Bargaining-Induced Transaction Demand for Fiat Money

Merwan Engineer and Shouyong Shi
Revised July, 1995

Abstract
This paper demonstrates that Nash bargaining can induce a transaction demand for fiat money in a random matching model. To generate this demand, we introduce divisible services into the Kiyotaki and Wright (1991,1993) framework and allow agents to bargain over bundles of goods, services and money in bilateral matches. Other sources of the demand for money (physical transaction costs, private information and the absence of double coincidence of wants) are removed from the model. Though individual bargains are efficient, the bargaining terms of trade in asymmetric barter matches generates an inefficiency in the general equilibrium. Agents barter too much. When barter is inefficient, fiat money may be valued. The role of money in the model derives solely from the fact that purchases with money yield better expected terms of trade than barter purchases. In contrast to other search models, money does not speed up trade.

Keywords: Bargaining, Money, Search.
JEL Classification: C78, E40.

Merwan Engineer, Department of Economics, University of Guelph and University of Victoria, PO Box 3050 MS 8532, Victoria, British Columbia, Canada, V8W 3P5.

Shouyong Shi, Department of Economics, Queen's University, Kingston, Ontario, Canada, K7L 3N6.

*We would like to thank Dan Bernhardt, Ken Burdett, Nobuhiro Kiyotaki, Asha Sadanand, Alberto Trejos, Randy Wright, and seminar participants at the Canadian Economics Theory Meetings, Canadian Economics Macro Study Group, Delhi School of Economics, Indian Statistical Institute (Delhi), University of Buffalo, University of Pennsylvania and University of Victoria for their valuable comments on an earlier version of this paper. The usual disclaimer applies. We are grateful to the Indian Statistical Institute and the University of Pennsylvania for providing stimulating research environments and to the Social Sciences and Humanities Research Council of Canada for financial support.
1. Introduction

In a series of influential papers, Kiyotaki and Wright (1989, 1991, 1993) have developed a search-theoretic approach to monetary economics. This approach articulates how Jevons' classic absence of the coincidence of wants problem can yield the existence of valued money as a rational expectations equilibrium.\(^1\) Recently, Williamson and Wright (1994) have shown that the search approach can generate valued money in an environment in which private information generates a recognizability problem. Of course, both problems have been used to motivate valued money in general equilibrium models in which Walrasian markets are decentralized.\(^2\) Nevertheless, the search approach is compelling because it models trades between agents and endogenously derives money as the object which is most liquid in equilibrium. The search approach is able to model the liquidity nature of money because it abandons the fiction of the Walrasian auctioneer coordinating exchanges.

This paper demonstrates that the non-Walrasian nature of the search approach also yields a new rationale for money as a medium of exchange. The rationale derives from the mechanism for determination of the terms of trade between agents. In particular, we explore the implications of the model when the terms of trade are determined by the Nash bargaining solution. This mechanism is the natural choice as agents are matched bilaterally and it is supported by both cooperative and non-cooperative models consistent with the Nash program. Using the Nash solution, we are able to isolate a bargaining-induced transaction demand for money. Agents demand money solely because it is a liquid asset which improves their terms of trade in exchange relative to barter.

The bargaining-induced demand for money is absent in the Kiyotaki and Wright models because the terms of trade are flexible to focus on other issues. An agent can hold either

\(^1\)This search approach builds on Jones' (1976) adaptive expectations search model. See Kiyotaki and Wright (1993) for references to papers using the search approach. Ostroy and Starr (1990) provide a survey of transaction money models.

one unit of an indivisible storable good or one unit of indivisible money in his possession, restricting exchange to one for one trades. To free up the terms of trade, we allow agents to exchange divisible units of perishable goods which we call services. Agents bargain for bundles of services and goods or money.

To isolate the bargaining-induced demand for money, our model has neither a recognizability problem nor an absence of the coincidence of wants in barter matches. All agents in our model produce goods which are highly valued by all other agents. The barter model has an universal double coincidence of wants in the sense that swapping and consuming production goods yields each agent in any bilateral match at least as much utility has the cost of replacing the production good in inventory.

The universal double coincidence of wants feature results in goods always being exchanged and consumed in the pure barter equilibrium. Nevertheless, this equilibrium is inefficient. The inefficiency stems from the presence of some goods which are more highly valued than other goods. Random matching results in asymmetric coincidence of wants matches. Nash bargaining in such matches yields terms of trade which lower expected utility relative to an alternative feasible exchange rule. In particular, the agent who values the other's good less dearly, the seller, is able to extract sufficient services from the other agent, the buyer, to drive the marginal utility of services below their marginal cost. Bargaining forces the wrong trades on agents. Though individual bargains are efficient, barter is inefficient in the general equilibrium because all agents have an equal chance of being assigned a buyer or seller in their next match. Hence, they would be better off if less services were produced on average. This creates room for valued at money.

Fiat money is valued when it enables buyers to get better terms of trade. This reduces the inefficient production of services in asymmetric matches relative to barter. The terms of trade depend on the value of holding money which is endogenous to the model. The Nash solution yields buyers better terms of trade the greater is the value of holding money. In addition to the assets agents hold, the bargaining power of agents depends on the surplus weights and threat points. When these favor the sellers in asymmetric barter matches, the value of holding money is greater and money improves welfare.
The role of money in this paper derives solely from the feature that money purchases yield better terms of trade on average. In contrast to other search models, money does not speed up trade. This is because all goods are valued highly and are always bartered and consumed in our model. Nevertheless, indirect exchange through money takes place. Without money, an agent faces the possibility of bartering for his preferred good at a disadvantage when matched with somebody who does not value his good as dearly.

Shi (1995) and Trejos and Wright (1995) also develop search-theoretic money models in which relative prices are determined in bargaining. In these models, the price level is determined by the amount of the seller's services that are exchanged for the buyer's indivisible unit of money. As in other search models, fiat money is valued because it speeds up the rate at which agents are matched to sellers who are willing to trade. Our paper differs from these papers in three important ways. First, we include storable goods as well as services in the model. Second, we allow money holder's to produce services which makes offers more flexible. Thirdly, we model barter in asymmetric matches which permits the isolation of the bargaining-induced demand for money.

The paper proceeds as follows. Section 2 details the model without money and bargaining. Section 3 develops the constrained optimum, and Section 4 compares the optimum to the pure barter equilibrium. Section 5 develops the monetary economy. Extensions are discussed in Section 6. Section 7 explains the strategic bargaining game that supports the Nash solution and discusses alternative bargaining specifications. Section 8 concludes.

2. The Model without Money and Bargaining

The model generalizes the Kiyotaki and Wright (1991, 1993) models in two respects. First, divisible services are included. Second, exchange costs are allowed to be zero. Both features permit a much wider range of barter transactions. A key difference with the Kiyotaki and Wright models is that all goods in our model are valued sufficiently highly that there is a universal double coincidence of wants in the sense that swapping and consuming production goods yields each agent in any bilateral match at least as much utility has the cost of replacing the production good in inventory.
Agents and the production of goods. The economy has a continuum of infinitely-lived agents indexed by $i$ on the unit interval. Each agent $i$ can produce a storable good $i$ in indivisible units of 1. Goods are produced instantaneously at disutility $\frac{3}{4} > 0$. However, an agent can only hold one unit in inventory at a time and holding a good precludes producing another good.\(^3\) Agents are initially endowed with a unit of their own good.

As in Kiyotaki and Wright (1993), we make the simplifying assumption $\frac{3}{4} = 0$ to avoid having to detail production decisions. We show later that agents always produce after consumption if goods are valued highly.

Preferences over goods. Agents derive utility from consuming all goods other than their own production good. Each agent partitions goods into two groups: preferred goods and mediocre goods. Eating an preferred good yields an agent utility $u > \frac{3}{4}$ whereas, eating a mediocre good yields utility $b(u; \frac{3}{4}) > 0$; with $0 < b < 1$: Thus, mediocre goods are valued at least at the cost of production but not as highly as preferred goods. This captures diverse tastes for valued goods in a simple way.

Each agent’s set of preferred goods represents a proportion $x > 0$ of all storable goods, and mediocre goods comprise the remaining $1 - x$ of goods (as the own production good is of measure 0). In addition, each good is a preferred good of an equal proportion $x$ of agents and the mediocre good of the remaining $1 - x$ of agents. Heterogeneity in the model is such that each agent and good can be treated symmetrical.

Matching and transaction costs. Agents are matched pairwise and randomly according to a Poisson process with constant arrival rate $\bar{\gamma} > 0$. There are three types of possible matches between good holders. A match in which both agents have the other’s preferred good is called an preferred good match (PGM). A mediocre good match (MGM) is when both agents has the other’s mediocre good. An asymmetric good match (AGM) match is when only one agent has the other’s preferred good. Asymmetric good matches are a natural reflection of the diversity in tastes.\(^4\)

\(^3\)Instead of limiting inventory to one unit as in Kiyotaki and Wright (1991), we could use the alternate assumption in Aiyagari and Wallace (1991) and Kiyotaki and Wright (1993) that agents cannot produce before they consume another agent’s good.

\(^4\)In Kiyotaki and Wright (1993) a subset of goods yield utility $u$ and other goods are not consumed so that there is only a coincidence of wants in matches involving the most wanted items. In contrast, an
As all goods are valued at least at the cost of production, agents have an optimal strategy to consume any good they receive in exchange. Consumption is instantaneous. After consumption the match is immediately dissolved. The agent can then produce a new good instantaneously and search for a new match.

Unlike Kiyotaki and Wright, we assume that there is a zero transaction cost to receiving a good in exchange, $^2 = 0$. With zero exchange costs, our barter model displays a universal double coincidence of wants in goods in all bilateral matches in the sense that trading for the partner's good yields at least as much utility as replacing the production good in inventory. We later show that the results are reinforced when $^2 > 0$:

Services. All good holders can produce, trade and consume services (or equivalently perishable goods). A agent is capable of instantaneously producing any amount of divisible units of a service at a constant unit cost of $c > 0$. The service can only be consumed by other agents and $q$ units of services yields the consumer utility $v(q) = c(1 - e)q$, where $0 < e < 1$. This specification is tractable and avoids the exchange of services for services, as $v^0(q) = c(1 - e) < c$. However, services may be produced to purchase goods. In Section 6.2 we demonstrate that the more general specification, in which there is a coincidence of wants in services and marginal utility diminishes (such that $v^0(0) > c$ and $lim_{q \to 1} v(q) = c(1 - e)$), yields similar results.

