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THE BIAS OF INEQUALITY MEASURES IN VERY SMALL SAMPLES: SOME ANALYTIC RESULTS*

David E. Giles

Department of Economics, University of Victoria
Victoria, B.C., Canada V8W 2Y2

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Abstract

We consider the class of generalized entropy (GE) measures that are commonly used to measure inequality. When used in the context of very small samples, as is frequently the case in studies of industrial concentration, these measures are significantly biased. We derive the analytic expression for this bias for an arbitrary member of the GE family, using a small-sigma expansion. This expression is valid regardless of the sample size, is increasingly accurate as the sampling error decreases, and provides the basis for constructing 'bias-corrected' inequality measures. We illustrate the application of these results to data for the Canadian banking sector, and various U.S. industrial sectors.

Keywords: Inequality indices, generalized entropy, bias, small-sigma expansion

JEL Classifications: C13; C16; D31

Author Contact:
David E. Giles, Dept. of Economics, University of Victoria, P.O. Box 1700, STN CSC, Victoria, B.C., Canada V8W 2Y2; e-mail: dgiles@uvic.ca; Phone: (250) 721-8540; FAX: (250) 721-6214

1. Introduction

Many different measures or indices of inequality have been proposed, and there is a vast literature on this topic. Seminal contributions include those of Atkinson (1970, 1983) and Sen (1973), and a useful overview is provided by Litchfield (1999). It is widely (although not universally) accepted that any useful inequality index should satisfy the following five axioms: anonymity, decomposability, income scale independence, the principle of population, and the Pigou-Dalton transfer principle. See Cowell (1985) for full details. Many of the standard inequality measures can be shown to be special cases of the class of generalized entropy (GE) measures. Such measures can be expressed as:

$$GE(\alpha) = \frac{1}{\alpha(\alpha-1)}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i}{\bar{y}}\right)^{\alpha} - 1\right] \quad ; \quad i = 1, 2, \ldots, n \; ; \quad \alpha \in [0, \infty) \qquad (1)$$

where $\bar{y} = (1/n)\sum_{i=1}^{n} y_i$. Cowell (1995) shows that any inequality measure that satisfies all of these axioms must be a member of the GE family, so this class of measures is of particular interest. Specific special members of this family include Theil's (1967) mean log deviation[1]:

$$GE(0) = \left[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\bar{y}}{y_i}\right)\right] \; ;$$

Theil's (1967) Index:

$$GE(1) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_i}{\bar{y}}\right)\log\left(\frac{y_i}{\bar{y}}\right)\right] \; ;$$

and half the squared coefficient of variation (CV):

$$GE(2) = \left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2\right] / (2\bar{y}^2) \, .$$

When any of these measures are constructed in practice, we have a sample estimator of the corresponding population measure. A question that then arises is, "to what extent are these sample statistics reliable estimators of their population counterparts?" Under mild conditions, these statistics are unbiased if the sample is infinitely large, but this is of limited comfort in practice. Recently, Breunig (2001) has addressed this question in relation to the squared sample

CV, by deriving approximations to its finite-sample bias and mean squared error in terms of a large-sample asymptotic expansion. Among other things, he showed that CV^2 is biased downwards if the underlying population distribution is positively skewed (as would usually be the case with income data, for example).[2] Deltas (2003) points out that when such measures of inequality are used to measure industrial concentration, there is likely to be a sizeable problem with bias, as typically such studies (*e.g.*, Hart and Prais, 1956; Ghosh, 1975; Adelaja, 1998) involve very small sample sizes. He provides Monte Carlo evidence regarding the magnitude of the downward bias of the Gini coefficient in samples of the size likely to be encountered in industry concentration studies. He also provides an interesting empirical application that illustrates that ignoring this bias can reverse one's conclusions in practice.

While Breunig's (2001) 'large-*n*' asymptotic expansion provides a useful approximation to the finite-sample bias of $2GE(2)$ for moderate sample sizes, by its very nature it is unlikely to yield an accurate measure of the bias when the sample size is very small. In such cases, an alternative approach is needed. In this paper we derive the approximate bias for *any* member of the GE class of inequality measures, using a 'small-σ' expansion, rather than a 'large-*n*' expansion. Under minimal assumptions, our bias calculations are increasingly accurate as the sampling error becomes smaller, regardless of the form of the underlying population distribution, and regardless of the sample size.

