

**Profit-Dependent Risk Aversion in a Market-Fraction Model:
Evidence from the DAX Index**

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A handwritten signature in black ink, appearing to read 'Tao Wang', written in a cursive style.

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Abstract

This paper develops a profit-dependent risk aversion (PDRA) market fraction model by allowing the risk aversion of trend followers to depend on their realized trading profits. When trend followers make profits, their risk aversion increases and their demand for the risky asset falls; when they make losses, their risk aversion decreases and their demand rises. This creates an additional feedback between past profits, risk aversion, and market demand. In addition to a symmetric PDRA specification, the paper considers an asymmetric extension in which the sensitivity of risk aversion to realized gains and losses is allowed to differ. The constant-risk-aversion benchmark, PDRA, and asymmetric PDRA models are calibrated to daily DAX 30 returns from December 1987 to July 2026 using differential evolution to match 180 autocorrelation moments of raw returns, squared returns, and absolute returns. The empirical results show that the PDRA and asymmetric PDRA models generally have a greater ability to reproduce the empirical features of the DAX than the benchmark model, while the asymmetric model shows some further improvement over the symmetric model. Both PDRA specifications provide a closer match to GARCH volatility persistence and FIGARCH long memory in conditional variance. Evidence from the broader long-memory tests is more mixed; the benchmark model performs better for raw returns, whereas the PDRA models perform better for squared and absolute returns in the Wald tests. Both PDRA models better reproduce the volatility asymmetry in the DAX, but the asymmetric model does not show a clear advantage over the symmetric model. The PDRA models also show a greater ability to reproduce the empirical tails under the moment estimator, particularly the asymmetric PDRA model. The observable validation provides stronger support for the asymmetric specification; both PDRA models reproduce the within-trend volatility patterns more closely than the benchmark model, while the asymmetric PDRA model provides the closest overall match and performs best across most robustness checks.

Keywords: Heterogeneous agent models; Long-range dependence; Power-law behavior; Time-varying risk aversion; Volatility clustering

JEL Classification: C15, D84, G12

Contents

1	Introduction	1
2	Related Studies	4
3	Profit-Dependent Risk Aversion Model	6
3.1	Market Environment	7
3.2	Fundamentalists	9
3.3	Trend Followers	10
3.4	Strategy Switching	12
3.5	Price Formation	13
4	Theoretical Properties	14
4.1	Existence of the Fundamental Equilibrium	14
4.2	Local Stability	15
5	Data and Calibration	16
5.1	Data and Stylized Facts	16
5.2	Calibration Procedure	19
6	Empirical Results	21
6.1	Calibrated Parameters	21
6.2	Conditional Volatility Persistence: GARCH Estimates	23
6.3	Long Memory in Conditional Volatility: FIGARCH Estimates	24
6.4	Long-Memory Properties: Wald Tests	25
6.5	Tail Behavior	28
6.6	Volatility Asymmetry Diagnostics	30
6.7	Sensitivity Analysis	32
6.8	Observable Validation	33
6.8.1	Observable Market-State Evidence	34
6.8.2	Cross-Model Comparison of Observable Within-Trend Indicators	36
6.8.3	Robustness across Alternative Empirical Targets	37
7	Conclusion	39

1 Introduction

Financial markets exhibit a number of empirical regularities that are difficult to explain using the standard asset pricing paradigm based on the efficient market hypothesis and rational expectations (Fama, 1970; Cont, 2001). In representative-agent models, investors are assumed to process information identically and therefore form a single market expectation (Lucas, 1978). Stock returns display heavy-tailed distributions, volatility clustering, weak autocorrelations of raw returns, and slowly decaying autocorrelations of squared and absolute returns. The slow decay of these autocorrelations is commonly referred to as long-range dependence, or power-law behavior (Ding et al., 1993; Cont, 2001). These stylized facts suggest that changes in fundamentals alone cannot fully account for observed asset price dynamics. Instead, interactions among heterogeneous investors with different beliefs and trading strategies play an important role in driving asset prices (Day and Huang, 1990; Chiarella, 1992; Brock and Hommes, 1998; Lux and Marchesi, 1999; Hommes, 2006; LeBaron, 2006). Understanding how these interactions generate realistic market dynamics has therefore become one of the central questions in financial economics.

Heterogeneous agent models (HAMs) provide a natural framework to study these issues. Unlike representative-agent models, HAMs allow different groups of investors to have heterogeneous expectations and to adjust their trading strategies over time (Brock and Hommes, 1998; Lux and Marchesi, 1999). The interaction between fundamentalists and trend followers, together with evolutionary strategy selection, can generate many of the stylized facts observed in financial markets (Brock and Hommes, 1998; Lux and Marchesi, 1999; He and Li, 2007). As a result, HAMs have become one of the most successful behavioral approaches to asset pricing. Although heterogeneous agent models have been extended in many directions, the treatment of investors' risk preferences has changed relatively little. Many heterogeneous agent models continue to assume that investors' risk aversion remains constant over time (Brock and Hommes, 1998; Lux and Marchesi, 1999; He and Li, 2015, 2017). Treating risk aversion as fixed simplifies the analytical structure of the model, but it ignores the possibility that investors' willingness to bear risk changes with their past trading performance. Because risk aversion plays a central role in determining investors' demand for risky assets, changes in risk preferences may create an additional feedback between past trading performance, risk attitudes, and asset demand, thereby affecting market dynamics.