3. The Constrained Optimum

Before looking at the equilibrium determination of the terms of trade it is useful to briefly develop a welfare benchmark for the optimal terms of trade in matches. Our benchmark is the rule a planner would choose to maximize the utilitarian welfare function constrained by the exogenous matching pattern.

The planner would have agents exchange and consume goods in all matches because all goods are valued at least at their replacement cost. On the other hand, the planner agent's valuation of a good varies continuously in Kiyotaki and Wright (1991) according to the distance that good is from his location on a circle. Matches with a coincidence of wants with respect to the most wanted items occur with probability zero. Nearly all barter matches are asymmetric in the sense that agents derive different utilities from consuming the other's good. However, not all matches involve goods being valued at least at the cost of production.
would forbid the exchange of services in matches because the production cost exceeds the consumption utility. As matching is exogenous and agents enter new matches with their production good, maximizing the utilitarian welfare function simply involves maximizing the sum of the utilities in each match. We have the following constrained optimum.\(^5\)

**Proposition 3.1.** Welfare is maximized by requiring that agents swap and consume goods in each match.

Since an agent encounters other agents at rate \( \bar{r} \) and the proportion \( x \) of agents have preferred goods and \( 1 - x \) have mediocre goods, an agent's expected utility under this rule is \( W^* = \bar{r}(xu + b(1 - x)u) = r \); where \( r \) is the discount rate.

The next section demonstrates that the barter equilibrium is inefficient relative to this benchmark because bargaining in AGMEs involves the excess trading of services.

### 4. Bargaining and the Pure Barter Equilibrium

#### 4.1. The Generalized Nash Bargaining Specification

The terms of trade in each bilateral barter exchange is determined as the outcome of a generalized Nash bargaining game which maximizes the weighted product of the surpluses from the exchange of goods and services

\[
\max_{q_i, q_j} S_i^i S_j^j \quad \text{s.t.} \quad S_i, S_j \geq 0; q_i, q_j \geq 0;
\]

where \( S_i \) and \( S_j \) are the surpluses from the exchange of the indivisible goods and divisible services and \( !; 0 \cdot ! \cdot 1 \), is the weight on \( i \)'s surplus. We allow the weight to differ depending on the match type. The surplus, \( S_i = U_i - \bar{U}_i \), is the difference in utility between the payo\( \overline{U}_j \) and the threat point, \( \bar{U}_j \). The threat points are taken exogenously by agents to be the expected utility of leaving the match without trading, and therefore depends on the inventory that an agent is holding going into a match. Thus, if agent \( j \) is holding his production good \( j \), then \( \bar{U}_j = V_j \); where \( V_j \) is the expected utility of holding

\(^5\)This benchmark can be also derived as the rule that a representative agent, not knowing his type behind a veil of ignorance, would choose in order to maximize ex ante utility given the exogenous matching pattern.
good j in search. Variants on this specification and the strategic game which supports this specification are detailed in Section 7 and Appendix B.

4.2. Bargaining in Barter Matches

In general the expected utility of holding different goods may differ depending on equilibrium expectations. This greatly complicates the analysis of bargaining. Like Kiyotaki and Wright, we impose the following assumption which allows us to concentrate on equilibria in which all goods are equally liquid in order to study at money.

Assumption 1: The expected utility of searching with any good is the same, $V_i = V_g$ for all i.

In both preferred good matches (PGMs) and mediocre good matches (MGMs), agents are in a symmetric position to their trading partner (because of the assumption $\mathcal{U}_i = \mathcal{U}_j = V_g$). Hence, we weight each agent’s surplus equally, $\frac{1}{2}$. The Nash solution picks a symmetric outcome in which no services are exchanged.

In PGMs it is straightforward to show that agents exchange only goods. The surpluses of agents i and j in an PGM are

$$S_i = U_i - \mathcal{U}_i = u + c(1 - e)q_i - cq_i + V_g$$
$$S_j = U_j - \mathcal{U}_j = u + c(1 - e)q_j - cq_j + V_g$$

Both surpluses are specified with each agent consuming the other's good. Services may be produced by each agent at a unit cost of $c > 0$. An agent derives utility $c(1 - e) > 0$ from a unit of his partner's service. As all agents enter and exit matches holding goods, the surplus only involves the utility from trade within the period.

Using the expressions for $S_i$ and $S_j$, the constraints $q_i > 0$ and $q_j > 0$ can be rewritten:

$$S_i \cdot (2i - e)u_i (1 - e)S_j$$
$$S_j \cdot (2j - e)u_j (1 - e)S_i$$

These two constraints form a kink at $(S_i; S_j) = (u; u)$. Since the level curve, $S_i^2 S_j^2 = \text{constant}$, is symmetric in $S_i$ and $S_j$, the Nash product is maximized at the kink. The quantities of services produced at the kink are $(q_i; q_j) = (0; 0)$. 7
The analysis of a MGM is identical to a PGM after replacing \( u \) with \( bu \) in the surplus equations. The solution is \((S_i; S_j) = (bu; bu)\) and \((q_i; q_j) = (0; 0)\). If \( b = 0 \); agents are indifferent to trading goods, and we assume that goods are exchanged and consumed.

In an AGM we dub the agent that wants the preferred good the buyer and the agent that has that good the seller. Let \( q_i (q_b) \) be service payments paid by the seller (buyer) to the buyer (seller). The surpluses are:

\[
S_s = bu + c(1 - e)q_b i - cq_b; \quad S_b = u + c(1 - e)q_b i - c q_b;
\]

Solving for \( q_b \) as functions of \( S_s \) and \( S_b \) yields:

\[
q_b = \frac{[(1 - e) + b]u i (1 - e)S_b - S_s}{ce(2 - e)} \quad \text{and} \quad q_b = \frac{[1 + (1 - e) b]u i (1 - e) S_s - S_b}{ce(2 - e)}.
\]

Thus, \( q_b = 0 \) and \( q_s = 0 \) are equivalent to

\[
S_s \cdot [(1 - e) + b]u i (1 - e)S_b \quad \text{and} \quad S_b \cdot [1 + (1 - e) b]u i (1 - e)S_s: \quad (4.1)
\]

These constraints are drawn in Figure 1.

(Figure 1 here.)

As agents are in nonsymmetric situations, the generalized Nash solution is used:

\[
\max_{w} f_{s}^{w} s_{i}^{w} : S_s \cdot 0; S_b \cdot 0; (4.1) g;
\]

where \(! = w\); with \( 0 \cdot w < 1 \); is the weight on the seller's surplus in an AGM. Define

\[
w', \quad \frac{(1 - e) b}{1 + b (1 - e)} \quad \text{and} \quad w', \quad \frac{b}{(1 - e) + b} > w;
\]

Lemma 4.1. The bargaining solution in an asymmetric good match (AGM) is as follows:

\[
\begin{align*}
&[0; \bar{w}]; \quad [1 + b (1 - e)](1 - w) u; \quad \frac{1 + (1 - e) b}{1 - e} w u; \quad 0; \quad w + (1 - w) b (1 - e) u \\
&[w; w]; \quad \frac{b}{1 - e} u; \quad \frac{1 + (1 - e) b}{1 - e} w u; \quad 0; \quad 0 \\
&(\bar{w}; 1); \quad \frac{1}{1 - e} (1 - w) u; \quad [(1 - e) + b] w u; \quad \frac{(1 - e) + b}{1 - e} w u; \quad 0
\end{align*}
\]

(4.2)
Proof. (i) $w < w$: Consider the following maximization:

$$\max_{(s_s, s_b)} \sum_{i=1}^{n} S_i^w S_j^b \text{ s.t. } S_b \cdot [1 + b(1 - e)]u_i (1 - e)S_s; \quad S_s \geq 0; \quad S_b \geq 0.$$ 

That is, disregard for a moment the constraint $q_s \geq 0$ (the first one in (4.1)). The maximization delivers the values $S_s$ and $S_b$ listed in (4.2). Then it can be shown that at the maximization point $q_s > 0$. (ii) $w \geq w$: Disregard for a moment the constraint $q_b \geq 0$ (the second one in (4.1)) and solve the maximization. The solution yields the values in (4.2). At the maximization point $q_b > 0$. (iii) $w \geq w$: The Nash product is maximized at the kink formed by the constraints (4.1). At the kink $q_b = q_s = 0$ and the values for $S_m$ and $S_g$ immediately follow. Q.E.D.

There are three cases for the solution depending on $w$ and $b$: We concentrate on the case in which the seller has sufficient bargaining power, $w > w$; to extract services from the buyer. This is the case drawn in Figure 1. Notice that as the value of mediocre goods diminishes, this case holds for a wider range of $w$: When mediocre goods are valued at their cost of production, $b = 0$, this case encompasses the entire range $w > w = 0$ and the buyer produces $q_b = wu = c$ of services for the seller.

We now show that the barter equilibrium is inefficient when the terms of trade involve the exchange of too many services.

4.3. The Pure Barter Equilibrium

Using the bargaining solutions above, the value function for a good holder satisfies:

$$rV_g = h x^2 u + x(1 - x)[S_s + S_b] + (1 - x)^2 u^i$$

All agents encounter each other at rate and trade in all matches. The probability that the match is an PGM is $x^2$ in which case the agent receives surplus $u$. The probability that the match is an AGM is $2x(1 - x)$ and an agent has an equal chance of being a buyer or a seller. The probability that a match is a MGM is $(1 - x)^2$.