The plan of the rest of the paper is as follows. In the next section we discuss small-sigma approximations and present the principal results that we use in our derivations. Section 3 presents the main theoretical results, and interprets their implications. Two empirical examples that illustrate these implications are provided in section 4, and section 5 concludes.

2. Small-Disturbance Approximations

As noted in the last section, under some mild conditions, the inequality measures under consideration are well behaved, asymptotically. By this, we mean that as the sample size becomes infinitely large the sample measures converge in probability to the corresponding population measures. In the same manner, we can evaluate the dispersion of the sample measures in terms of their asymptotic distribution. However, in practice we are concerned with their behaviour in *finite* samples.

The determination of the *exact* finite-sample bias of $GE(\alpha)$ is confounded by the fact that the expression in (1) is highly non-linear in the random ('*y*') data. In such cases, we can consider various analytic approximations to this bias. One possible approach is that followed by Breunig (2001) in the case of CV^2 – we can use an approximation based on an analytic expansion whose accuracy improves as '*n*', the sample size, grows. Such approximations, proposed by Nagar (1959) in the econometric context, involve an expansion of the sampling error such that the successive terms are in decreasing order of '*n*', in probability. When used to determine the moments of an estimator, this approach yields the moments of the Edgeworth expansion of that estimator's distribution (Ullah, 2004, p.29).

Another option, first proposed by Kadane (1971), is to approximate the finite-sample moments of the estimator by using an expansion of the sampling error such that successive terms are in decreasing order of the population standard deviation, σ, in probability. These 'small-disturbance', or 'small-σ', approximations have been found to be extremely valuable for a number of problems in econometrics. They are valid for any sample size, and they have the additional merit that they do not require any additional assumptions about the behaviour of the sample moments of the data as *n* increases. Assumptions of the latter type are required for the validity of 'large-*n*' expansions, and can be difficult to verify in practice. A good recent discussion of 'small-disturbance' expansions is given by Ullah (2004, pp.36-45).

Now, consider the following data-generating process:

$$y_i = \mu + \sigma u_i \quad ; \qquad i = 1, 2, \ldots, n \tag{2}$$

where $\mu \neq 0$ and the u_i's are independently and identically distributed with

$$E(u_i) = 0 \quad ; \quad E(u_i^2) = 1 \quad ; \quad E(u_i^3) = \gamma_1 \quad ; \quad E(u_i^4) = \gamma_2 + 3 . \tag{3}$$

So, the skewness of the population distribution is γ_1, and γ_2 is its excess kurtosis. The $GE(\alpha)$ inequality measures in (1) are highly non-linear functions of the data. If they are constructed from a sample of data from this population, they will be biased estimators of their population counterparts. Our primary objective is to determine the magnitude of this bias, under very mild assumptions about the population distribution, to $O(\sigma^4)$, and we use the following result.

Lemma 1 (Ullah, 2004, p.38.)

Let 'y' be an n-element random vector, with $y = \mu + \sigma u$, where 'u' satisfies the conditions in (3) above, and the non-zero mean, μ, is a function of a parameter vector, θ. Let $\hat{\theta} = h(y)$ be an estimator of θ, where $h(y)$ and its derivatives exist in a neighborhood of μ. Then[3]

$$E(\hat{\theta} - \theta) = \sigma^2 \Delta_2 + \sigma^3 \gamma_1 \Delta_3 + \sigma^4 (\gamma_2 \Delta_4 + 3\Delta_{22}) \ , \tag{4}$$

where, for $s = 2, 3, 4$:

$$\Delta_s = \frac{1}{s!} \sum_{k=1}^{n} \left[\frac{\partial^s h(y)}{\partial y_k^s} \right]_{y=\mu} \quad ; \quad \Delta_{22} = \frac{1}{4!} \sum_{k=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial^4 h(y)}{\partial y_k^2 \partial y_j^2} \right]_{y=\mu} .$$

3. Bias Results

Our main result, the proof of which is given in the Appendix, now follows.

