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Almost Transferable Utility, Changes in Production Possibilities, and the Nash Bargaining and the Kalai-Smorodinsky Solutions

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Abstract

Consider a two-person economy in which allocative efficiency is independent of distribution but cardinality of the agents' utility functions precludes transferable utility. I show that both agents either benefit or lose with any change of production possibilities under the Nash Bargaining and the Kalai-Smorodinsky solutions.

Keywords: Axiomatic bargaining, resource monotonicity, transferable utility, risk aversion

JEL Classifications: C71, D13, D63

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1 Introduction

An expansion of the utility possibility set (UPS) due to an increase in one of many goods in a two-person economy does not in general increase both agents' utilities under the Nash Bargaining solution (NBS) or the Kalai-Smorodinsky solution (KSS) (Chun and Thomson, 1988). If, however, utility is transferable, both agents benefit with an increase in one of many goods in a two-person economy under NBS or KSS (Bergstrom, 1989).

Bergstrom (1989, p.1147) defines transferable utility (TU) as the requirement that any redistribution of utilities among agents along the utility possibility frontier (UPF) leads to a utility vector that sums up to the same constant as the sum of utilities did before the redistribution. A necessary condition for TU is that allocative efficiency must be independent of the distribution of welfare.[1]

I show that in a two-person economy with an arbitrary number of goods produced, any change in the production possibility set (PPS) impacts both agents' utilities in the same direction under the NBS or the KSS, if agents' preferences are such that allocative efficiency is independent of distribution of welfare but agents' utility functions do not yield TU. That means I consider utility functions that share the ordinal characteristics with utility functions that lead to TU. For example, if agents are strictly risk averse the cardinality of their utility functions precludes TU. Cardinality of agents' utility functions also matters in welfare economics because bargaining solutions and the maximization of social welfare functions typically lead to different consumption bundles for individuals depending on which cardinal representation of their ordinal preferences are used.[2]

The next section introduces the model, section 3 presents the main result, and section 4 concludes and discusses applications of my result.

2 The Model

Two agents, Ava ($i = A$) and Bob ($i = B$), produce $l \geq 2$ goods to consume. The PPS is given by Y, where $Y \in \mathbb{R}^l_+$ is a convex set.

The consumption bundle of agent i is denoted by x_i. The set of feasible consumption bundles of the two agents is given by X. It contains at least one private good and may contain public goods and it is constrained by Y. Ava's utility function is given by $u_A(x_A)$, and Bob's utility function is given by

[1] Preferences must allow the indirect utility representation of the Gorman Polar Form in an economy with only private goods (Bergstrom and Varian, 1985) and a form dual to the Gorman Polar form in an economy with public and private goods (Bergstrom and Cornes, 1981, 1983).

[2] In the context of bargaining solutions, Nash (1950) explicitly states that agents have von Neumann-Morgenstern utility functions. In Rubinstein et al. (1992)'s restatement of the Nash Bargaining problem in terms of preferences, choice under uncertainty plays an essential role. However, Kalai and Smorodinsky (1975) do not mention that utility functions are of the von Neumann-Morgenstern type, and neither do textbooks when they introduce the Nash bargaining problem (e.g., Moulin, 1988; MasColell et al., 1995).

$u_B(x_B)$. Agents' utility functions are assumed to be continuous, concave and twice differentiable.

Definition 1 *Let function $f_i(.)$ be increasing, twice differentiable and convex. There is* almost transferable utility *if agents' preferences are such that the utility possibility set is given by the convex set*

$$U = \{(u_A, u_B) \in \mathbb{R}^2 : f_A(u_A) + f_B(u_B) \leq \lambda\}$$

for all Y and where λ is a constant.

In what follows I assume that the assumption of almost TU is satisfied. Note that for any Y, resource and technological constraints only play a role in the size of λ, but $f_A(u_A) + f_B(u_B)$ are independent of these constraints.[3]

Definition 2 *Bargaining solutions. Let $d = (d_A, d_B)$ be the threatpoint in the bargaining problem, where $d \in U$, and $f_A(d_A) + f_B(d_B) < \lambda$. (i) The Nash bargaining solution is the unique utility vector u_N that maximizes $(u_A - d_A)(u_B - d_B)$ s.t. $f_A(u_A) + f_B(u_B) = \lambda$. (ii) Let $\bar{u}_A$ ($\bar{u}_B$) be the highest utility Ava (Bob) receives if Bob (Ava) receives d_B (d_A). The Kalai-Smorodinsky solution is the unique utility vector u_K that equalizes relative utility gains: $(u_A - d_A)/(\bar{u}_A - d_A) = (u_B - d_B)/(\bar{u}_B - d_B)$ and $f_A(u_A) + f_B(u_B) = \lambda$.*

3 Results

Let $v_i(x_i) = f_i(u_i(x_i))$, that is, $v_i(x_i)$ is a monotonic transformation of $u_i(x_i)$.