Risk aversion has long been recognized as one of the central behavioral mechanisms in the broader asset-pricing literature (Lucas, 1978; Campbell and Cochrane, 1999). Prospect theory implies that risky choices are reference dependent and that gains and losses can affect decision making asymmetrically (Kahneman and Tversky, 1979; Tversky and Kahneman,

1992). In dynamic settings, prior gains and losses can in turn affect subsequent risk taking (Thaler and Johnson, 1990; Coval and Shumway, 2005; Imas, 2016). Recent empirical evidence also indicates that aggregate risk aversion varies over time and contains information about subsequent stock returns (Guiso et al., 2018; Guidolin et al., 2025). Building on this idea, behavioral asset-pricing models have shown that endogenous changes in risk preferences can significantly influence asset prices and market dynamics (Barberis et al., 2001; Bekaert et al., 2022). Despite its importance in the broader behavioral finance literature, endogenous risk aversion has received much less attention in heterogeneous agent models. One notable exception is Park (2014), who introduces endogenous time-varying risk aversion into the adaptive beliefs system of Brock and Hommes (1998). However, Park (2014) primarily studies the nonlinear dynamics of the adaptive beliefs system, including bifurcation behavior and volatility dynamics, and evaluates the proposed mechanism through theoretical analysis and numerical simulations. The study shows that the endogenous time-varying risk-aversion mechanism generates richer nonlinear dynamics and improves several statistical properties of simulated returns. However, an important empirical question remains: does allowing risk aversion to respond to realized trading profits actually improve the empirical performance of a heterogeneous-agent model relative to a constant-risk-aversion benchmark? And if so, does allowing risk aversion to respond differently to gains and losses provide any additional improvement?

By contrast, the empirically calibrated market fraction framework developed by He and Li (2017) provides a quantitative framework for evaluating alternative behavioral mechanisms. Through calibration to financial market data while preserving analytical tractability, it enables the empirical contribution of a behavioral mechanism to be assessed against a common benchmark. This provides a suitable setting for examining whether the endogenous profit-dependent risk aversion mechanism can improve the empirical performance of a heterogeneous agent model while preserving its ability to reproduce the stylized facts of financial returns. Accordingly, this paper examines whether introducing an endogenous profit-dependent risk aversion mechanism into the empirically calibrated market fraction model improves its ability to reproduce the statistical properties of observed financial returns, and whether allowing risk aversion to respond asymmetrically to gains and losses provides any further improvement. The proposed models are calibrated to DAX 30 daily returns and evaluated against the original market fraction model with constant risk aversion under the same empirical calibration framework. This allows the empirical contribution of profit-dependent risk aversion to be assessed through a direct comparison with the benchmark model, and the role of asymmetric responses to gains and losses through a direct comparison between the two PDRA specifications.

Motivated by the extension of [Park \(2014\)](#), this paper uses the empirically calibrated framework of [He and Li \(2017\)](#) to evaluate the contribution of profit-dependent risk aversion and gain-loss asymmetry. Specifically, this paper develops a market fraction model in which the risk aversion of trend followers evolves with their realized trading profits. It introduces an additional behavioral feedback between trading performance, risk preferences, asset demand, and price dynamics. In addition to the symmetric PDRA specification, the paper considers an asymmetric extension in which realized gains and losses are allowed to affect trend followers' risk aversion with different sensitivities. The symmetric PDRA model is obtained as the restricted case in which these sensitivities are equal. The paper also derives the fundamental equilibrium and analyzes the local stability of the deterministic system, showing that the proposed behavioral mechanism can be incorporated without sacrificing the analytical tractability of the original model. The constant-risk-aversion benchmark, PDRA, and asymmetric PDRA models are calibrated using the same DAX 30 data and the same calibration framework. The results show that introducing profit-dependent risk aversion improves the empirical performance of the model relative to the original benchmark market fraction model of [He and Li \(2017\)](#), which assumes constant risk aversion, and the asymmetric specification further improves the fit in some dimensions. Looking across the individual diagnostics, both PDRA specifications more closely reproduce the GARCH volatility persistence and FIGARCH long memory observed in the DAX 30 than the benchmark model. The Wald tests provide a more mixed picture; the benchmark model performs better for raw returns, whereas the PDRA specifications perform better for squared and absolute returns. The tail-index results similarly favor the PDRA specifications under the moment estimator, with the asymmetric PDRA model showing the closest overall agreement with the empirical return tails. Volatility-asymmetry diagnostics also show that both PDRA models outperform the benchmark, although the additional improvement from allowing different responses to gains and losses is modest and not robust across diagnostics. Stronger support for the asymmetric specification comes from the observable validation results. Both PDRA models reproduce the within-trend volatility patterns more closely than the benchmark, while the asymmetric PDRA model generally performs best across alternative target definitions and robustness checks.