Theorem 1

Viewing $GE(\alpha)$ in (1) as an estimator of the corresponding underlying population inequality measure,

$$Bias[GE(\alpha)] = \frac{\sigma^2 (n-1)}{2n\mu^2} + \frac{\sigma^3 \gamma_1 (n-1)}{6n^2 \mu^3} \left[\alpha(n-2) - 2(n+1) \right]$$
$$+ \frac{\sigma^4 (n-1)}{24 n^3 \mu^4} \left(\gamma_2 [\alpha^2 (n^2 - 3n + 3) - \alpha(5n^2 - 3n - 9) + 6(n^2 + n + 1)] + 3n[\alpha(\alpha - 5)(n-1) + 6(n+1)] \right) \tag{5}$$

■

Interpreting this result, first note that, to $O(\sigma^2)$, for *any* population distribution

$$Bias[GE(\alpha)] = \frac{\sigma^2 (n-1)}{2n\mu^2} \approx \upsilon^2 / 2 \ ,$$

where $\upsilon = (\sigma / \mu)$ is the *population* coefficient of variation. So, to this rather crude order of approximation, $GE(2)$ is 100% upward-biased! This expression is independent of *α*, and is strictly positive. Let us now consider more accurate bias expressions associated with some important choices of *α*. The following results emerge immediately from (5):

Corollary 1

To $O(\sigma^4)$,

$$Bias[GE(0)] = \frac{\sigma^2(n-1)}{12n^3\mu^4}\left(6n^2\mu^2 - 4n(n+1)\sigma\mu\gamma_1 + 3\sigma^2[\gamma_2(n^2+n+1)+3(n+1)]\right) \quad (6)$$

$$Bias[GE(1)] = \frac{\sigma^2(n-1)}{12n^3\mu^4}\left(6n^2\mu^2 - 2\sigma n(n+4)\mu\gamma_1 + \sigma^2[\gamma_2(n^2+3n+9)+3n(n+5)]\right) \quad (7)$$

$$Bias[GE(2)] = \frac{\sigma^2(n-1)}{2n^3\mu^4}\left(n^2\mu^2 - 2\sigma n\mu\gamma_1 + 3\sigma^2(\gamma_2+n)\right) \quad (8)$$

■

Differentiating these expressions with respect to γ_1, it is easily shown that in these three cases the bias is a decreasing function of skewness if $\mu > 0$. This is consistent with Deltas's (2003) Monte Carlo evidence for the Gini coefficient in very small samples. Many distributions (*e.g.*, the log-normal) that may be of interest in the context of the distribution of incomes are leptokurtic. Now suppose that the underlying population is leptokurtic or mesokurtic, so $\gamma_2 \geq 0$. It follows immediately from (6) – (8) that in these cases, to $O(\sigma^4)$, $GE(\alpha)$ will be biased *upwards* if the data are *negatively* skewed. This is consistent with Breunig's (2001) 'large-*n*' result for CV^2. However, when measuring inequality, in practice a more interesting case is when $\gamma_1 > 0$.

Corollary 2

If the population distribution has a positive mean and is positively skewed, then the biases in Theorem 2 will be *positive*, to $O(\sigma^4)$, if and only if:[4]

$Bias[GE(0)]$: $$\gamma_1 < \left(\frac{3n}{2\upsilon(n+1)}\right) + \left(\frac{3\upsilon}{4n(n+1)}\right)\left[\gamma_2(n^2+n+1)+3(n+1)\right]$$

$Bias[GE(1)]$: $$\gamma_1 < \left(\frac{3n}{\upsilon(n+4)}\right) + \left(\frac{\upsilon}{2n(n+4)}\right)\left[\gamma_2(n^2+3n+9)+3n(n+5)\right]$$

$$Bias[GE(2)]: \qquad \gamma_1 < \left(\frac{n}{2\upsilon}\right) + \left(\frac{3\upsilon}{2n}\right)[\gamma_2 + n] \ .$$

To illustrate these results further, note that under the parameterization in (3), for the log-normal distribution $\upsilon = (e-1)^{1/2} = 1.311$, $\gamma_1 = (e-1)^{1/2}(e+2) = 6.185$, and $\gamma_2 = (e^4 + 2e^3 + 3e^2 - 3) = 113.936$. It is readily verified that for this distribution, the conditions in Corollary 2 are satisfied for all real 'n', so each of these three inequality measures is *upward*-biased when the underlying population is log-normal. In contrast, Deltas (2003) shows that the Gini coefficient has a substantial *downward* bias in small samples for several population distributions, including the log-normal, and Breunig (2001, p.17) reaches the same conclusion for CV^2 with log-normal data, on the basis of his 'large-n' asymptotic approximation.[5]

