectively. That is, the regularity condition in Assumption 2.2 is not satisfied only when $\phi_{11} = 0.9$. It is of interest, throughout this section, to see how simulation results vary according to cases where the regularity condition is fulfilled or not.

This section comprises three sub-sections. Section 4.1 presents quantile-quantile plots of the $\log QLR$ test statistic to check the importance of the regularity condition in conducting asymptotic inferences. Section 4.2 then gives recursive rejection frequencies of the test in order to see how fast size distortions reduce as the number of observations increases. Recursive rejection frequencies are defined as the rates of the null hypothesis being rejected as the sample size increases step by step; thus they can be interpreted as recursively-recorded empirical sizes. Finally, Section 4.3 applies the Bartlett correction and wild bootstrapping to the $\log QLR$ test statistic with a view to reducing size distortions.

4.1 Quantile-Quantile plots

This sub-section inspects quantile-quantile (QQ) plots of the $\log QLR$ test statistic according to several values of spectral radius. In order to check the validity of Theorem 3.1 in an asymptotic context, the number of effective observations, denoted by T, needs to be large and so $T = 500$.

Figure 2 plots various empirical quantiles of the $\log QLR$ test statistic against quantiles of a $\chi^2(1)$ distribution. The vertical axis corresponds to the empirical quantiles, while the horizontal axis to the quantiles of the χ^2 distribution. Empirical QQ plots are expressed by various dotted lines, whereas the baseline asymptotic QQ plots, based on the χ^2 distribution, are displayed by the thick straight line of 45 degree. The empirical QQ plots are based on the four different values of ϕ_{11}: 0.7, 0.8, 0.85, and 0.9, as described above.

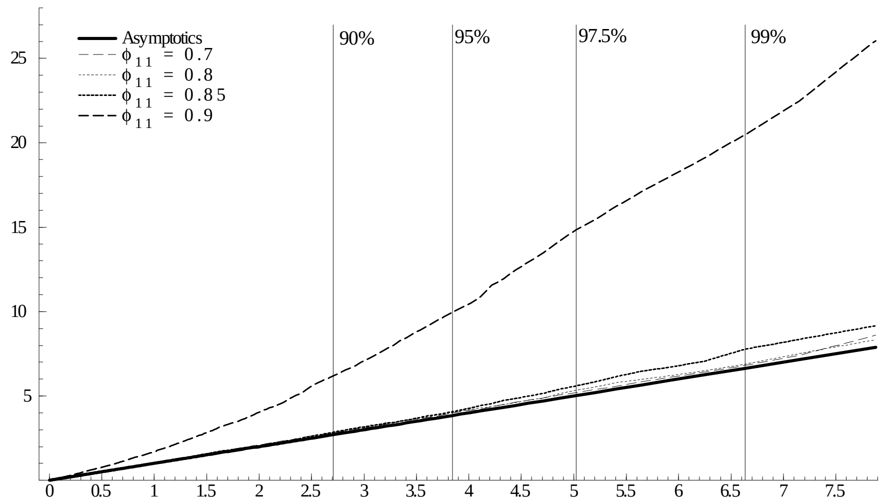

Figure 2: Quantile-Quantile plots ($T = 500$)

Size distortions are fairly small, according to Figure 2, when ϕ_{11} takes the first three values or the spectral radius is smaller than 1.0. In stark contrast, when $\phi_{11} = 0.9$ so the spectral radius does not satisfy the regularity condition, size distortions are enormously large, giving rise to the problem of serious misleading inferences. The importance of the regularity condition in asymptotic inferences is manifest in Figure 2. These results are consistent with the asymptotic argument presented in the previous section.

4.2 Recursive rejection frequencies

Next, this sub-section examines recursive rejection frequencies of the log QLR test statistic in question. A nominal size is 5% and T increases from 50 to 500.

As shown in Figure 3, size distortions are rather large in all the cases when T is small. The distortions, however, steadily decrease and empirical sizes approach to 5% as T increases, when the regularity condition in Assumption 2.2 is satisfied. An exceptional case is again when $\phi_{11} = 0.9$ so the spectral radius does not meet the regularity condition. Size distortions in this case are very large regardless of the numbers of observations, thereby yielding the problem of serious misleading inferences. These features are in line with Figure 2.