The remainder of the paper is organized as follows. Section 2 reviews related studies. Section 3 presents the profit-dependent risk-aversion model and its symmetric and asymmetric specifications. Section 4 provides theoretical analysis of the fundamental equilibrium and local stability. Section 5 describes the data and calibration procedure. Section 6 reports the empirical results and compares the benchmark, PDRA, and asymmetric PDRA models. Section 7 concludes the paper. All technical proofs are provided in the appendix.

2 Related Studies

Instead of assuming a representative investor, HAMs allow investors to hold heterogeneous expectations and to adapt their trading behavior over time. Most models distinguish between two groups of traders: fundamentalists, who believe that prices will revert towards their fundamental values, and trend followers, who extrapolate recent price movements. Asset prices are determined by the interaction between these competing beliefs through market clearing, while investors may switch between trading strategies according to their relative performance. These interactions create nonlinear feedback between expectations, strategy selection, excess demand, and price dynamics (Brock and Hommes, 1998; Lux and Marchesi, 1999; Hommes, 2006; He and Li, 2008; He et al., 2019). Over the past three decades, heterogeneous agent models have therefore evolved from a theoretical alternative to the representative-agent paradigm into one of the main frameworks for studying nonlinear financial market dynamics.

As heterogeneous agent models became increasingly successful in reproducing stylized facts, the focus of the literature gradually shifted. Early studies mainly asked whether heterogeneous interactions could generate realistic market behavior (Day and Huang, 1990; Chiarella, 1992; Brock and Hommes, 1998; Lux and Marchesi, 1999). Once this had been established, subsequent research focused on improving the behavioral realism and empirical performance of these models. Recent studies have developed existing heterogeneous agent models by introducing richer forms of investor heterogeneity, more detailed trading environments, greater emphasis on empirical evaluation, and additional behavioral mechanisms (Chiarella and Iori, 2002; Chiarella et al., 2009; Franke and Westerhoff, 2012; He and Li, 2015, 2017; Dieci and He, 2018; Schmitt, 2021). In general, these developments can be grouped into four complementary directions.

One direction enriches investor heterogeneity itself. Earlier heterogeneous agent models usually distinguished only two representative trader types (Brock and Hommes, 1998; Lux and Marchesi, 1999; Hommes, 2006), whereas more recent studies allow investors to differ continuously in their beliefs, information, wealth, or forecasting rules (Dieci and He, 2018; Zhao and Zhang, 2022). These extensions allow heterogeneous agent models to better reflect the diversity of real-world investors without changing their basic modelling framework (Dieci and He, 2018; Cavalli et al., 2021; Zhao and Zhang, 2022). Another direction focuses on improving the market environment in which heterogeneous investors interact. Earlier heterogeneous agent models typically assume a single risky asset traded through a simple market-maker mechanism. More recent studies extend these models to multi-asset settings and incorporate more realistic trading environments, including matching-based pricing mechanisms, market frictions, and limit-order trading (Chiarella and Iori, 2002; Chiarella et al., 2009; Dieci and

He, 2018). Such developments incorporate important features of market microstructure into heterogeneous agent models and allow these models to capture the effects of institutional features on market dynamics (Chiarella and Iori, 2002; Chiarella et al., 2009; Dieci and He, 2018; Fu et al., 2024).

A further direction places increasing emphasis on empirical validation and quantitative evaluation. While early heterogeneous agent models were mainly evaluated by their ability to reproduce stylized facts resembling financial markets, more recent studies increasingly require behavioral models to match observed data quantitatively. Calibration, simulation-based estimation, moment matching, and formal statistical testing have therefore become important tools for evaluating heterogeneous agent models (Gilli and Winker, 2003; Franke and Westerhoff, 2012; Dieci and He, 2018; Kim et al., 2021). This development reflects a broader shift in the literature from demonstrating that heterogeneous interactions can generate stylized facts to evaluating alternative behavioral mechanisms using empirical evidence. Since different behavioral assumptions may generate similar empirical patterns, empirical validation has therefore become essential for distinguishing between competing behavioral mechanisms. Among studies following this direction, the market fraction model developed by He and Li (2015, 2017) is particularly relevant to the present study. Instead of relying solely on qualitative simulations, they calibrate the model directly to DAX 30 data by matching the autocorrelation structure of returns, squared returns, and absolute returns. Their results show that the model successfully reproduces many important stylized facts of financial returns, especially the long-memory properties of return volatility. In addition, the model remains analytically tractable, so its deterministic dynamics, fundamental equilibrium, and local stability conditions can be studied explicitly. These features make it a useful benchmark for examining how alternative behavioral assumptions affect the model’s empirical performance.