Approximate bias-corrected variants of the GE inequality measures can be constructed by replacing μ, σ, γ_1 and γ_2 in (5) with their sample counterparts, $\bar{y}$, s, g_1 and g_2, and then subtracting this estimated bias from $GE(\alpha)$. For example, for $GE(2)$, the corresponding (approximately) bias-corrected measure is:

$$G\hat{E}(2) = \left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]/(2\bar{y}^2) - \frac{s^2(n-1)}{2n^3\bar{y}^4}\left(n^2\bar{y}^2 - 2sn\bar{y}g_1 + 3s^2(g_2 + n)\right)$$

where[6]

$$s = \left[\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]^{1/2}$$

$$g_1 = \frac{1}{ns^3}\sum_{i=1}^{n}(y_i - \bar{y})^3$$

$$g_2 = \left(\frac{1}{ns^4}\sum_{i=1}^{n}(y_i - \bar{y})^4\right) - 3 \ .$$

4. Empirical Examples

We consider two empirical illustrations of the above results, each relating to industrial concentration in the case of very small samples. Our first example involves the size distribution of the five largest charter banks in Canada, where size is measured in terms of total assets.[6] These five banks dominate the Canadian banking sector, and are the focus of attention in the context of potential bank mergers. The raw and bias-corrected values for $GE(0)$, $GE(1)$ and $GE(2)$ for the years 2001 to 2003 appear in Table 1. In each case, the inequality measures are biased upwards by a substantial margin. In both value and percentage terms, these biases are very similar for $GE(0)$ and $GE(2)$. In all cases, it is clear that neglecting the bias that arises here due to the small sample and the small amount of variation in the data, would be most unwise.

Our second example is for the distributions of the market capitalization of the five largest firms in selected U.S. industries.[7] Interest in the degree of concentration with respect to the largest four or five firms in an industry arises in antitrust cases, for example.[8] The results in Table 2 again illustrate the substantial biases that can arise when the sample is this small. Also shown in Tables 1 and 2 are the corresponding percentage biases that would be inferred if Breunig's (2001) 'large-*n*' bias approximation were used. From Breunig's equation (5), the bias of $GE(2)$ to $O(n^{-1})$ is

$$Bias_n[GE(2)] = [GE(2)^{3/2} / n][6GE(2)^{1/2} - 2^{3/2}\gamma_1] .$$

These estimated biases are dramatically different in magnitude, and often in sign, from our own results. This underscores the potential danger of using a 'finite sample' approximation, based on an asymptotic expansion that is valid only as the sample size grows, when in fact the sample size is fixed at a very small value. In this case, our 'small-σ' approximation may be especially useful.

The magnitudes of the biases in these examples may seem surprising. However, by way of an example, Deltas (2003) finds comparable biases for the Gini coefficient in similar circumstances. His study involves 101 early twentieth century shipping cartels, each comprising between 2 and 24 firms, with an average of 6.2 firms per cartel. For these data the raw Gini coefficients for the distribution of firm size within the cartels are all biased downwards, on average by 40.1%, and by degrees ranging from 4.3% to 100%. Indeed, 20% of the biases are effectively 100% in magnitude.[9] In addition, Breunig (2001, p.18) reports that his 'large-*n*' bias corrections result in a change of over 5% for the squared coefficient of variation measure in a sample of 2,400 Kenyan households.

5. Conclusions

The general entropy family provides a rich and appealing group of measures that are suitable for measuring inequality in income or industrial concentration. When these measures are calculated in practice, they yield estimates of their underlying population counterparts, and this naturally raises the question of their quality in this respect. In this paper we focus on the bias of such measures, especially when the sample size is very small. This situation is especially pertinent in the context of evaluating industrial concentration, for example. In this case, approximations to the bias that rely on infinitely large, or increasingly large, sample sizes are of little use.

We provide analytic expressions for the bias of members of the generalized entropy family, based on an analytic expansion whose accuracy relies on increasingly small sampling error, rather than increasingly large sample size. We also show how these results can be used to construct (approximately) bias-adjusted inequality measures, and we illustrate their application with industrial concentration data for Canada and the U.S.A.. These empirical examples provide a graphic illustration of the extent to which bias corrections based on large sample approximations can be misleading in certain practical situations.