4.3 Bartlett correction and wild bootstrapping

Size distortions due to the breakdown of the regularity condition in Assumption 2.2, as observed in Figures 2 and 3, will be very difficult to remedy, for the corresponding limiting

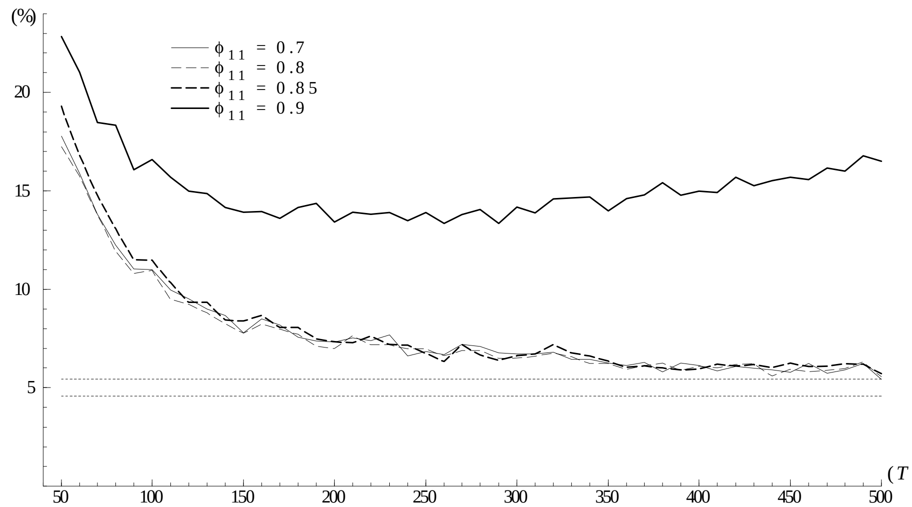

Figure 3: Recursive rejection frequencies

distribution may not exist. Size distortions when the regularity condition is satisfied, however, are expected to be moderated by the Bartlett correction and bootstrapping. This is because the distortions in this case stem from small T and tend to diminish as T increases, as shown in Figure 3.

This sub-section, using these techniques, aims to address the issue of size distortions when T is small. First, let us briefly explain the Bartlett correction, based on Bartlett (1937) and Johansen (2000). The idea is to correct a logged likelihood ratio test statistic, $\log LR$, by its finite-sample expectation for a given parameter point θ under the null hypothesis. In practice, however, it is not possible to derive the expectation explicitly. One can instead find its approximation

$$\mathsf{E}_{\theta}\left(\log LR\right) = f\left(1 + \frac{BC}{T}\right) + O_p\left(T^{-\frac{3}{2}}\right),$$

where f is the expectation of the limiting distribution, corresponding to the degree of freedom for a χ^2 distribution in conventional cases, and BC represents how the remaining parameters under the null hypothesis distort the expectation. The term BC is said to be the Bartlett correction factor; this is because a corrected $\log LR$ test defined as

$$\frac{\log LR}{1 + T^{-1}BC},$$

has expectation $f + O_p\left(T^{-\frac{3}{2}}\right)$, thus probably leading to a distribution closer to the asymptotic distribution than the original $\log LR$ test statistic. Under the assumption of Gaussian errors, Johansen (2000) derives BCs for $\log LR$ tests for various hypotheses on the cointegrating vectors; one of them is for a $\log LR$ test in which the null hypothesis corresponds to (5), as given in Corollary 6 in Johansen (2000). The idea of the Bartlett correction may also be applied to (6) in such a way as

$$\frac{\log QLR(\mathsf{H}_0\,|\mathsf{H}\,(r))}{1 + T^{-1}BC^{*}}, \tag{7}$$

where BC^* denotes the Bartlett correction factor under the assumption of GARCH innovations. Since the test statistic $\log QLR$ is also asymptotically χ^2-distributed, as shown in Theorem 3.1 above, the correction factor BC^* may be close to the factor BC derived under the Gaussian assumption, apart from the error covariance matrix. Thus, Corollary 6 in Johansen (2000) may lead to the following expression of BC^*:

$$BC^* = \left[\frac{1}{2}(p+s-r+3)+kp\right] + \frac{1}{r}\left[(2p+s-3r+1)\,\upsilon(\alpha)+2c(\alpha)\right],$$

where

$$\begin{aligned}
\upsilon(\alpha) &= tr\left\{V_\alpha\right\}, \\
c(\alpha) &= tr\left\{P\left(I_{r+(k-1)p}+P\right)^{-1}V_\alpha\right\} \\
&\quad + tr\left\{\left[P\otimes\left(I_{r+(k-1)p}-P\right)V_\alpha\right]\left[I_{r+(k-1)p}\otimes I_{r+(k-1)p}-P\otimes P\right]^{-1}\right\},
\end{aligned}$$

for

$$\begin{aligned}
V_\alpha &= \left(\alpha'\Sigma^{-1}\alpha\right)^{-1}\Sigma_{\beta\beta}^{-1}, \\
P &= \begin{pmatrix} I_r+\beta'\alpha & \beta'\Gamma_1 & \ldots & \beta'\Gamma_{k-2} & \beta'\Gamma_{k-1} \\ \alpha & \Gamma_1 & \ldots & \Gamma_{k-2} & \Gamma_{k-1} \\ 0 & I_p & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I_p & 0 \end{pmatrix},
\end{aligned}$$

and $\Sigma_{\beta\beta}$ represents variance of $\beta^{*\prime}X_t^*$ conditional on $\Delta X_t, ..., \Delta X_{t-k+2}$. Note that Ω in Johansen (2000) is replaced by Σ in BC^* defined above, due to the presence of GARCH errors. It is expected that the Bartlett-corrected $\log QLR$ test statistic, (7), performs better in small sample than the standard $\log QLR$ test statistic.

Let us turn to wild bootstrapping. Davidson and Flachaire (2001) investigate a wild bootstrap method in the presence of heteroskedastic errors. Gonçalves and Kilian (2004), and Cavaliere, Rahbek and Taylor (2008) then demonstrate the validity of the method in drawing accurate finite-sample inferences in time series context. Following Cavaliere *et al.* (2008) and Fachin (2000), a residual-based wild bootstrap algorithm in the cointegrated VAR model is formulated as follows:

1. Fit a cointegrated VAR model with no restriction on its cointegrating vector, and find unrestricted coefficients and residuals $\widehat{\xi}_t$. Next, fit a cointegrated VAR model with a restriction H on its cointegrating vector as in (5), and calculate the test statistic (6), which is denoted by $\log QLR$.

2. Generate T wild bootstrap residuals $\widehat{\xi}_t^b$ from

   $$\widehat{\xi}_t^b = \widehat{\xi}_t w_t, \quad \text{for} \quad t = 1, ..., T,$$

   where w_t is an independent $N(0,1)$ scalar sequence and $\widehat{\xi}_t$ is sampled with replacement and equal probabilities.

3. Using the unrestricted coefficients and wild bootstrap residuals, construct pseudo data recursively from

$$\Delta X_t^b = \widehat{\alpha}\widehat{\beta}^{*\prime} X_{t-1}^{*b} + \sum_{i=1}^{k-1} \widehat{\Gamma}_i \Delta X_{t-i}^b + \widehat{\xi}_t^b, \quad \text{for} \quad t = 1, ..., T,$$

   where the initial values are $X_i^b = X_i$ for $i = -k+1, ..., 0$.

4. Fit cointegrated VAR models with and without the restriction H using the pseudo data, and calculate a wild bootstrap $\log QLR$ test statistic, which is denoted by $\log QLR^b$.

5. Repeat 2 - 4 in B times.

6. Compute a wild bootstrap p-value by $p^b = B^{-1} \sum_{i=1}^{B} 1\left(\log QLR_i^b > \log QLR\right)$ and reject the null hypothesis at a significance level α^b if $p^b < \alpha^b$.