A final direction enriches the behavioral foundations of heterogeneous agent models. Recent studies increasingly recognize that realistic market dynamics depend not only on heterogeneous expectations and trading strategies, but also on the psychological mechanisms underlying investors’ decision-making. Behavioral extensions have therefore incorporated adaptive learning, market sentiment, bounded rationality, overconfidence, ambiguity, social interactions, and other behavioral biases to provide a more realistic description of investor behavior (Hommes, 2013; Cavalli et al., 2021). These developments have improved the ability of heterogeneous agent models to explain persistent market fluctuations and changing market conditions. Compared with other behavioral extensions, endogenous profit-dependent risk aversion has received remarkably little attention in the heterogeneous-agent literature. One notable exception is Park (2014), who introduces endogenous time-varying risk aversion into the Brock and Hommes (1998) framework, allowing investors’ risk preferences to evolve with

realized trading performance. A related behavioral literature suggests that the effect of past outcomes on subsequent risk attitudes need not be symmetric across gains and losses. Risky choices are reference dependent, so economically comparable gains and losses can have different effects on subsequent behavior (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). In dynamic settings, prior gains and prior losses can generate distinct changes in subsequent risk taking, including house-money and break-even effects (Thaler and Johnson, 1990). This gain-loss asymmetry has also been documented in financial-market settings, where positive and negative investment outcomes can have different effects on subsequent risk taking (Zhang and Semmler, 2009; Coval and Shumway, 2005; Imas, 2016). Related heterogeneous-agent models have similarly incorporated different risk attitudes towards gains and losses as a source of asymmetric price dynamics (Zhang and Li, 2011). These findings motivate the two profit-dependent risk-aversion models considered in this paper, with symmetric and asymmetric responses to gains and losses.

Relative to the existing literature, the paper makes two main contributions. It introduces profit-dependent risk aversion into the market fraction model of He and Li (2015, 2017). In the original framework, past profits enter market dynamics through the strategy-switching mechanism, while risk aversion remains fixed. The proposed extension adds another channel by allowing past trading performance to change trend followers' willingness to hold the risky asset. It also allows this response to differ between gains and losses. The paper also compares the empirical performance of the different risk-aversion specifications in the market fraction model. The constant-risk-aversion benchmark and the two profit-dependent specifications are calibrated to the same DAX 30 data and examined using the same empirical procedure. This allows their performance to be compared across volatility persistence, long memory, tail behavior, volatility asymmetry, and observable within-trend volatility patterns. The results generally favor the two profit-dependent models over the benchmark, with the asymmetric model showing its clearest advantage in the observable within-trend volatility tests.

3 Profit-Dependent Risk Aversion Model

This section presents the profit-dependent risk aversion (PDRA) specification within the market fraction framework of He and Li (2015, 2017). The main contribution is that the risk aversion of trend followers is allowed to vary with their realized trading profits. We first describe the market environment and derive the general asset-demand function under CARA preferences. We then specify the behavior of fundamentalists and trend followers, including the profit-dependent risk-aversion mechanism, before presenting the strategy-switching mechanism and the price-adjustment process.

3.1 Market Environment

We consider a financial market with one risky asset and one risk-free asset. The risky asset pays a stochastic dividend, while the risk-free asset earns a constant gross return R each trading period. The market consists of two types of boundedly rational investors: fundamentalists and trend followers. At each trading period, investors choose their holdings of the risky asset by maximizing expected utility. The two trader types differ in their expectations of future payoffs, perceptions of risk, and attitudes towards risk, leading to different demands for the risky asset.

Let p_t denote the price level of the risky asset at time t , and let D_t denote the dividend paid between $t - 1$ and t . Investors may also allocate their wealth to a risk-free asset, which yields a constant gross return R per trading period. The gross risk-free return per trading period is assumed to be $R = 1 + r/K$, where $r = 0.05$ is the constant annual risk-free interest rate, following [He and Li \(2015, 2017\)](#), and $K = 250$ is the number of trading days per year. The value of r is kept fixed to maintain consistency with the benchmark model. The sensitivity of this assumption is examined in Subsection 6.7 using alternative values of r . This implies a daily gross risk-free return of $R = 1.0002$. Investors choose their portfolio holdings by maximizing a constant absolute risk aversion (CARA) utility function,

$$U_h(W) = -\exp(-a_{h,t}W), \quad h \in \{1, 2\}, \quad (3.1)$$

where W denotes investors' wealth and $a_{h,t} > 0$ is the coefficient of absolute risk aversion for trader type h . A larger value of $a_{h,t}$ indicates that investors are more risk averse and therefore choose a smaller position in the risky asset for a given level of expected return and perceived risk. In the benchmark market fraction model, the coefficient of risk aversion is assumed to be constant for each trader type. In the proposed models, however, the risk aversion of trend followers is allowed to evolve over time according to their realized trading profits, while the risk aversion of fundamentalists remains constant. The specification of $a_{h,t}$ is presented in the following subsections. CARA preferences are widely used in heterogeneous-agent models because they provide a simple representation of risk-averse behavior ([Brock and Hommes, 1998](#); [Lux and Marchesi, 1999](#); [He and Li, 2015, 2017](#)). Combined with the assumption of normally distributed future asset payoffs, they allow the investors' portfolio optimization problem to be solved analytically. The optimal demand for the risky asset can be derived directly.