Table 1. Inequality Measures and Biases: Assets of Canadian Charter Banks

Year*	Raw Measures**			Bias-Corrected Measures		
	$GE(0)$	$GE(1)$	$GE(2)$***	$G\hat{E}(0)$	$G\hat{E}(1)$	$G\hat{E}(2)$
2001	0.00843	0.00853	0.00869	0.00195	0.00180	0.00183
	(76.90)	(78.50)	(78.02) [1.04]			
2002	0.00972	0.01012	0.01061	0.00244	0.00545	0.00239
	(74.88)	(76.34)	(75.13) [1.28]			
2003	0.01363	0.01451	0.01555	0.00375	0.01018	0.00369
	(72.48)	(74.55)	(72.84) [1.87]			

* As at 31 October.

** Percentage biases appear in parentheses.

*** Percentage biases based on Breunig's (2001) 'large-*n*' approximation appear in square brackets.

Table 2. Inequality Measures and Biases: Capitalized Values in U.S. Industries

Industry*	Raw Measures**			Bias-Corrected Measures		
	$GE(0)$	$GE(1)$	$GE(2)$***	$\hat{GE}(0)$	$\hat{GE}(1)$	$\hat{GE}(2)$
AM	0.23872	0.24076	0.27632	0.16381	0.01932	0.03376
	(31.38)	(85.98)	(74.44) [-1.65]			
CE	0.05430	0.04622	0.04096	0.01460	0.00381	0.00587
	(73.11)	(87.30)	(92.26) [14.71]			
CH	0.17926	0.17100	0.17378	0.12340	0.09328	0.03387
	(31.16)	(80.19)	(82.24) [11.56]			
DM	0.06338	0.05749	0.05430	0.02086	0.01777	0.00950
	(67.08)	(83.48)	(86.13) [9.09]			
EU	0.01644	0.01570	0.01511	0.00368	0.00570	0.00237
	(77.61)	(84.97)	(84.20) [5.34]			
MCB	0.07263	0.07132	0.07279	0.02902	0.04177	0.01489
	(60.03)	(79.12)	(78.97) [1.68]			
MP	0.08533	0.08857	0.09621	0.03864	0.07028	0.02090
	(54.72)	(76.41)	(73.45) [-4.85]			
OG	0.08762	0.09026	0.10099	0.04095	0.07028	0.02031
	(53.27)	(77.50)	(72.72) [-5.55]			
SD	0.55431	0.43084	0.41461	0.63033	0.13052	0.06538
	(-13.71)	(84.82)	(92.36) [34.64]			
SP	0.02256	0.02324	0.02420	0.00645	0.01601	0.00534
	(71.41)	(76.98)	(76.12) [-3.47]			
TS	0.19120	0.16738	0.16001	0.11369	0.05752	0.02835
	(40.54)	(83.06)	(86.91) [15.27]			

* AM = Automobile Manufacturers; CE = Communication Equipment; CH = Chemicals (Major Diversified); DM = Drug Manufacturers; EU = Electric Utilities; MCB = Money Center Banks; MP = Meat Products; OG = Oil & Gas (Major Integrated); SD = Soft Drinks; SP = Shipping; TS = Telecom Services (Domestic).

** Percentage biases appear in parentheses.

*** Percentage biases based on Breunig's (2001) 'large-*n*' approximation appear in square brackets.

Appendix: Proof of Theorem 1

Let $h(y) = GE(\alpha) = \left[\frac{1}{n}\bar{y}^{-\alpha}\sum_{i=1}^{n} y_i^{\alpha} - 1\right] / \left[\alpha(\alpha-1)\right]$, and let $h^{(m)}(y) = \frac{\partial^m h(y)}{\partial y_k^m}$.

Some tedious but straightforward partial differentiation yields the following results:

$$h^{(1)}(y) = \left[\bar{y}^{-\alpha} y_k^{\alpha-1} - \frac{1}{n}\bar{y}^{-(\alpha+1)}\sum_{i=1}^{n} y_i^{\alpha}\right] / \left[n(\alpha-1)\right] \tag{A.1}$$

$$h^{(2)}(y) = \left[(\alpha-1)\bar{y}^{-\alpha} y_k^{\alpha-2} - \frac{2\alpha}{n}\bar{y}^{-(\alpha+1)} y_k^{\alpha-1} + \frac{\alpha+1}{n^2}\bar{y}^{-(\alpha+2)}\sum_{i=1}^{n} y_i^{\alpha}\right] / \left[n(\alpha-1)\right] \tag{A.2}$$