Allocative efficiency is given by $max_{x_A,x_B} v_A(x_A) + v_B(x_B)$ s.t. $(x_A, x_B) \in X(Y)$ (Bergstrom and Cornes, 1983 and Bergstrom and Varian, 1985).[4]

Lemma (UPS expansion and contraction): Given almost TU, any change in production possibilities of the economy results in an expansion or contraction of the UPS.

Proof. With almost TU, a change in the production possibility set Y and therefore a change in X can only impact λ. Since $\lambda = max_{x_A,x_B} v_A(x_A) + v_B(x_B)$ s.t. $(x_A, x_B) \in X(Y)$, λ increases or decreases with a change in Y. Hence the old UPS is either entirely contained in the new UPS or the new UPS is entirely contained in the old UPS. ■

Proposition Under the NBS or KSS, any change in production possibilities either benefits both agents or makes both agents worse off if and only if agents' utility functions yield almost TU.

[3] For example preferences yielding a utility possibility frontier given by $U = \{(u_A, u_B) \in \mathbb{R}^2 : \lambda_B u_A + \lambda_A u_B = \lambda_A \lambda_B\}$, where λ_i is equal to the highest utility of agent i if she receives all the resources available, violate the assumption, even if for a particular Y, $\lambda_A = \lambda_B$.

[4] Since allocative efficiency is a purely ordinal concept, maximizing $v_A(x_A) + v_B(x_B)$ subject to technological and resource constraints leads to an efficient allocation.

Proof. For sufficiency, Chun and Thomson (1988) show that if agents have concave utility functions over one good only, and this good's supply increases, both agents benefit under NBS and KSS. Express $u_i = f_i^{-1}(v_i)$, where $f_i^{-1}(.)$ is a concave function. With this notation the NBS is found by $\max_{v_a,v_B}(f_A^{-1}(v_A)-d_A)(f_B^{-1}(v_B)-d_B)$ s.t. $v_A+v_B=\lambda$, and the KSS by solving for v_A and v_B given $\frac{f_A^{-1}(v_A)-d_A}{f_A^{-1}(\lambda-f_B(d_B))-d_A} = \frac{f_B^{-1}(v_B)-d_B}{f_B^{-1}(\lambda-f_A(d_A))-d_B}$ and $v_A+v_B=\lambda$. Presented in this way a change in λ plays the same role as a change in the only good provided in the one-good economy of Chun and Thomson (1988) and thus their proof applies. For completeness the proof is available on my webpage using the same notation as in this note and accounting for $d \neq (0,0)$.[5]

For necessity, note first that if the UPS changes such that the old and the new UPF intersect, both NBS and KSS can lead to one agent being better off and the other agent being worse off. Figure 1 illustrates this case.[6] Second, I show that such a change in the UPS can be the result of a change in the PPS even if agents have identical utility functions, but allocative efficiency is not independent of distribution. Suppose Ava and Bob have well-behaved preferences over two private goods and the Inada conditions hold. Moreover, $u(0,x_{i2}) = u(x_{i1},0) = 0$ for all $x_{i1}, x_{i2} \geq 0$. In order to find the intercepts of the UPF, Ava's indifference curve must be tangent to the production possibility frontier (PPF). See Figure 2, point A. If both agents have equal utility on the UPF, both agents must receive the same amount of both goods. Figure 2 shows that point A is not efficient if both agents receive equal utility, because Ava's marginal rate of substitution (MRS) when she receives half of the total amount of each good is not the same as the marginal rate of transformation (MRT) at point A. Thus changing the product mix by moving down along the PPF leads to another efficient allocation associated with the point on the UPF at which both agents have equal utility. In order to make the old and the new UPF intersect, now change the PPF so that point A is on the new PPF and has the same slope at point A as Ava's indifference curve has if she receives half of the amounts of both goods. Point A on the new PPF is now the efficient allocation when Ava's and Bob's utility are equal. This implies that the point on the new UPF at which both agents enjoy equal utility lies inside the old UPS. This also means that the new PPF cuts the old PPF from above in point A and would therefore yield a new UPF with higher intercepts than the old PPF just like the two UPFs drawn in Figure 1. ■

3.1 Conclusion

I focus on a class of agents' preferences such that allocative efficiency is independent of distribution, a necessary condition for TU, but allow for agents to be

[5] The result extends to n individuals under the class of bargaining solutions given by $\max_{u_1,\ldots,u_n} \sum_{i=1}^{n} g_i(u_i-d_i)$ s.t. $\sum_{i=1}^{n} f_i(u_i)=\lambda$, where g_i is a concave function (Chun and Thomson, 1988). NBS is a special case of this class.

[6] Figure 1 is drawn using KSS, but NBS leads to a qualitatively similar graph.

strictly risk averse or to have cardinal utility functions that preclude TU. Given this class of utility functions, for any change in the PPS the NBS and KSS will either increase or decrease both agents' utilities.

The Nash Bargaining solution has become increasingly important in the analysis of impacts of policy changes on intrafamily distribution. The results presented here establish under which conditions policy changes that lead to a change in production possibilities of a family change each family member's welfare in the same direction provided the threatpoint remains the same.