Let $z_{h,t}$ denote the number of units of the risky asset held by trader type h at time t , where a positive value indicates a long position and a negative value indicates a short position.

Given this portfolio choice, next-period wealth is

$$W_{h,t+1} = RW_{h,t} + z_{h,t}(p_{t+1} + D_{t+1} - Rp_t), \quad (3.2)$$

where the first term is the value of current wealth invested in the risk-free asset, and the second term is the excess payoff from holding the risky asset. Specifically, an investor who purchases one unit of the risky asset at price p_t receives the future price p_{t+1} and the dividend D_{t+1} , instead of earning the risk-free return Rp_t . Therefore, $p_{t+1} + D_{t+1} - Rp_t$ represents the excess payoff from holding one unit of the risky asset over the corresponding investment in the risk-free asset.

To determine investors' optimal portfolio choice, the distribution of future payoffs must be specified. Following [He and Li \(2015, 2017\)](#), the dividend process is assumed to satisfy $D_t \sim \mathcal{N}(\bar{D}, \sigma_D^2)$, where $\bar{D}$ denotes the constant expected dividend, so that $\mathbb{E}_t(D_{t+1}) = \bar{D}$, $\sigma_D^2 = q\sigma_1^2$, and σ_1^2 denotes the baseline conditional variance of future prices over one trading period, which is normalized to one throughout the analysis following the implementation of [He and Li \(2015, 2017\)](#). The expected dividend is related to the fundamental price through $\bar{D} = (R - 1)\bar{p}$, where $\bar{p}$ denotes the fundamental price, also consistent with the specification in [He and Li \(2015, 2017\)](#). The expected dividend is therefore fully determined by the fundamental price and the risk-free return. The dividend variance is assumed to be proportional to this baseline price variance, with the proportionality parameter given by $q = r^2$. This specification links dividend uncertainty to price uncertainty without introducing an additional free parameter into the model. The derivation showing how $\sigma_D^2 = q\sigma_1^2$ implies $q = r^2$ is provided in [Appendix A.1](#). Since the future payoff from holding the risky asset consists of both the future price and the dividend, the assumption of normally distributed dividends, together with the assumption of normally distributed price forecasts, implies that future asset payoffs are conditionally Gaussian. Combined with CARA preferences, this transforms the expected utility maximization problem into a mean-variance optimization problem, allowing the investors' optimal demand for the risky asset to be derived analytically.

Maximizing the expected CARA utility therefore yields the optimal demand for the risky asset,

$$z_{h,t} = \frac{\mathbb{E}_{h,t}(p_{t+1} + D_{t+1}) - Rp_t}{a_{h,t}V_{h,t}(p_{t+1} + D_{t+1})}, \quad (3.3)$$

where $\mathbb{E}_{h,t}(\cdot)$ and $V_{h,t}(\cdot)$ denote trader type h 's conditional expectation and conditional variance of future asset payoffs, respectively. Equation (3.3) shows that investors demand more of the risky asset when its expected excess payoff increases, but reduce their holdings when either perceived risk or risk aversion becomes larger. The optimal demand therefore

depends on three components: investors' expectations of future payoffs, their perceived risk, and their degree of risk aversion. The following subsections specify how these three components are determined for fundamentalists and trend followers.

3.2 Fundamentalists

Fundamentalists believe that asset prices fluctuate around their fundamental value and regard deviations from this value as temporary. They therefore expect market prices to move towards the next-period fundamental value. Following [He and Li \(2015, 2017\)](#), their price expectation is given by

$$\mathbb{E}_{1,t}(p_{t+1}) = p_t + (1 - \alpha)(p_{t+1}^* - p_t), \quad \alpha \in (0, 1), \quad (3.4)$$

where p_{t+1}^* denotes the next-period fundamental value of the asset. Equivalently, $\mathbb{E}_{1,t}(p_{t+1}) - p_t = (1 - \alpha)(p_{t+1}^* - p_t)$. Hence, $(1 - \alpha)$ determines the expected speed of adjustment towards the fundamental value. A larger value of α places more weight on the current market price and therefore implies a slower expected adjustment, whereas a smaller value of α implies a faster adjustment.