$$\begin{aligned} h^{(3)}(y) = [&(\alpha-1)(\alpha-2)\bar{y}^{-\alpha} y_k^{\alpha-3} - \frac{3\alpha(\alpha-1)}{n}\bar{y}^{-(\alpha+1)} y_k^{\alpha-2} + \frac{3\alpha(\alpha+1)}{n^2}\bar{y}^{-(\alpha+2)} y_k^{\alpha-1} \\ &- \frac{(\alpha+1)(\alpha+2)}{n^3}\bar{y}^{-(\alpha+3)}\sum_{i=1}^{n} y_i^{\alpha}] / \left[n(\alpha-1)\right] \end{aligned} \tag{A.3}$$

$$\begin{aligned} h^{(4)}(y) = [&(\alpha-1)(\alpha-2)(\alpha-3)\bar{y}^{-\alpha} y_k^{\alpha-4} - \frac{4\alpha(\alpha-1)(\alpha-2)}{n}\bar{y}^{-(\alpha+1)} y_k^{\alpha-3} \\ &+ \frac{6\alpha(\alpha-1)(\alpha+1)}{n^2}\bar{y}^{-(\alpha+2)} y_k^{\alpha-2} - \frac{4\alpha(\alpha+1)(\alpha+2)}{n^3}\bar{y}^{-(\alpha+3)} y_k^{\alpha-1} \\ &+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{n^4}\bar{y}^{-(\alpha+4)}\sum_{i=1}^{n} y_i^{\alpha}] / \left[n(\alpha-1)\right]. \end{aligned} \tag{A.4}$$

Also, define $h_j^{(2)}(y) = \frac{\partial^3 h(y)}{\partial y_k^2 \partial y_j} = \frac{\partial h^{(2)}(y)}{\partial y_j}$, and $h_{jj}^{(2)}(y) = \frac{\partial^4 h(y)}{\partial y_k^2 \partial y_j^2} = \frac{\partial h_j^{(3)}(y)}{\partial y_j}$, and note that for the case where $j = k$, we have $h_j^{(2)}(y) = h^{(3)}(y)$ and $h_{jj}^{(2)}(y) = h^{(4)}(y)$. Now, for $j \neq k$, we have:

$$\begin{aligned} h_j^{(2)}(y) = [&-\alpha(\alpha-1)\bar{y}^{-(\alpha+1)} y_k^{\alpha-2} + \frac{\alpha(\alpha+1)}{n}\bar{y}^{-(\alpha+2)}(2y_k^{\alpha-1} + y_j^{\alpha-1}) \\ &- \frac{(\alpha+1)(\alpha+2)}{n^2}\bar{y}^{-(\alpha+3)}\sum_{i=1}^{n} y_i^{\alpha}] / [n^2(\alpha+1)] \end{aligned} \tag{A.5}$$

$$\begin{aligned} h_{jj}^{(2)}(y) = [&\alpha(\alpha-1)\bar{y}^{-(\alpha+2)}(y_k^{\alpha-2} + y_j^{\alpha-2}) - \frac{2\alpha(\alpha+2)}{n}\bar{y}^{-(\alpha+3)}(y_k^{\alpha-1} + y_j^{\alpha-1}) \\ &+ \frac{(\alpha+2)(\alpha+3)}{n^2}\bar{y}^{-(\alpha+4)}\sum_{i=1}^{n} y_i^{\alpha}][(\alpha+1)/(n^3(\alpha-1))] \end{aligned} \tag{A.6}$$

From (A.2) – (A.4),

$$\Delta_2 = \frac{1}{2!}\sum_{k=1}^{n}\left[h^{(2)}(y)\right]_{y=\mu}$$

$$= \frac{1}{2}\sum_{k=1}^{n}\frac{1}{n(\alpha-1)}[(\alpha-1)\mu^{-\alpha}\mu^{\alpha-2} - \frac{2\alpha}{n}\mu^{-(\alpha+1)}\mu^{\alpha-1} + \frac{(\alpha+1)}{n^2}\mu^{-(\alpha+2)}n\mu^{\alpha}]$$

$$= \left(\frac{n}{2}\right)\left(\frac{(\alpha-1)\mu^{-2} - 2\alpha n^{-1}\mu^{-2} + (\alpha+1)n^{-1}\mu^{-2}}{n(\alpha-1)}\right)$$