As small-sample properties are of interest, Monte Carlo experiments in this sub-section focus on two cases where $T = 50$ and $T = 100$. With regard to a set of experiments involving wild bootstrapping, the number of wild bootstrap replications is 500 or $B = 500$, while the number of Monte Carlo replications is $5,000$, so that the total number of loops is $500 \times 5,000$. For a comparative analysis, a set of experiments using the standard $\log QLR$ test is also conducted.

The results of the experiments are presented in Table 1. Empirical sizes of the standard $\log QLR$ test statistic are provided, then followed by those of the Bartlett-corrected and wild bootstrap $\log QLR$ test statistics. All the estimates are rounded to one decimal place. According to the table, both of the Bartlett-corrected and wild bootstrap $\log QLR$ tests lead to much smaller size distortions than the standard $\log QLR$ test. The wild bootstrap tests, in particular, exhibit only negligible distortions when the regularity condition is satisfied with $T = 100$. In contrast, the breakdown of the regularity condition gives rise to noticeable size distortions even when these techniques are used, although the distortions are smaller than those in the standard case. The experiments demonstrate the validity of these methods in addressing size distortions when T is small.

5 Summary and conclusion

A standard cointegrated VAR model assumes innovation errors to be independently and identically Gaussian distributed. This assumption, however, is not always fulfilled in practice. Residuals from vector autoregression using financial and market data tend to show some temporal dependence, behaving like ARCH or GARCH processes. Rahbek *et al.* (2002) thus investigate effects of multivariate ARCH innovations on inferences in a cointegrated VAR model. Rahbek *et al.* (2002) are mainly concerned with a $\log QLR$ test for the determination of the cointegrating rank. This note, based on Rahbek *et al.* (2002), inspects impacts of multivariate GARCH innovations on hypothesis testing for

Standard $\log QLR$ Test

($T = 50$)				
Level $\backslash\ \phi_{11}$	0.7	0.8	0.85	0.9
10%	26.8	26.5	28.9	31.8
5%	17.8	17.3	19.3	22.9
1%	6.1	6.3	7.5	10.5

($T = 100$)				
Level $\backslash\ \phi_{11}$	0.7	0.8	0.85	0.9
10%	18.2	17.5	18.6	23.8
5%	11.0	11.0	11.5	16.6
1%	3.3	3.7	4.1	7.4

Bartlett-Corrected $\log QLR$ Test

($T = 50$)				
Level $\backslash\ \phi_{11}$	0.7	0.8	0.85	0.9
10%	13.2	13.1	14.2	17.6
5%	7.0	7.1	8.0	11.3
1%	1.8	2.0	2.3	4.0

($T = 100$)				
Level $\backslash\ \phi_{11}$	0.7	0.8	0.85	0.9
10%	9.6	9.9	11.1	19.1
5%	5.0	5.2	6.2	13.0
1%	1.1	1.3	1.9	5.8

Wild Bootstrap $\log QLR^b$ Test

($T = 50$)				
Level $\backslash\ \phi_{11}$	0.7	0.8	0.85	0.9
10%	10.9	10.8	12.8	16.4
5%	6.0	5.9	6.7	9.6
1%	1.2	1.7	1.8	3.3

($T = 100$)				
Level $\backslash\ \phi_{11}$	0.7	0.8	0.85	0.9
10%	10.4	9.3	9.9	14.3
5%	4.7	4.6	5.3	8.6
1%	1.0	1.1	1.2	3.0

Table 1: Bartlett correction and wild bootstrapping

cointegrating vectors. It is reviewed that the $\log QLR$ test statistic for such restrictions is asymptotically χ^2-distributed, provided that the required regularity condition is satisfied. Monte Carlo experiments are then conducted on the test statistic of interest. The experiments highlight a crucial role played by the regularity condition in drawing reliable statistical inferences. The Bartlett correction and wild bootstrapping are also used in the experiments, and both of the methods have proved to be useful in improving the small-sample performance of the test statistic.

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