Consistent with [He and Li \(2015, 2017\)](#), fundamentalists are assumed to perceive a constant level of risk, reflecting the view that they rely on fundamental information instead of recent price fluctuations when forming their expectations. Their conditional variance of future asset payoffs is therefore given by

$$V_{1,t}(p_{t+1} + D_{t+1}) = (1 + q)\sigma_1^2. \quad (3.5)$$

The variance of future asset payoffs consists of the variance of the future price and the variance of the dividend. Since the future price variance is given by σ_1^2 and the dividend variance satisfies $\sigma_D^2 = q\sigma_1^2$, the baseline variance of future asset payoffs is $(1 + q)\sigma_1^2$. Following [He and Li \(2015, 2017\)](#), the proportionality parameter q is determined by the annual risk-free interest rate, yielding $q = r^2$. The derivation is provided in [Appendix A.1](#). This assumption implies that fundamentalists regard the uncertainty surrounding future asset payoffs as constant over time and therefore do not revise their assessment of risk in response to changing market conditions. The risk aversion of fundamentalists is also assumed to remain constant, $a_{1,t} = a_1$. This reflects the assumption that their willingness to bear risk does not change over time. Changes in their demand for the risky asset are driven solely by changes in expected returns, since both their perceived risk and risk aversion remain constant.

Substituting the fundamentalists' price expectation, the expected dividend, perceived risk,

and risk-aversion coefficient into (3.3) gives

$$z_{1,t} = \frac{p_t + (1 - \alpha)(p_{t+1}^* - p_t) + (R - 1)\bar{p} - Rp_t}{a_1(1 + r^2)\sigma_1^2}.$$

Collecting terms in the numerator yields

$$p_t + (1 - \alpha)(p_{t+1}^* - p_t) + (R - 1)\bar{p} - Rp_t = (1 - \alpha)(p_{t+1}^* - p_t) - (R - 1)(p_t - \bar{p}).$$

Therefore,

$$z_{1,t} = \frac{(1 - \alpha)(p_{t+1}^* - p_t) - (R - 1)(p_t - \bar{p})}{a_1(1 + r^2)\sigma_1^2}. \quad (3.6)$$

The demand function consists of two components. The first term reflects fundamentalists' expectation that the market price will move towards the fundamental value; they increase their demand when $p_t < p_{t+1}^*$ and reduce it when $p_t > p_{t+1}^*$. The second term captures the opportunity cost of holding the risky asset relative to the risk-free asset. When the market price exceeds the long-run fundamental price $\bar{p}$, this term reduces their demand; when the market price is below $\bar{p}$, it increases their demand. Fundamentalists' trading behavior therefore generally tends to move the market price towards its fundamental value.

3.3 Trend Followers

Trend followers form expectations by extrapolating recent price movements. They expect existing price trends to continue in the near future and therefore increase their demand after price increases while reducing their demand after price declines. Their expectation is given by

$$\mathbb{E}_{2,t}(p_{t+1}) = p_t + \gamma(p_t - u_t), \quad \gamma > 0, \quad (3.7)$$

where $u_t = \delta u_{t-1} + (1 - \delta)p_t$ is an exponentially weighted moving average of past prices that serves as the reference level against which trend followers evaluate current price movements. It provides a smoothed estimate of the recent price trend, allowing trend followers to compare the current price with this reference level when forming their expectations. The parameter $\delta \in (0, 1)$ determines how quickly past prices are discounted when forming the moving average. Larger values of δ place greater weight on historical prices, whereas smaller values allow recent prices to have a stronger influence. The parameter γ measures the strength of trend extrapolation. A larger value of γ implies that trend followers respond more aggressively to deviations of the current price from the moving average.

Trend followers perceive risk as varying over time. Their conditional variance is specified

as

$$V_{2,t}(p_{t+1} + D_{t+1}) = (1 + q + bv_t)\sigma_1^2, \quad (3.8)$$

where the first component $(1 + q)\sigma_1^2$ represents the baseline level of uncertainty about future asset payoffs. The second component $bv_t\sigma_1^2$ allows perceived risk to vary with recent price fluctuations. The variable v_t evolves according to

$$v_t = \delta v_{t-1} + \delta(1 - \delta)(p_t - u_{t-1})^2, \quad (3.9)$$

where the first term carries forward information about past market volatility and the second term incorporates the squared deviation of the current price from the previous period's moving average. This specification allows perceived volatility to respond to recent price movements while retaining information from earlier periods. The deviation is squared so that movements above and below the reference level both increase measured volatility, with larger deviations receiving greater weight. Thus, v_t measures recent market volatility by tracking the squared deviation of the current price from its moving average over time. The parameter $\delta \in (0, 1)$ determines the persistence of the volatility measure, while $b > 0$ measures how strongly recent market volatility affects perceived risk. Larger deviations of the current price from its moving average increase v_t , leading trend followers to perceive greater uncertainty about future asset payoffs and therefore reduce their demand for the risky asset.