$$= \frac{(n-1)}{2n\mu^2} \tag{A.7}$$

$$\Delta_3 = \frac{1}{3!}\sum_{k=1}^{n}\left[h^{(3)}(y)\right]_{y=\mu}$$

$$= \frac{1}{6}\sum_{k=1}^{n}\frac{1}{n(\alpha-1)}[(\alpha-1)(\alpha-2)\mu^{-\alpha}\mu^{\alpha-3} - \frac{3\alpha(\alpha-1)}{n}\mu^{-(\alpha+1)}\mu^{\alpha-2}$$

$$+ \frac{3\alpha(\alpha+1)}{n^2}\mu^{-(\alpha+2)}\mu^{\alpha-1} - \frac{(\alpha+1)(\alpha+2)}{n^3}\mu^{-(\alpha+3)}n\mu^{\alpha}]$$

$$= \frac{(n-1)[\alpha(n-2) - 2(n+1)]}{6n^2\mu^3} \tag{A.8}$$

$$\Delta_4 = \frac{1}{4!}\sum_{k=1}^{n}\left[h^{(4)}(y)\right]_{y=\mu}$$

$$= \frac{1}{24}\sum_{k=1}^{n}\frac{1}{n(\alpha-1)}[(\alpha-1)(\alpha-2)(\alpha-3)\mu^{-\alpha}\mu^{\alpha-4} - \frac{4\alpha(\alpha-1)(\alpha-2)}{n}\mu^{-(\alpha+1)}\mu^{\alpha-3}$$

$$+ \frac{6\alpha(\alpha-1)(\alpha+1)}{n^2}\mu^{-(\alpha+2)}\mu^{\alpha-2} - \frac{4\alpha(\alpha+1)(\alpha+2)}{n^3}\mu^{-(\alpha+3)}\mu^{\alpha-1}$$

$$+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{n^4}\mu^{-(\alpha+4)}n\mu^{\alpha}]$$

$$= \frac{(n-1)[\alpha^2(n^2-3n+3) - \alpha(5n^2-3n-9) + 6(n^2+n+1)]}{24n^3\mu^4} \tag{A.9}$$

Further, distinguishing between the cases where $j = k$ and $j \neq k$, and using (A.6) and the results immediately before (A.5):

$$\Delta_{22} = \frac{1}{4!}\sum_{k=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial^4 h(y)}{\partial y_k^2 \partial y_j^2}\right]_{y=\mu}$$

$$= \Delta_4 + \frac{1}{4!}\sum_{j=1}^{n}\ \sum_{\substack{k=1\\k\neq j}}^{n}[h_{jj}^{(4)}(y)]_{y=\mu}$$

$$= \frac{(n-1)[\alpha^2(n^2-3n+3)-\alpha(5n^2-3n-9)+6(n^2+n+1)]}{24n^3\mu^4} + \frac{(n-1)(\alpha+1)(2\alpha n-3\alpha-6)}{24n^3\mu^4}$$

$$= \frac{(n-1)}{24n^2\mu^4}[\alpha^2(n-1)-5\alpha(n-1)+6(n+1)] \quad . \qquad \text{(A.10)}$$

Substituting (A.7) – (A.10) into equation (4) completes the proof of Theorem 1.

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Footnotes

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1. The results for $GE(0)$ and $GE(1)$ are obtained by applying l'Hôpital's rule.
2. This result involves an approximation to $O(n^{-1})$, where 'n' is the sample size.
3. This result was obtained for non-Normal distributions by Ullah *et al*. (1995).
4. Note that it makes sense to measure inequality only if $n > 1$. These inequalities are preserved in the relatively uninteresting case where $\mu < 0$ if the coefficient of variation is re-defined as $\upsilon = (\sigma / |\mu|)$ to preserve a positive dispersion measure.
5. It should be noted that his result for the case where $\gamma_1 > 0$ does not depend on γ_2 or 'n'.
6. Source: www.etrade.ca
7. Market capitalizations are at the end of trading on the NYSE on 26 July 2005. Source: www.yahoo.com
8. Of course, other measures of concentration are also commonly used. The Herfindahl index is popular in this context, and Hall (2005) provides the exact bias for this measure in the context of count data.
9. These biases are not reported explicitly by Deltas, but have been calculated from the data that he kindly supplied.