As all agents are good holders, the equilibrium value of holding a good in the pure
barter equilibrium, denoted by $V_g(B)$; is just given by the value function.\textsuperscript{6}

**Proposition 4.2.** A pure barter always exists in which agents enter into trade with each agent they meet in search. The value of holding a good in search is $V_g(B) = V_g$:

The pure barter utility can be compared to the welfare benchmark. Substituting $S_s$ and $S_b$ into the value function and rearranging yields

$$W^\pi_i \cdot V_g(B) = (1 - x) ce(q_b + q_b) = R; \text{ where } R = \frac{r}{x};$$

It follows that an ine\textsuperscript{ciency arises from the excess trading of services in AGMs.

**Proposition 4.3.** If either buyers or sellers have su\textsuperscript{cient bargaining power, $w < \bar{w}$ or $w > \bar{w}$, the barter equilibrium is ine\textsuperscript{ciency relative to the constrained optimum: $W^\pi_i \cdot V_g(B) > 0$: Otherwise, $w \in [\bar{w}; \bar{w}]$; the barter equilibrium is e\textsuperscript{ciency.}

In the rest of the paper we concentrate on the case where sellers have su\textsuperscript{cient bargaining power, $w > \bar{w}$: This is the natural case to concentrate on when $b$ is small; that is, when there is a substantial asymmetry in the demand for preferred goods relative to mediocre goods. Then the ine\textsuperscript{ciency is

$$W^\pi_i \cdot V_g(B) = (1 - x) (ewu_i \cdot \frac{be(1 - w)}{1 - e}u) = R > 0;$$

As our leading example we consider the model when the demand asymmetry is the greatest, $b = 0$; so that the mediocre good is valued at the cost of production. As long as sellers have some bargaining power, $w > 0$, buyers must produce services to make the purchase and the pure barter equilibrium is ine\textsuperscript{ciency, $V_g(B) < W^\pi$: Relative to a straight swap of goods, bargaining reduces the sum of the surpluses by $ewu$ in each AGM. Nash bargaining in asymmetrical matches results in the production of services because it maximizes the weighted product of the surpluses in each match rather than the sum of the surpluses. Only when sellers have no bargaining power, $w = 0$; are buyers able to purchase preferred goods without producing services yielding an e\textsuperscript{ciency pure barter equilibrium.

\textsuperscript{6}When $b = 0$ agents are indifferent to swapping goods in AGMs so other equilibria exist in which goods are not swapped in such matches. However, the value of holding a good remains $V_g(B)$ in equilibrium.
The ine\(ci\)ciency characterized above does not refer to bargaining in individual matches. Given \(V_g\), the bargaining outcome in any match is e\(ci\)cient as it lies on the match's utilities possibility frontier. Rather the ine\(ci\)ciency arises because each agent has an equal chance of being a buyer or a seller in their next match. A social contract to reduce the exchange of services in all matches would raise \(V_g\). However, such a contract is not enforceable.

Throughout we have been careful to say that the ine\(ci\)ciency arises because of the exchange of too many services. This is because the ine\(ci\)ciency is not attributable to services being undesirable per se, \(v^0(0) = c(1 + e) < c\). In Section 6.2 we show that the ine\(ci\)ciency remains under a more general specification in which \(v^0(0) > c\) and \(\lim_{q \to 1} v(q) = c(1 + e)\). In this case it is optimal to exchange some services in all matches. Nevertheless, the ine\(ci\)ciency remains because the bargaining terms of trade drive marginal utility of services below their marginal cost.

In the next section we show that the ine\(ci\)ciency that arises from bargaining in AGM\(es\) is necessary for valued at money.

5. The Monetary Economy and Equilibrium

We now introduce a new storable item into the economy which is neither produced or consumed and determine if independent treatment results in it being valued in equilibrium.\(^7\) This exercise addresses a classic question in monetary theory: When can intrinsically useless objects, at money, have value as a medium of exchange? We show that at money may have value when the terms of trade are determined by Nash bargaining.

In the monetary economy, initially a fraction \(M\) of agents are endowed with an indivisible unit of money and the remaining \(1 - M\) agents possess their production good. Agents can only hold one unit of a storable item in inventory, so they can not hold both money and a good simultaneously. Agents can not produce money and can not produce goods while they hold money. Like good holders, money holders can produce and consume services.

\(^7\)An alternative exercise is to consider the case where all goods are not equally liquid to examine the possibility of the emergence of commodity money. However, the analysis of the production and consumption of commodity money is very complicated in the current model, and we leave this to future research. The examination of the existence of valued money is an easier and more interesting exercise.
To simplify matters, the analysis is restricted in two ways. First, we only examine our leading case where mediocre goods are valued at the cost of production, \( b = 0 \): (Section 6.1 establishes that the main results extend to \( b > 0 \).) Second, we concentrate on finding equilibria in which the value of searching with intrinsically useless money (\( V_m \)) is greater than the value of searching with an intrinsically valuable good, \( V_m > V_g \).

5.1. Bargaining in Money Matches

The bargaining solutions between good holders is unaffected by the presence of money in the economy and so we only have to examine matches in which there is money. First consider matches in which the good holder has the money holder's preferred good. Let \( q_m \) be the quantity of services produced by the money holder and \( q_g \) be the quantity of services produced by the good holder. A trade involves money and \( q_m \) for the good and \( q_g \). The money holder consumes the good and produces a production good to become a good holder and the good holder becomes a money holder. This trade yields surplus \( S_m \) to the money holder and a surplus \( S_g \) to the good holder:

\[
S_m = u + c(1 - e)q_g - csq_m + Y \\
S_g = c(1 - e)q_g - csq_m + Y
\]

where \( Y = V_m - V_g \). Both \( V_g \) and \( V_m \) and hence \( Y \) are taken as exogenous by agents. Solving for \( q_m \) and \( q_g \) yields:

\[
q_m = \frac{u + sM - (1 - e)S_g}{ce(1 - e)} \\
q_g = \frac{(1 - e)S_m - sG}{ce(1 - e)}
\]

Thus, \( q_m \geq 0 \) and \( q_g \geq 0 \) are equivalent to

\[
S_m \cdot u + sM - (1 - e)S_g \geq e \cdot Y \\
S_g \cdot (1 - e)S_m - sG \geq e \cdot Y
\]

These constraints are drawn in Figure 2.

(Figure 2 here.)

We set the weight on the good holder's surplus in a monetary match equal to the weight on the seller's surplus in an AGM, \( w = w \). The Nash solution is given by

\[
\text{max } f_S w S_m^w : S_g, 0; S_m, 0. (5.1) g
\]
Lemma 5.1. Given $Y \geq 0$, the bargaining solution in a monetary match is as follows:

$$
\begin{align*}
Y &> 0; (1 - e) w (1 - e) (u + e Y); \frac{1 - e}{1 - e + e w}; Y; 0; 0; 0; \\
(1 - e) w (1 - e) (u + e Y) &> \frac{1 - e}{1 - e + e w}; (1 - e) w (1 - e) (u + e Y); \frac{1 - e}{1 - e + e w}; Y; 0; 0; 0; \\
0 &> \frac{1 - e}{1 - e + e w}; (1 - e) w (1 - e) (u + e Y); \frac{1 - e}{1 - e + e w}; 0; 0; 0; 0.
\end{align*}
$$

Proof. For $Y > u = e$, there is no trade as no non-negative $S_m$ and $S_g$ satisfy (5.1). Following the proof to Lemma 4.1, it can be shown: if $Y \geq \frac{w u}{1 - e + e w}$, the solution lies on the segment of the surplus frontier to the right of the kink in Figure 2; if $Y \geq \frac{1 - e}{1 - e + e w}$, the solution is on the frontier to the left of the kink, and if $Y \geq \frac{w u}{1 - e + e w}$, the solution is at the kink. Q.E.D.

Remark 1. Matches in which the good holder has the money holder's mediocre good are a special case of Lemma 5.1 in which $u = 0$. There is no trade when $Y > 0$.

Monetary trade only occurs for the preferred good. There are three possible outcomes given exogenous reservation utilities $V_m$ and $V_g$. Either no services are produced or one of the agents produces services for the other. In the next subsection, we find that a good holder never produces services in equilibrium.

5.2. Monetary Equilibrium

The value functions for good holders and money holders satisfy:

$$
\begin{align*}
\bar{r}V_g &= - (1 - M) x^2 u + x (1 - e) [w (1 - e) (u + (1 - w) u)] + M x S_g; \\
\bar{r}V_m &= - (1 - M) x S_m
\end{align*}
$$

The value function for the good holder is simply altered to reflect encounters with money holders. A good holder encounters a money holder who wants to trade at rate $M x$ and immediately consummates the trade receiving surplus $S_g$. Conversely, a money holder
encounters an agent holding his preferred good at rate \( (1 - M) x \) and immediately consummates the trade receiving surplus \( S_m \). Formal derivation of equations like (5.3) and (5.4) can be found in Kiyotaki and Wright (1991).

Equation (5.3) reveals that holding money does not increase the probability of trading for the preferred good unlike in other search models of money. This is because good holders not only encounter agents with their preferred good in PG Mes but also as buyers in AG Mes. Hence, the rate at which good holders are able to barter for their preferred good is \( (1 - M) [x^2 + x(1 - x)] = (1 - M) x \). This is the same rate that money holders encounter agents with whom they want to trade. In fact, since good holders trade also as sellers they actually engage in more trades than money holders. Thus, if money is valued, it is valued not because it speeds up the procurement of the preferred good but because it improves the terms of trade for agents.

The analysis is restricted to finding symmetric stationary pure strategy equilibria.

Definition 5.2. An equilibrium is a joint solution of (5.2), (5.4) and (5.3) for \( q_m; q_g; V_m; \) and \( V_g \).