The results are also useful for economists who combine elements of non-cooperative behavior and cooperative bargaining in their models. For example, suppose agents have agreed to apply a bargaining solution to distribute goods once they are produced by agents' actions in the economy, under what conditions do agents have an incentive to produce an efficient amount of goods? I show under what conditions on agents' utility functions any change in production possibilities can only shrink or expand the UPS. Hence an expansion of the UPS is in everybody's best interest and agents' actions, although chosen non-cooperatively, will lead to Pareto efficiency.[7]

4 Figures

[7] Bergstrom (1989) is concerned with a similar question in the context of Gary Becker's "Rotten Kid theorem." His results rely on TU.

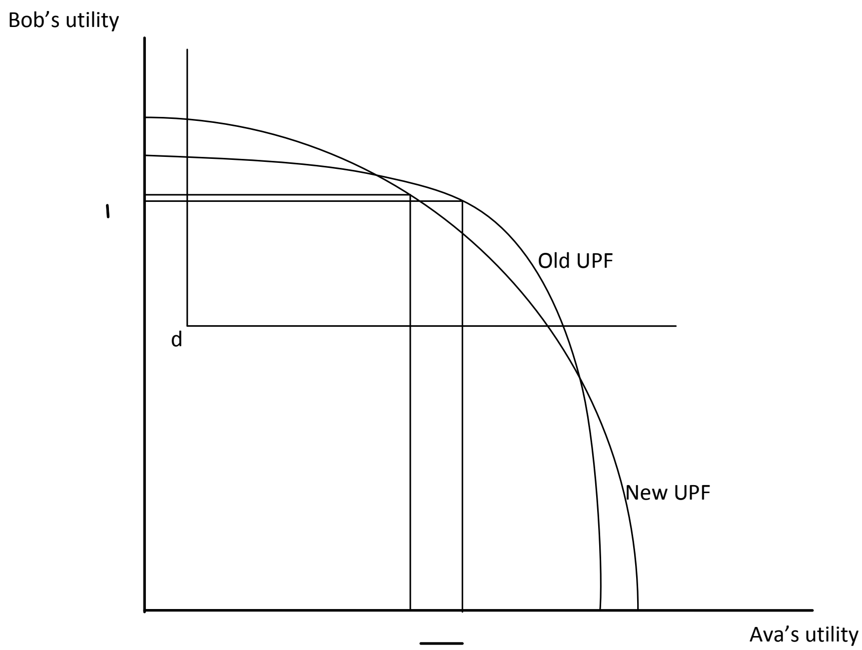

Figure 1: Change in UPS makes Bob better off and Ava worse off.

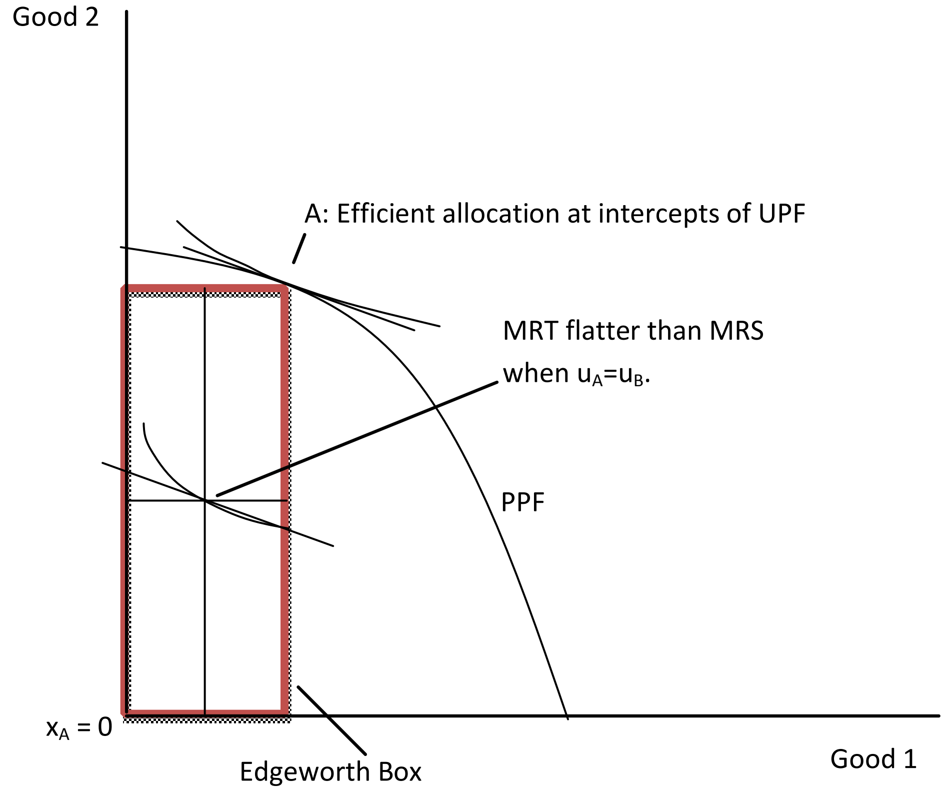

Figure 2

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