We allow the risk aversion of trend followers to evolve according to their realized trading profits. Specifically, the most general specification considered in this paper is given by

$$a_{2,t} = \begin{cases} (1 - \omega)a_2 \exp(s_{\text{gain}}\pi_{C,t-1}) + \omega a_{2,t-1}, & \pi_{C,t-1} \geq 0, \\ (1 - \omega)a_2 \exp(s_{\text{loss}}\pi_{C,t-1}) + \omega a_{2,t-1}, & \pi_{C,t-1} < 0, \end{cases} \quad (3.10)$$

where $\pi_{C,t-1} = z_{2,t-1}(p_t + D_t - Rp_{t-1})$ denotes the realized trading profit earned by trend followers between periods $t - 1$ and t . It equals the position held in the previous period multiplied by the realized excess payoff from holding one unit of the risky asset relative to the risk-free asset. The exponential specification ensures that risk aversion remains strictly positive for all realized trading profits while allowing trading performance to affect risk aversion in a nonlinear manner. The parameter $a_2 > 0$ denotes the benchmark level of risk aversion, while $\omega \in [0, 1]$ determines the persistence of risk aversion over time. Larger values of ω place greater weight on past risk aversion, whereas smaller values make current risk aversion more responsive to recent trading performance. The parameters $s_{\text{gain}} \geq 0$ and $s_{\text{loss}} \geq 0$ measure the sensitivities of risk aversion to realized gains and realized losses, respectively. The responses of risk aversion to gains and losses need not be symmetric.

The symmetric PDRA model is obtained by imposing the restriction $s_{\text{gain}} = s_{\text{loss}} = s$, so that the same profit-sensitivity parameter applies to both gains and losses. The resulting risk-aversion dynamics are given by

$$a_{2,t} = (1 - \omega)a_2 \exp(s\pi_{C,t-1}) + \omega a_{2,t-1}. \quad (3.11)$$

Treating the symmetric PDRA model as a restricted version of the general specification makes it possible to compare the two specifications directly and assess the effect of allowing different responses to gains and losses. Furthermore, setting $s = 0$ in (3.11) reduces the model to the constant-risk-aversion specification of He and Li (2017), provided that the initial risk-aversion level equals a_2 . The proposed framework therefore nests both the symmetric PDRA model and the benchmark model, facilitating direct comparisons among all three specifications.

Substituting the trend followers' price expectation in (3.7), perceived variance in (3.8), and risk-aversion specification in (3.10) into (3.3) gives

$$z_{2,t} = \frac{\gamma(p_t - u_t) - (R - 1)(p_t - \bar{p})}{a_{2,t}(1 + r^2 + bv_t)\sigma_1^2}. \quad (3.12)$$

This expression shows that the demand of trend followers is jointly determined by their expected excess returns, perceived risk, and time-varying risk aversion. Realized trading performance affects future demand through its influence on risk aversion.

3.4 Strategy Switching

The market consists of both fixed and switching investors. Fixed investors always follow the same trading strategy, whereas switching investors choose between the fundamentalist and trend-following strategies according to their relative profitability. This evolutionary mechanism reflects the idea that investors do not know in advance which strategy will perform best. Instead, they adapt their choices over time by allocating a larger fraction of switching investors to strategies with higher recent profitability.

Let $n_0 \in [0, 1]$ denote the fraction of investors who never change their trading strategy, and let $m_0 \in [-1, 1]$ describe the composition of this fixed group. A larger value of m_0 implies that a greater proportion of fixed investors are fundamentalists, whereas a smaller value implies that more fixed investors are trend followers. Let $m_t \in [-1, 1]$ denote the relative preference of switching investors for the fundamentalist strategy over the trend-following strategy. It is determined by

$$m_t = \tanh \left[\frac{\beta}{2} (z_{1,t-1} - z_{2,t-1}) (p_t + D_t - Rp_{t-1}) \right], \quad (3.13)$$

where $\beta > 0$ measures the intensity of strategy switching. The expression $(z_{1,t-1} - z_{2,t-1})(p_t + D_t - Rp_{t-1})$ measures the difference in realized trading profits between fundamentalists and trend followers during the previous period. The first term, $(z_{1,t-1} - z_{2,t-1})$, represents the difference in their holdings of the risky asset, while the second term, $(p_t + D_t - Rp_{t-1})$, is the realized excess payoff from holding one unit of the risky asset. Their product therefore gives the difference in realized trading profits between the two strategies. Larger values of β imply that switching investors respond more strongly to differences in profitability, leading to faster adjustments in market composition. This specification reflects the assumption that investors adapt their strategy choices according to recent trading performance, so that strategies with higher realized profits attract a larger proportion of switching investors. The hyperbolic tangent function maps the switching signal into the interval $[-1, 1]$, ensuring that m_t remains bounded while allowing market shares to adjust smoothly over time.