The examination of monetary equilibrium \( V_m > V_g \) can be simplified to finding a fixed point for \( Y = V_m \), \( V_g > 0 \). Subtracting (5.3) from (5.4) (using the definitions of \( S_m \) and \( S_g \)) yields:

\[
Y = f c(1 - e(1 - M))q_g, c(1 - eM)q_m + e w(1 - M)(1 - x)u_g = (1 + R): \tag{5.5}
\]

Let

\[
Q = f\left(\frac{q_g}{c(1 - e(1 - M))}\right) f(1 + R)Y = e w(1 - M)(1 - x)u_g: \tag{5.6}
\]

Then from (5.5) we have:

\[
Q = f(Y) \left(\frac{1}{c(1 - e(1 - M))}\right) f(1 + R)Y = e w(1 - M)(1 - x)u_g: \tag{5.7}
\]

The bargaining outcomes for \( q_m \) and \( q_g \) can also be used to express \( Q \) in terms of \( Y \) because
q_m and q_g only depend on Y in Lemma 5.1:

\[
Q = Q(Y) = \begin{cases} 
8 + \frac{1_i \cdot e M - (1_i \cdot ew)Y_i (1_i) e w_u}{c(1_i) e} & \text{if } Y 2 [0; (1_i \cdot e w)] \\
(1_i e w) w_u & \text{if } Y > \frac{1_i e w}{1_i e w} \\
0; & \text{if } Y 2 \frac{1_i e w}{1_i e w} \text{ and } w_u > \frac{1}{e} \\
\text{no trade;} & \text{if } Y > \frac{1}{e}.
\end{cases}
\]

A monetary equilibrium is a solution to \(Q(Y) = f(Y)\). Once \(Y\) is found, Lemma 5.1 and the definition of \(Q\) can be used to find the equilibrium values for \(q_m\) and \(q_g\).

(Figure 3 here.)

Figure 3 plots \(f(Y)\) and \(Q(Y)\). The intersections \(E^s\) and \(E^w\) are defined as follows:

\[
E^s: \quad q_g = q_m = 0; \quad Y = ewu(1_i \cdot M)(1_i \cdot x) = (1 + R);
\]

\[
E^w: \quad q_g = 0; \quad q_m = \frac{wu (1_i \cdot e w)(1_i \cdot M)(1_i \cdot x) j (1_i \cdot e)(1+R)}{c(1_i \cdot e M)(1_i \cdot M)(1_i \cdot x)}; \\
Y = (1_i \cdot e) wu \frac{e(1_i \cdot e w)(1_i \cdot M)(1_i \cdot x)}{(1_i \cdot e)} 1^{-1} R^n.
\]

In \(E^s\) no services are rendered by the money holder whereas in \(E^w\) service payments are positive. Hence, the relative value of holding money as opposed to a good, \(Y\), is greater in \(E^s\). In \(E^w\); \(q_m = 0;\) if and only if

\[
R \cdot \frac{e(1_i \cdot e w)(1_i \cdot M)(1_i \cdot x)}{(1_i \cdot e)} 1^{-1} R^n.
\]

This expression turns out to be a condition for the existence of a monetary equilibrium. Notice that it is satisfied for \(w < 1\) and sufficiently close to 1.

Proposition 5.3. A monetary equilibrium in which holding money is more valuable than holding a good, \(V_m > V_g\), exists if and only if \(R \cdot R^n\) and \(w > 0\), in which case \(E^s\) and \(E^w\) are monetary equilibria.

Proof. The following features of Figure 3 can be directly verified:

(i) There is no intersection between \(f(Y)\) and \(Q(Y)\) which gives a positive \(Q\) for \(Y \cdot u = e\).

(ii) When \(w > 0\), the intersection between \(f(Y)\) and the horizontal axis lies in \((0; wu(1_i \cdot e w))\). This intersection is on the left of \((1_i \cdot e w) w_u \cdot (1_i \cdot e w) \cdot R > R^n\). The intersection between \(f(Y)\) and the vertical axis is always above the intersection between \(Q(Y)\) and the vertical axis, \(Q(0) < f(0) < 0;\) thus, the intersection \(E^w\) exists if \(R \cdot R^n\).
If these conditions are satisfied, \( E^w \) and \( E^s \) are equilibria. When \( R = R^\alpha \), point \( E^w \) coincides with point \( E^s \) on the horizontal axis and the two equilibria coincide.

(iii) When \( w = 0 \); there is no \( Y > 0 \) such that \( f(Y) \) and \( Q(Y) \) intersect. \( \text{Q.E.D.} \)

Corollary 5.4. The condition for a monetary equilibrium, \( R \cdot R^\alpha \), implies:

\[
e > \frac{1}{w} \frac{\frac{p}{1-e^w}}{1 - \frac{1}{2}}; \quad x < \frac{2e^w}{1 - \frac{1}{e^w}}; \quad M < \frac{1}{w} \frac{1}{e^w} \frac{1}{x(1 - \frac{1}{e^w})}.
\]

Proof. The condition for \( M \) comes directly from the requirement \( R^\alpha > 0 \); which represents the lower bound of \( R \). For that restriction to be satisfied by some positive \( M \), in turn, \( x \) has to satisfy the condition in the Corollary. For the condition on \( x \) to be satisfied by some number in \((0,1)\), it is required that \( e > (1 - \frac{2}{w}) \).

As long as sellers or buyers do not have all the bargaining power, \( w \not\in (0,1) \), there is a parameter region for which a monetary equilibrium \( Y > 0 \) exists. If such a monetary equilibrium exists, then equilibrium \( E^s \) coexists with \( E^w \). In both equilibria, the quantity of services exchanged in a monetary trade is smaller than in an AGM, \( q_m < q_b \), and the surplus is greater, \( S_m > S_b \). It also can be verified that the good holder is better off than a seller, \( S_g > S_s \). This is because money received by the good holder improves that agent's future terms of trade. Were it possible to isolate an AGM without upsetting the monetary equilibrium, both the buyer and the seller would benefit from the buyer's good being replaced with a unit of money. The inefficiency stemming from AGMs is necessary for valued at money.

When \( w = 1 \), no monetary equilibrium exists because sellers are able to extract all the surplus from money holders; this implies that money is valueless by (5.4). At the other extreme, where buyers have all the bargaining power, \( w = 0 \); it can be shown that the equilibria coincide at \( Y = 0 \): Holding money can not dominate holding a good, because buyers in AGMs receive all the surplus. Money is valued but only because good holders are indifferent to accepting it.

Finally, notice that as \( R \not\in \mathbb{R} \), monetary equilibria exist in economies with very rapid encounters between agents (arbitrarily large \( \alpha \) ). This is in contrast to other search models.
in which money speeds up the rate of exchange. Money remains valuable in our model because it improves the terms of trade in matches.

5.3. Comparing Equilibria

When \( w > 0 \) the monetary equilibria differ in terms of the purchasing power of money, as indicated by \( q_m \), and the relative value of holding money, \( Y \). As both are greater in \( E_s \), we refer to \( E_s \) as the strong equilibrium and refer to \( E_w \) as the weak equilibrium. In \( E_s \) buyers produce no services. It can be readily shown that \( S_m(E_s) > S_m(E_w) \) and \( S_g(E_s) > S_g(E_w) \) so that both buyers and sellers are better off in monetary trades in which money is more valuable. Money holders are able to secure better terms of trade when money is highly valued, and money is more valuable in equilibrium when it improves the buyer’s terms of trade.\(^8\)

The following proposition follows immediately from equations (5.4) and (5.3) and the fact that the surpluses in monetary trades are greater in the strong equilibrium.

Proposition 5.5. The strong equilibrium \( E_s \) Pareto dominates the weak equilibrium \( E_w \):

Now consider the optimum quantity of money for the strong monetary equilibrium employing a utilitarian social welfare function, \( W = M V_m + (1_i \ M)V_g \). Substituting in \( V_m(E_s) \) and \( V_g(E_s) \) yields:

\[
W = (1_i \ M)[1_i \ ew(1_i \ M)(1_i \ x)]u=R:
\]

The optimal quantity of money, \( M^o \), is straightforwardly derived by maximizing this function subject to the equilibrium constraint \( R \cdot R^x \):

Proposition 5.6. The optimal quantity of money in the strong equilibrium is:

\[
M^o = \begin{cases} 
1_i \ \frac{1}{\left(R+1\right)(1_i \ e)} & \text{if } w_L < w < \frac{1}{1+2\left(R+1\right)(1_i \ e)} \ \wedge \ \frac{1}{\left(R+1\right)(1_i \ e)} \ \wedge \ W_H \ \cdot \ 1; \\
0; & \text{otherwise.}
\end{cases}
\]

\(^8\)Interestingly, the flexible price models of Shi (1993) and Trejos and Wright (1993) also display an even number of monetary equilibria which coexist and differ in the purchasing power of money. This suggests that multiple monetary equilibria are a feature of search and bargaining money models.
Money improves welfare when the buyer has little bargaining power \((w > w_L > \frac{1}{2})\) and must produce more services for the seller in AGMes. By replacing barter in an AGM, money reduces the exchange of little valued services \((e > 2(R + 1) = (3 + 2R - 2x) > 2 = 3)\) and increases the surplus by \(S_m(E^S) + S_g(E^S); (S_j + S_i) = ewu\): Of course, when the buyer has too little bargaining power \((w , w_H)\) money is not valued so it can not be used to improve welfare. \(M = 0\) corresponds to the pure barter equilibrium: \(W = V_g(B)\).

For the monetary economy we have ignored barter equilibria and also the possibility of monetary equilibria with \(V_m < V_g\): If agents can discard their money and receive their production good, then no monetary equilibrium exists with \(V_m < V_g\) and there is a pure barter equilibrium in which the value of searching with a good is \(V_g(B)\):

Finally, notice that the model displays the classic relationship that the monetary equilibrium only exists when the pure barter equilibrium is inefficient. Bargaining outcomes which yield the excess production of services are necessary for valued \(^{-}\) at money. Though money may improve welfare, it can not achieve the constrained optimum.

5.4. Production and Transaction Costs

In the above analysis we set \(\frac{1}{2} = 0\) and \(2 = 0\) to show that production and transaction costs were not generating the results. A good holder’s utility declines in the stock of money at rate \(i^{-}xu[x + (1_i x)(1_i w) + (1_i x)(1_i e) w] = i^{-}xu[1_i (1_i x) ew]\) and a money holder’s utility at rate \(i^{-}xu\), ignoring the effects of \(M\) on \(Y\) (which cancel out). The positive effect of money comes from replacing good holders with money holders which reduces the number of AGMes, \(i^{-}x \times (1_i x) ewu\). Thus, the positive effect only dominates at \(M = 0\) when \(w > w\). The large negative effect comes from money displacing goods in the economy. This is an implication of the inventory assumption (made for tractability reasons) that an agent cannot hold money and a good at the same time.

Under the assumption that agents cannot discard money, such a monetary equilibrium exists when \(R \in R^+\). A barter equilibrium exists when \(R < R\), where \(V_g = (1_i M)V_g(B)\) and \(V_m = 0\). The barter equilibrium and the monetary equilibrium are mutually exclusive when \(V_m < V_g\): This is because as \(e\) becomes small it is pro \(^{-}\)table to exchange in monetary matches on the basis of services alone. As \(R < R\), a barter equilibrium coexists with the monetary equilibrium in Proposition 5.3. When agents can discard money, the monetary equilibrium with \(V_m < V_g\) is eliminated and so is the barter equilibrium in favour of the pure barter equilibrium.

In Engineer and Shi (1994), a positive \(^{-}\)xed transaction cost of receiving services, \(c^{\circ} < u\); is included in the model. This feature increases service payments in AGMes. In the extreme case of \(c^{\circ}\) arbitrarily close to \(u\), the buyer will render up to \(u = c^{\circ}\) services to secure the seller’s good which destroys all the surplus in the exchange. The fact that more surplus is destroyed results in the condition for the monetary equilibrium being less restrictive.
by replacing \( u \) with \( u \odot \frac{3}{4} > 0 \), provided that it is pro\textsuperscript{\textregistered}table to produce. It is pro\textsuperscript{\textregistered}table to produce after consuming as long as \( V_g > \frac{3}{4} \). This condition is satisfied if \( u \) or \( R \) are sufficiently large.\(^{12}\)

Introducing a positive transaction cost for receiving a good, \( \delta > bu \), eliminates trades in MGM\( s \). It does not however eliminate the exchange of goods in AGM\( s \) as long as transferring the buyer's good is sufficient relative to providing extra services, \( b(u \odot \frac{3}{4} + \frac{3}{4} \delta^2 > \frac{3}{4}(1 \odot e) \): When \( e \) is large, a large range of \( \delta > bu \) satisfy this requirement. In this range, positive transaction costs result in larger service payments in AGM\( s \) when \( w < 1 = (2 \odot e) \); exacerbating the barter inef\textsuperscript{\textregistered}ciency. As there are no exchange cost to accepting money, the condition for monetary equilibrium is less restrictive.

6. Monetary Equilibrium with a Stronger Coincidence of Wants

6.1. Monetary Equilibrium with \( b > 0 \)

We now briefly demonstrate that monetary equilibria may exist when agents derive strictly positive net utility \( bu > 0 \) from consuming a mediocre good. Monetary equilibria exist because the model is continuous in \( b \) above \( b = 0 \).

Outcomes in matches between money holders and agents holding their preferred good are still given by Lemma 5.1. Matches between a money holder and those holding their mediocre good yields a solution given by replacing \( u \) with \( bu \) in Lemma 5.1 (see Remark 1). The analysis is made straightforward if we only consider monetary equilibria in which money holders do not trade for mediocre goods. Such trades do not take place if \( Y > bu = e \), as the nonnegative surplus constraint is violated. Thus, when \( Y > bu = e \) only equation (5.3) has to be modified:

\[
rV_g = -(1 \odot M) x^2 + (1 \odot x)^2 b + x(1 \odot x)(1 + \frac{b}{1 \odot e})(1 \odot ew) u + \sim M x S_g \tag{6.1}
\]

\(^{12}\)There is no production when \( V_g < \frac{3}{4} \). This occurs when \( u \) or \( R \) are sufficiently small. Provided that \( u \) or \( R \) are not too small, there may also exist a threshold bargaining weight \( w_0(M = 0) < 1 \) beyond which there is no production. In this case, the barter economy can be thought to close down as the result of the presence of the bargaining inefficiency. It can be shown that there exists parameters (for each monetary equilibrium) such that the threshold bargaining weight is increasing in the money supply, \( w_0(0) > 0 \). Hence, the introduction of money can resuscitate the economy in some circumstances.
The corresponding function $f(Y; b)$ is continuous in $b$. Since $Q(Y)$ is unchanged, it follows that a monetary equilibrium exists for $b$ arbitrarily small when $R < R^\omega$.

Appendix A analyzes the model with $b > 0$ and establishes that a "strong" and a "weak" monetary equilibrium coexist under conditions which are narrower than Proposition 5.3 but are always satisfied for $e$ close to 1 and $b$ close to 0.

6.2. Monetary Equilibrium with a Coincidence of Wants in Services

We now show that bargaining can give rise to valued money in a more general model in which the exchange of services is efficient. In particular, suppose that the utility derived from services is $v(q) = c(1 - e)q + (h/k) \ln(kq + 1)$, where $k > 0$. We assume that $h > ce$ so that the marginal utility of services is initially greater than the marginal cost of production, $v^0(0) > c$. Hence, all agents will have a coincidence of wants in services. However, as $k \to 1$, the service utility converges to the original specification: $\lim_{k \to 1} v(q) = c(1 - e)q$.

In both a MGM and a PGM $q^\omega > 0$; where $q^\omega$ is such that $v^0(q^\omega) = c$, services are exchanged. In an AGM the seller produces $q_s < q^\omega$ services for the buyer. The buyer however, produces $q_b > \max\{wu = c; q^\omega\}$ services. To make the trade the buyer must produce too many services relative to the welfare maximizing transfer $q^\omega$. With diminishing marginal utility, bargaining in AGMs drives the marginal utility of services to the seller below the marginal cost of producing services. Hence the pure barter equilibrium is inefficient.

Does the excessive production of services in AGMs still provide a role for money? In an appendix, available from the authors on request, we provide a limiting argument to show that a monetary equilibrium exists for large $k$. For each of the match types it can be shown that as $k \to 1$ the allocations converge continuously to those in the basic model. It follows that the corresponding functions $f(Y; k)$ and $Q(Y; k)$ are continuous in $k$ and converge to the functions in the text: $\lim_{k \to 1} f(Y; k) = f(Y)$ and $\lim_{k \to 1} Q(Y; k) = Q(Y)$: Hence a monetary equilibrium exists for $k$ sufficiently large when $R < R^\omega$.

This extension only makes the natural assumption that the marginal utility of consuming services falls, albeit quickly, below the cost of production. Thus, the basic model's
linear specification is not generating the results. In fact, in this extension the linear utility component provides a positive base level for marginal utility.

7. Strategic Bargaining and Alternative Bargaining Specifications

7.1. The Strategic Bargaining Game

It is well known that the Nash cooperative bargaining solution to problems in which the surplus is fixed can be derived from non-cooperative sequential bargaining games (see Binmore (1987) and Osborne and Rubinstein (1990)). This is also the case for our model. The equivalence of the solutions obtains because the search values $V_g$ and $V_m$ are taken as fixed by agents in individual matches. Appendix B develops a sequential bargaining game which yields the generalized Nash solutions used in the text. In the sequential game, the surplus weight $\lambda$ corresponds to the probability of "sellers" moving "rst in each round of bargaining, and the reservation utility depends on the exogenous separation of agents.

The sequence of moves in a monetary match is illustrated in Figure 4. After the money holder meets a good holder with his preferred good, nature chooses the good holder with probability $w$ and the money holder with probability $1 - w$ to "rst propose a pair $(q_g; q_m)$, where $q_g (q_m)$ is the quantity of services which the good holder (money holder) renders. The respondent either immediately accepts or rejects the proposal. If the proposal is accepted, the money holder immediately exchanges a unit of money and $q_m$ services for his preferred good and $q_g$ services from the good holder. Then the two depart and can no longer recognize each other. If the proposal is rejected, both agents encounter new agents during the interval at rate $\lambda$. If either of the two "nds a new "desirable" partner, the current trade relation is dissolved. If neither agent "nds a new desirable partner (probability $1 - a \lambda$), bargaining continues according to the above sequence which begins with nature choosing a new proposer.

Between bargaining rounds agents search for new desirable partners. The money holder leaves the existing match if he encounters another good holder who has his preferred good. This occurs with probability $\xi(1 - M)x$. The good holder leaves the existing match if he runs into another money holder who wants his preferred good or into any good holder.
This occurs with probability \( a \cdot (1 - M) x \). The \( X \)'s in Figure 4 are the expected payoffs of separating with a new partner.\(^{13}\) Agents do not separate if they meet an agent with whom there is no basis for trade.

The criterion for leaving an existing match between bargaining rounds is that the agent encounter a new partner with whom he would have entered into bargaining in the absence of being attached in an existing match. Thus, we restrict the analysis to strategies that are independent of the history of matching. This yields a simple and intuitive criterion for separation which does not require that an agent know anything about his new partner's possible existing trading partner.\(^{14}\)

The exogenous separation of agents during rounds of bargaining yields the threat point as a credible threat. In a recent paper, Trejos and Wright (1995) also analyze a money model in which agents exogenously separate and describe it as search while bargaining.\(^{15}\)

7.2. Bargaining Weights

The bargaining weights in money matches and AGMs were chosen to be equal in the model to demonstrate that the demand for money was not coming from that source. However, there is no reason to suppose the weights are the same, and as might be suspected, increasing the relative weight of money holders increases the value of money.

Denote the weight on the seller's surplus in an AGM as \( w_g \) and the weight on the money holder's surplus in a monetary trade as \( w_m \). The bargaining solutions are given by replacing \( w \) with the appropriate weight in Sections 4 and 5, and the functions \( f(Y; w_g) \) and \( Q(Y; w_m) \) take the same form as in Section 5 but with the appropriate weight replacing

\(^{13}\)The appendix details the good holder's separation expected payoffs for each match type. The possibility that both the money holder and the good holder find new partners is ignored as it is of second-order importance when \( c \) is small.

\(^{14}\)The criterion also has the merit of not always requiring separation. A money holder only separates when he meets another good holder with his preferred good. In this case he is better off separating because his old partner may have separated. The good holder will also prefer to separate when he meets another money holder. However, when he meets a good holder things are less clear cut. The criterion requires that the good holder always separate when meeting a good holder because they always trade. Restricting the good holder to not separate in say a PGM or an AGM, decreases the good holders reservation utility and gives a solution which further benefits the money holder.

\(^{15}\)Wolinsky (1987) characterizes a market equilibrium in a model with sequential bargaining and search while bargaining. See Osborne and Rubinstein (1990) for a survey of related nonmonetary models.
Suppose \( w_g = w_m \) and \( R = R^* \) so that \( f \) intersects \( Q \) at the kink closest to the origin in Figure 3. A decrease in \( w_g \) shifts the \( f \) function upwards so that the curves do not intersect and no monetary equilibrium exists. On the other hand, an increase in \( w_g \) shifts the \( f \) function downward so that there are two distinct equilibria. Now consider a change in \( w_m \) around \( w_g = w_m \) and \( R = R^* \): An increase in \( w_m \) shifts where \( Q \) intercepts the horizontal axis to the right so that no monetary equilibrium exists; whereas, a decrease in \( w_m \) shifts the intercept to the left expanding the range of equilibria. Hence, the range of parameters over which money is valued increases with the bargaining power of money holders (1 - \( w_m \)) as well as the bargaining power of sellers in AGM es. This result highlights that money is valued because holding it improves the buyer’s terms of trade.

7.3. Threat Points

The bargaining power of agents is also affected by the choice of threat points. Consider the more general specification \( \mathcal{U}_g = \mu_g V_g \) and \( \mathcal{U}_m = \mu_m V_m \); where \( \mu_g \geq 0 \) and \( \mu_m \geq 0 \) are parameters. Using these terms, the previous analysis corresponds to \( \mu_g = \mu_m = 1 \):

With the more general specification, lowering \( \mu_g \); ceteris paribus, reduces service payments in AGM es. This reduces the inefficiency of barter and hence reduces the value of money. On the other hand, lowering \( \mu_m \) improves the bargaining outcome for money holders in money matches making money more valuable. The combined effect of decreases in the threat points can be analyzed by looking at small changes in the threat points around \( \mu_g = \mu_m = 1 \). A small equal proportionate decrease in the threat points results in a narrower range for monetary equilibrium. In Engineer and Shi (1994) we examine the model for the extreme case in which threat points are zero for the specific bargaining parameter \( w = \frac{1}{2} \). In this case, the strong monetary equilibrium exists only when \( V_m = V_g \) in contrast to the basic model. This is because service payments are reduced substantially in AGM es.

---

16Lowering the threat points to zero corresponds to agents not searching between rounds of bargaining in the sequential game. As the seller does not leave the match, the buyer is able to get better terms of trade.

17In this case, an inefficient barter equilibrium exists but under a narrower set of parameter values. Monetary equilibria in which holding money is preferred, \( V_m > V_g \) can be generated by introducing a small preferential bargaining weight for money holders or by introducing a small exchange cost.
Changing the threat points empathizes that it is the buyer's lack of bargaining power in AGMes which gives rise to a bargaining-induced demand for fiat money.

8. Conclusion

Using the search-theoretic approach to monetary economics, this paper identifies a bargaining-induced transaction demand for fiat money. To isolate this demand, we allow agents to derive utility from all commodities so that there is no absence of the coincidence of wants in matches between good holders in the model. Nor is there a recognizability problem|agents have complete information about their trading partner once matched. Nevertheless, the barter equilibrium is inefficient. Bargaining in matches in which demands are asymmetric yields terms of trade which lower ex-ante utility relative to an alternative feasible exchange rule. The demand for money originates from this bargaining inefficiency.

To demonstrate that bargaining is generating the results, the basic model considers the case most conducive to barter|when there are zero physical transaction costs. The seller in an asymmetric match is no worse off swapping goods one for one without receiving services. If these exchanges occurred, money would not be valued. The reason they do not occur is because the buyer dearly wants the preferred good and seller is able to use his bargaining power to get better terms of trade. In particular, the seller is able to extract sufficient services from the buyer to drive the marginal utility of services well below the marginal cost of production.

In the monetary equilibrium, searching with an intrinsically useless object {fiat money} is preferred to searching with a valued good because buyers receive better terms of trade in asymmetric matches exchanging money than bartering their production good. In turn, money yields better terms of trade because it is more valuable in equilibrium. Money's greater value gives the holder generalized bargaining power, independent of the trading partner's preferences. Money also has the property of generalized purchasing power as it yields the same terms of trade in all matches.

\[^{18}\text{Of course, the search model implicitly presumes imperfect information with respect to particular future matches. As agents with preferred goods cannot be costlessly located, there is an absence of the double coincidence of wants with respect to preferred goods.}\]
As in all money models, the value of money depends on beliefs. In our model the Nash solution yields the buyer better terms of trade the more highly valued is money. There are multiple monetary equilibria. Both the value of money and the terms of trade are greater in the strong equilibrium and it Pareto dominates the weak equilibrium.

Of course, the value of money also depends on the specific bargaining game. When the surplus weights and threat points favor the seller, the value of money is greater. This is because the barter inef ciency is greater and using money gives the buyer better expected terms of trade. If the monetary equilibrium results in a large reduction in service payments, money improves welfare. On the other hand, if the seller has no bargaining power, there is no inef ciency and no monetary equilibrium exists. The model displays the classic relationship that the barter equilibrium must be inef cient for valued money.

Many avenues remain to be explored. We intend to demonstrate that a bargaining-induced transaction demand for commodity money will arise in a search model where goods are treated asymmetrically. This is a more complicated project because goods and agent categories must be tracked. The bargaining-induced demand for money should also appear in more complicated matching structures in which buyers and sellers are organized according to markets as in Burdett et. al. (1993). Both of these search models have promise for reducing the negative effect of "at money displacing goods that is present in the current model. Finally, we intend to explore whether other monetary models can be modified to generate a bargaining-induced transaction demand for money.
References


**APPENDIX**

**A. Monetary Equilibrium with b>0**

This appendix proves that a monetary equilibrium may exist when agents derive positive net utility $bu > 0$ from consuming a mediocre good (beyond the cost of producing another good). Section 6.1 details the outcomes in the various matches and changes to the value functions. The analysis is restricted to finding symmetric pure strategy equilibria in which money holders only trade for preferred goods.

**Definition A.1.** A monetary equilibrium in which money holders only trade for preferred goods is a joint solution of (5.2), (5.4) and (6.1) for $q_m, q_g; V_m$ and $V_g$ for which $Y > V_m, V_g > bu=e$.
The condition $Y > b_u e$ precludes exchanges between money holders and those holding mediocre goods. The examination of existence simplifies to finding a fixed point for $Y > b_u e$. Subtracting (6.1) from (5.4) yields:

$$Y = \frac{c(1_i e(1_i M) q_b i, c(1_i e M q_m + (1_i M)(1_i x)^h e w j [1_x + e(1_l w)]^i b u}{1 + R}$$

Using the definition of $Q$ we have:

$$Q = f(Y; b) - f(Y) + \frac{(1_i M)(1_i x)}{c(1_i e M)} \frac{x}{x} + e(1_l w) b u,$$

The bargaining outcome can be used to express $Q(Y)$ exactly as in the text because $q_m$ and $q_b$ only depend on $Y$ in Lemma 5.1.

A monetary equilibrium in which money holders only trade for their preferred good is a solution to $Q(Y) = f(Y; b)$: As before there is a strong and a weak monetary equilibrium. Define:

$$E^s_b: \quad q_m = q_b = 0; \quad Y = \frac{(1_i M)(1_i x)}{1+R} \frac{h e w j [1_x + e(1_l w)]^i b u}{c(1_i e M)}$$

$$E^w_b: \quad q_b = 0; \quad q_m = \frac{w u i}{c(1_i e M)} \frac{x}{x} + e(1_l w) b u,$$

$$E^w_\pi: \quad (1_i e M); (1_i x)(1_i w) \frac{x}{x} + e(1_l w) b u,$$

In $E^w_b$: $q_m = 0$ if and only if

$$R \cdot R^\pi_b \cdot R^\pi_i (1_i M)(1_i x) \frac{x}{x} + e(1_l w) b u,$$

This condition is satisfied for $e$ close to 1 (then $R^\pi > R$) and $b$ close to 0 when $w > 0$. It can be alternatively expressed

$$b \cdot b^\pi \cdot \frac{e w(1_l e w)(1_i e M)(1_i x)(1_i e w)(1+R)}{(1_i M)(1_i x)(1_i e w) \frac{x}{x} + e(1_l w)}$$

There are three other conditions restricting $b$: The condition $Y > b_u e$ implies upper-bound values for $b$ with respect to $E^s_b$ and $E^w_b$:

$$b^{s_u} = \frac{(1_i M)(1_i x) e^2 w}{1 + R + e(1_i M)(1_i x) \frac{x}{x} + e(1_l w)}$$

$$b^{w_u} = \frac{e(1_i e w)(1_i e M)(1_i x)(1_i e)(1+R) + e(1_i M)(1_i x)(e(1_i w) + (1_i e) x)}{(1_i e w)(1_i e M)(1_i x)(1_i e)(1+R) + e(1_i M)(1_i x)(e(1_i w) + (1_i e) x)}.$$
Finally, the condition for buyers making service payments in AGMes, \( w > \mathbb{w} \) requires \( b < (1 \ i \ e)w = (1 \ i \ w) \). It can be shown that \( b^* < b^{\mathbb{w}} < (1 \ i \ e)w = (1 \ i \ w) < b^{\mathbb{v}_n} \); so that if \( R \cdot R^w \) is satisfied the other conditions are also satisfied.

**Proposition A.2.** A monetary equilibrium in which agents only exchange money for their preferred goods exists if and only if \( R \cdot R^w \); in which case \( E^w \) and \( E^w_b \) are monetary equilibria.

**Proof.** The following features can be verified with reference to Figure 3:
(i) There is no intersection between \( f(Y; b) \) and \( Q(Y) \) which gives a positive \( Q \) as \( f(Y; b) \) lies above \( f(Y) \) which lies above \( Q(Y) \) in the interval \([wu=(1 \ i \ ew); u=e]\) as shown in the proof of Proposition 5.3.
(ii) The intersection between \( f(Y; b) \) and the horizontal axis lies in \((0; wu=(1 \ i \ ew))\) as \( f(Y; b) \) lies above \( f(Y) \) and \( R \cdot R^w \) implies \( f(0; b) < 0 \). This intersection is in the interval \([ (1 \ i \ e)wu=(1 \ i \ ew); wu=(1 \ i \ ew)) i \& R \cdot R^w \): Thus, \( f(Y; b) \) and \( Q(Y) \) intersect \( i \& R \cdot R^w \). \( E^w \) coincides with \( E^w_b \) on the horizontal axis when \( R = R^w_b \). Otherwise, the intersections are distinct. Q.E.D.

**B. The Sequential Bargaining Model**

This appendix derives the Nash solution from a non-cooperative sequential bargaining game. We only prove this result for bargaining between a money holder matched to a good holder who has the money holder's preferred good in inventory. The proofs for the other types of bargaining matches are similar. For a description of the game and the sequence of moves we refer the reader to Section 7.

To solve for the value of the equilibrium proposals, it is useful to express the proposals in terms of the utility values each agent obtains.

**Lemma B.1.** For a given value \( z \) to the good holder, the money holder can at most propose a value \( \hat{A}(z) \) for himself where

\[
\hat{A}(z) = \begin{cases} 
  u + V_g + (1 \ i \ e)(V_m \ i \ z) ; & \text{if } 0 \cdot z \cdot V_m \\
  u + V_g + \frac{V_m \ i \ z}{1 \ i \ e} ; & \text{if } V_m \cdot z \cdot V_m + (1 \ i \ e)(u + V_g) ;
\end{cases}
\]

(B.1)
Similarly, for a given value $z$ to the money holder, the good holder can at most propose a value $\mathcal{C}(z)$ for himself where

$$
\mathcal{C}(z) = f \left( \frac{V_m + (1 - e)(u + V_g)z}{V_m + \frac{u + V_g - z}{1 - e}} \right) \quad \text{if} \quad 0 \cdot z \cdot u + V_g \cdot z \cdot (1 - e) V_m + u + V_g;
$$

(B.2)

Proof. Given a value $z$ to the good holder, the money holder's maximization problem is

$$
\hat{A}(z) = \max_{(q_g; q_m)} [u + V_g + (1 - e) c q_g + c q_m : V_m + (1 - e) c q_m + c q_g \cdot z, q_g \cdot 0; q_m \cdot 0];
$$

(B.3)

As the objective function and the constraints are linear in $(q_g; q_m)$; a simple diagram can show that the maximum is $\hat{A}(z)$ specified in (B.1) and is achieved with the following maximizers:

$$
(q_g; q_m) = f \left( (V_m \cdot z = c; 0) \right) \quad \text{if} \quad 0 \cdot z \cdot V_m \quad (0; z \leq V_m) = \{(1 - e) c \}; \quad \text{if} \quad V_m < z \cdot V_m + (1 - e)(u + V_g);
$$

(B.4)

Similarly, the good holder's maximization problem yields the function $\mathcal{C}(z)$. Q.E.D.

Remark 2. For any value $z \in [0; V_m + (1 - e)(u + V_g)]$; $\mathcal{C}(\hat{A}(z)) = z$ and hence $\mathcal{C}^{-1} = \hat{A}$.

Now denote by $G_m$ ($G_g$) the subgame where nature has chosen the money holder (good holder) to be the first proposer. Let $W_m$ ($W_g$) be the equilibrium value which the money holder (good holder) proposes for himself in $G_m$ ($G_g$). We will later verify that the equilibrium values of the proposals are given as follows:

$$
W_m = \hat{A}((1 - r \xi) B_g); \quad W_g = \mathcal{C}((1 - r \xi) B_m)
$$

(B.5)

where $r$ is the instantaneous discount rate, $\xi$ is the interval between bargaining rounds, and $B_m$ ($B_g$) is the expected value to the money holder (good holder) after a rejection of a proposal by either agent at the end of the interval. The key to determining the equilibrium values is to determine $(B_g; B_m)$.

The values of $(B_g; B_m)$ are determined by agents' search outcomes. During search, the money holder (good holder) accepts the new match if he expects at least a value $V_m$ ($V_g$). To calculate these values let $X_m$ be the value a money holder expects to obtain after being matched to an agent who has his preferred good; $X_g$ be the value a good holder expects
from such a match. Let \( X_p \) be the value a good holder expects to obtain from a PGM and \( X_n \) is the value from an AGM. Let \( X_s \) (\( X_b \)) be the value a seller (buyer) expects from an AGM. Then

\[
rV_m = \bar{V}_m + (1 - \bar{M}) X_m \; \text{and} \; rV_g = \bar{V}_g + (1 - \bar{M}) (X_g + (1 - \bar{x}) X_s + (1 - \bar{x}) X_b) + (1 - \bar{x})^2 X_n.
\]

Agents take all the \( V \)'s and \( X \)'s as given when bargaining.

Given \( (B_g; B_m) \), the bargaining between the money holder and the good holder breaks down with the probability \( a \bar{x} \) after a rejection of a proposal, where

\[
a \bar{x} = \bar{x} (x + 1 - M) = \bar{x} f x (1 - M) + x M + (1 - M) [x^2 + 2x(1 - x) + (1 - x)^2] g;
\]

In particular, \( \bar{x} (1 - M) \) is the probability with which the money holder finds a new partner who has his preferred good. Therefore

\[
B_m = \bar{x} (1 - M) X_m + \bar{x} [a \bar{x} (1 - M)] V_m + (1 - \bar{x} a \bar{x}) (1 - w) W_m + w \bar{A}(W_g);
\]

The first term on the right-hand side is the expected value to the money holder when he finds a new partner; the second term is the value when he is left without any partner; the last term is the value when bargaining continues with the same partner. Since \( \bar{A}^1 = \bar{A} \) and \( X_m = (1 + r f (1 - M) x) V_m \), the above equation becomes

\[
B_m = \bar{x} (r + a) V_m + (1 - \bar{x} a \bar{x}) [(1 - w) W_m + w \bar{A}(W_g)]; \tag{B.6}
\]

Similarly,

\[
B_g = \bar{x} (r + a) V_g + (1 - \bar{x} a \bar{x}) [w W_g + (1 - w) \bar{A}(W_m)]; \tag{B.7}
\]

Denote

\[
Y_1 = \frac{[w + (1 - w) (r + a) \bar{x}] (1, e + [(2 e) (1 - w)] \bar{x}^2) \bar{c} V_g}{1, e + [(w + 1 - w) (1 - w)] \bar{x}^2 (r + a) \bar{c} (1 - w)};
\]

\[
Y_2 = \frac{[w + (1 - w) (r + a)] [(1 - w) \bar{x} + (2 e) (1 - w)] \bar{x}^2 \bar{c} V_g}{1, e \bar{x} + [(1 - w) \bar{x} + (2 e) (1 - w)] \bar{x}^2 (r + a) \bar{c}};
\]

\[
Y_3 = \frac{[w + (1 - w) (r + a)] [(1 - w) \bar{x} + (2 e) (1 - w)] \bar{x}^2 \bar{c} V_g}{e \bar{x} + [(1 - w) \bar{x} + (2 e) (1 - w)] \bar{x}^2 (r + a) \bar{c}};
\]
Lemma B.2. When \( \xi \) is sufficiently small, there is a unique pair \((W_g; W_m)\) which solves (B.5). The value of \( W_g \) is given as follows (ignoring some of the higher order terms of \( \xi \)):

(i) if \( Y \in [0; Y_1] \), then

\[
W_g = V_g + \left[ w + (1 - e) \right] \left[ \frac{r^2}{1 + a} \right] W_g + [w + e(1 - e)](r + a) + (1 - e)w \frac{r^2}{1 + a} \xi Y; \tag{B.8}
\]

(ii) if \( Y \in [Y_1; Y_2] \), then

\[
W_g = V_m + \frac{\xi e(1 - e)}{(r + a - ra \xi)^2} \left[ \frac{r^2}{1 + a} \right] W_g
\]

(iii) if \( Y \in [Y_2; Y_3] \), then

\[
W_g = V_g + (1 - e) \left[ w + (1 - e) \right] \left[ \frac{r^2}{1 + a} \right] W_g + [w + e(1 - e)](r + a) + (1 - e)w \frac{r^2}{1 + a} \xi Y \tag{B.9}
\]

If \( Y \) lies in none of the above regions, there is no trade between the two agents. Proof.

Note that \( \xi = \hat{A} \). Substituting (B.6) and (B.7) into (B.5), we have:

\[
\hat{A}(W_g) = 1_m; \quad \xi(W_m) = 1_g \tag{B.11}
\]

where

\[
\begin{align*}
1_g(W_g) &= \frac{(1 - r \xi)(r + a)W_g + w(1 - e)W_m}{w(1 - e)(1 + a \xi)W_g}; \\
1_m(W_m) &= \frac{(1 - r \xi)(r + a)W_m + w(1 - e)W_m}{w(1 - e)(1 + a \xi)W_m}.
\end{align*}
\]

Since \( W_m = \hat{A}(1_g(W_g)) \), it suffices to determine \( W_g \). Eliminating \( W_m \) from (B.11) yields:

\[
\text{LHS}(W_g) = \left[ 1_i \left( w(1 - e)(1 - a \xi) \right) \hat{A}(W_g) \right] i \left( w(1 - e)(1 - a \xi) \right) \xi (r + a) V_m + (1 - e)w(1 - e) \hat{A}(1_g(W_g)) = 0 \tag{B.12}
\]

Note that \( W_g > 1_g \) because \( W_g > V_g \) \( \forall Y \). Also, \( 1_g > V_m i \hat{A} \)

\[
W_g > A \left( \frac{1_i (1 - e)(1 - a \xi)}{w(1 - e)(1 - a \xi)} V_m i \frac{\xi (r + a)}{w(1 - e)(1 - a \xi)} W_g \right)
\]

We divide the proof into three cases.

Case (i): \( A < W_g \cdot V_m + (1 - e)(u + V_g) \). In this case, \( 1_g \) \( 2 \) \( V_m \cdot V_m + (1 - e)(u + V_g) \) for sufficiently small \( \xi \), and \( \hat{A}(z) = u + V_g + (V_m i \cdot z) \equiv (1 - e) \) for \( z = W_g ; 1_g \). Substituting
these expressions into (B.12) and simplifying, one obtains (B.8). The corresponding range of $Y$ for this case in Lemma B.2 is obtained from $Y \geq 0$ and $W_g \geq A$.

Case (ii): $W_g \geq [V_m; A]$. In this case, $V_g < V_m$. Hence $\bar{\Lambda}(1 - g) = u + V_g + (1 - e)(V_m - V_g)$ and $\bar{\Lambda}(W_g) = u + V_g + (V_g - W_g)$. Substituting these expressions into (B.12) and simplifying the result yield (B.9). In this case we ignore only the terms of orders higher than $\epsilon^2$ because the range of $W_g; A \leq V_m$, is in the order of $\epsilon$. (Ignoring the terms of the order $\epsilon^2$ will result in an imprecise range for $Y$ when we take the limit $\epsilon \to 0$ of the result.) The corresponding range of $Y$ is such that $W_g \leq [V_g; V_m]$.

Case (iii): $V_g \geq W_g < V_m$. Since $W_g < V_m$, $V_g < V_m$. Thus $\bar{\Lambda}(z) = u + V_g + (1 - e)(V_m - z)$ for $z = W_g; V_m$. Substituting these expressions into (B.12) and ignoring the higher order terms of $\epsilon$, one can verify that $W_g$ is given by (B.10). The corresponding range of $Y$ for this case is obtained from $W_g \geq [V_g; V_m]$. Q.E.D.

Remark 3. When $\epsilon \to 0$, $W_g \to S_g + V_g$ and $W_m \to S_m + V_m$ where $S_g$ and $S_m$ are the surpluses given by Lemma 5.1 in the text. Also, the values of $q_g$ and $q_m$ induced by $W_g$ and $W_m$ approach the ones given by Lemma 5.1.

Proposition B.3 below states that the values of $W_g$ and $W_m$ in Lemma B.2 are generated by the unique subgame perfect equilibrium of the sequential bargaining game. Therefore, Proposition B.3 and the above remark establish an equivalence between the Nash bargaining solution in Lemma 5.1 and the limit of the sequential bargaining result.

Proposition B.3. For sufficiently small $\epsilon$, there is a unique subgame perfect equilibrium in a monetary trade in which $(W_g; W_m)$ satisfy (B.5) and $(B_g; B_m)$ are given by (B.6) and (B.7). The equilibrium strategies are as follows. The good holder always proposes $W_g$ for himself and $(1 - r \epsilon)B_m$ for the partner; accepts proposals which give him no less than $(1 - r \epsilon)B_m$. rejects any other proposals and searches for a new partner. The money holder’s strategies are given similarly by switching subscript $g$ with $m$. The rst proposal is accepted immediately.

To prove Proposition B.3, we rst established the following lemma:
Lemma B.4. For sufficiently small $\xi$, we have $W_g > (1 \xi r \xi)B_g$ and $W_m > (1 \xi r \xi)B_m$.

Proof. We only show $W_g > (1 \xi r \xi)B_g$. The proof for $W_m > (1 \xi r \xi)B_m$ is similar. Since $\gamma(W_m) = \gamma_g$ by (B.11), then $W_g > (1 \xi r \xi)B_g$

$$W_g > (1 \xi r \xi)B_g$$

Because $V_g \cdot W_g$ and $(1 \xi r \xi)\xi(r + a) = r + a \xi r \xi$ is decreasing in $\xi$, the last inequality holds. Q.E.D.

Proof of Proposition B.3: We first show that the strategies in the proposition describe a subgame perfect equilibrium (SPE for short). Consider the subgame $G_g$. Since the money holder rejects any proposal which gives him less than $(1 \xi r \xi)B_m$, the most the good holder can propose for himself in any acceptable proposal is $\gamma((1 \xi r \xi)B_m) = W_g$ (see (B.5)). Such a proposal is feasible. Also, if the good holder makes an unacceptable proposal, the rejection results in $(1 \xi r \xi)B_g$ for the good holder. As $W_g > (1 \xi r \xi)B_g$ by Lemma B.4, the proposal $(W_g; (1 \xi r \xi)B_m)$ is the best proposal that the good holder can make. Given this proposal, the money holder obtains $(1 \xi r \xi)B_m$, accepting which is as good as rejecting. Thus the specified strategies are a Nash equilibrium in $G_g$. Similarly, the strategies are a Nash equilibrium in $G_m$ and hence a subgame perfect equilibrium.

To establish that the strategies form the unique SPE, we follow Osborne and Rubinstein (1990, pp. 56-58) and show that all possible SPE's give the same payoffs as the SPE in Proposition B.3. For this purpose, denote by $P_m$ ($P_g$) the supremum of payoffs to the money holder (good holder) generated by all SPE's of $G_m$ ($G_g$). Denote $p_m$ and $p_g$ as the corresponding in ma. We show $p_m = P_m = W_m$ and $p_g = P_g = W_g$.

Step 1: Some inequities. Consider $G_m$. If a proposal is rejected in the first period of $G_m$, the good holder expects to obtain at least $(1 \xi r \xi)\gamma_g$ where

$$\gamma_g = \gamma(r + a)V_g + (1 \xi a \xi)[wp_g + (1 \xi w)\gamma(P_m)]$$

(B.13)

(B.13) is obtained in a similar way to (B.7). After a rejection, the money holder can at
most expect \((1_i r g)\) \(\pi_m\) where

\[
\pi_m = \phi (r + a)V_m + (1_i a g)(1_i w)P_m + w\hat{A}(p_g)\tag{B.14}
\]

Thus

\[
P_m \cdot \max((1_i r g)\pi_m; \hat{A}((1_i r g), g)) \tag{B.15}
\]

Similarly, define

\[
\pi_g = \phi (r + a)V_g + (1_i a g)(1_i w)P_g + (1_i w)\hat{A}(p_m)\tag{B.16}
\]

Then

\[
P_g \cdot \max((1_i r g)\pi_g; \hat{A}((1_i r g), g)) \tag{B.17}
\]

Since the payoffs in Proposition B.3 are SPE payoffs, then

\[
\phi P = W_m \cdot P_m, \quad \phi P = W_g \cdot P_g \tag{B.21}
\]

Step 2: For sufficiently small \(\phi\), \(\hat{A}((1_i r g), m)\), \(\phi((1_i r g)\pi_m)\) and \(\hat{A}((1_i r g), g)\).

Proof. We only show the first inequality. The proof for the second inequality is similar. Suppose the first inequality is violated, i.e., \(\hat{A}((1_i r g), m) < (1_i r g)\pi_m\). Then (B.20)

\[
P_g \cdot (1_i r g)\pi_g.\] Substituting \(\pi_g\) from (B.17) and rearranging terms yields:

\[
p_m \cdot \hat{A} \frac{(1_i w)(1_i r g)(1_i a g)}{(1_i w)(1_i r g)(1_i a g)} P_g \phi \frac{(r + a)}{(r + a)} V_g \tag{B.22}
\]

On the other hand, (B.19) \(p_m \cdot \hat{A}(1_i r g)\pi_g\). Substituting \(\pi_g\) from (B.17) implies

\[
p_m \cdot \hat{A}(1_i g(P_g)) \tag{B.23}
\]

where \(1_g(\phi)\) is defined in the proof of Lemma B.2. The right-hand side of (B.23) must be less than or equal to the right-hand side of (B.22). This requirement can be simplified to

\[
P_g \cdot \frac{(r + a)(1_i r g)}{r + a} V_g.
\]
However, in the proof of Lemma B.4, the right-hand side of the above inequality is shown to be strictly less than \( W_g \), and hence strictly less than \( P_g \) by (B.21). A contradiction. Q.E.D.

Step 3: \( P_g \cdot W_g \).

\[
p_m \cdot \frac{1}{(1_i w)(1_i \bar{r} \xi)(1_i a \xi)} \hat{A}(P_g) \frac{\xi (r + a)}{(1_i w)(1_i a \xi)} V_m:
\]

This inequality and (B.23) imply \( \text{LHS}(P_g) \leq 0 \), where \( \text{LHS}(z) \) is defined at (B.12). Since \( \text{LHS}(W_g) = 0 \), it suffices to show that \( \text{LHS}(P_g) \) is a decreasing function. To show this, care should be taken because \( \text{LHS}(z) \) is not differentiable when either \( z = V_m \) or \( ^1 g(z) = V_g \).

However, for any \( z \in [0; V_m + (1_i e)(u + V_g)] \) and positive \( \pm \lim_0 \hat{A}^q z_i \), \( \lim_0 \hat{A}^q (z + \pm) \) and both limits exist. Using the facts that \( P_g > ^1 g(P_g) \) and that \( \hat{A} \) is a decreasing function, one can verify that

\[
\lim_0 \hat{A}^q (P_g i) \cdot \lim_0 \hat{A}^{q \pm} (P_g + \pm) < 0; \text{ for } \pm > 0:
\]

With this, one can further verify that \( \text{LHS}(P_g) \) is indeed decreasing in \( P_g \). Q.E.D.

Step 4: \( p_g = W_g = P_g \) and \( p_m = W_m = P_m \).

Since \( W_g \cdot P_g \) by (B.21), Step 3 implies \( P_g = W_g \). Then (B.23) implies \( p_m \cdot \hat{A}(^1 g(W_g)) = W_m \). Since \( p_m \cdot W_m \) by (B.21), then \( p_m = W_m \). Similarly, one can show that \( P_m = W_m \) and \( p_g = W_g \). This completes the proof for Step 4 and the proof for Proposition B.3. Q.E.D.