The corresponding market fractions are

$$q_{1,t} = \frac{1 + m_0}{2}n_0 + \frac{1 + m_t}{2}(1 - n_0), \text{ and } q_{2,t} = 1 - q_{1,t}, \quad (3.14)$$

where $q_{1,t}$ and $q_{2,t}$ denote the market fractions of fundamentalists and trend followers, respectively. The first term in $q_{1,t}$, $\frac{1+m_0}{2}n_0$, represents the fraction of all investors who are fixed fundamentalists. The second term, $\frac{1+m_t}{2}(1 - n_0)$, represents the fraction of switching investors who currently follow the fundamentalist strategy. Since m_t depends on the difference in realized profitability between the two strategies, the market composition changes over time as switching investors move towards the better-performing strategy.

3.5 Price Formation

The market price is determined by a market maker who adjusts prices according to aggregate excess demand. The market maker increases prices when aggregate excess demand is positive and decreases prices when it is negative. This price-adjustment mechanism is widely adopted in heterogeneous-agent models because it provides a simple link between individual trading decisions and aggregate price dynamics.

The market maker updates the market price according to

$$p_{t+1} = p_t + \mu (q_{1,t}z_{1,t} + q_{2,t}z_{2,t}) + \sigma_\delta \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, 1), \quad (3.15)$$

where $\mu > 0$ measures the speed at which the market maker adjusts prices in response to aggregate excess demand, and $\sigma_\delta > 0$ determines the magnitude of random liquidity shocks. Larger values of μ imply that prices respond more strongly to aggregate excess demand,

whereas smaller values lead to more gradual price adjustment. The first term is the current market price. The second term adjusts the price according to aggregate excess demand generated by the two trader groups. The final term captures random liquidity shocks that are independent of investors' expectations and trading strategies.

Finally, the fundamental value of the risky asset is assumed to follow a geometric random walk,

$$p_{t+1}^* = p_t^* \exp\left(-\frac{1}{2}\sigma_\epsilon^2 + \sigma_\epsilon\eta_t\right), \quad \eta_t \sim \mathcal{N}(0, 1), \quad (3.16)$$

where $\sigma_\epsilon > 0$ controls the volatility of changes in the fundamental value, and η_t is an independent standard normal shock representing new information about economic fundamentals. This specification assumes that changes in the fundamental value are entirely driven by random shocks and cannot be predicted using past information. The exponential form ensures that the fundamental value remains positive, while the drift adjustment $-\frac{1}{2}\sigma_\epsilon^2$ ensures that $\mathbb{E}_t(p_{t+1}^*) = p_t^*$, so that the fundamental-value process has no systematic drift in levels. The fundamental value therefore fluctuates randomly around its current level.

4 Theoretical Properties

This section investigates the theoretical properties of the PDRA model. In particular, we focus on two questions. Does the model still admit a well-defined fundamental equilibrium? If the equilibrium exists, does the risk-aversion mechanism affect its local stability? We address these questions by analyzing the deterministic version of the model and examining the local dynamics around the fundamental equilibrium.

4.1 Existence of the Fundamental Equilibrium

To study the existence of the fundamental equilibrium and its local stability, we consider the deterministic system obtained by setting $\varepsilon_t = \eta_t = 0$, fixing $p_t^* = \bar{p}$, and setting $D_t = \bar{D} = (R-1)\bar{p}$ for all t . This removes all exogenous sources of randomness and allows the intrinsic dynamics and equilibrium properties of the model to be analyzed analytically. The state variables of the model are the market price p_t , the trend reference price u_t , the perceived variance v_t , and the time-varying risk aversion of trend followers $a_{2,t}$. The deterministic system can therefore be written as $\mathbf{x}_{t+1} = \mathbf{F}(\mathbf{x}_t)$, where $\mathbf{x}_t = (p_t, u_t, v_t, a_{2,t})^\top$.

Proposition 1. *The deterministic system admits a fundamental equilibrium $\mathbf{x}^* = (\bar{p}, \bar{p}, 0, a_2)^\top$ at which the market price equals the fundamental value, the trend reference coincides with the market price, perceived variance is zero, and the risk aversion of trend followers equals its baseline level.*

Proposition 1 establishes the fundamental equilibrium of the model. At this equilibrium, both trader types have zero demand for the risky asset, so trend followers earn zero realized trading profits. The profit-dependent term in the risk-aversion updating rule therefore drops out, leaving risk aversion at its steady-state level.

4.2 Local Stability

We next examine the local stability of the fundamental equilibrium. The question is whether prices return to the fundamental value after a small disturbance. If the equilibrium is locally stable, such deviations gradually disappear; otherwise, they may persist over time. The boundary case $\omega = 1$ is excluded from the following analysis. As shown in Appendix A.2, the deterministic system then admits a continuum of fundamental equilibria because the risk-aversion state is not pinned down by the updating equation. In this case, $a_{2,t} = a_{2,t-1}$, so trend followers' risk aversion remains fixed at its initial value. A perturbation to the risk-aversion state is therefore permanent. For example, if the system starts from $(\bar{p}, \bar{p}, 0, a_2)^\top$ and risk aversion is perturbed to $a_2 + \varepsilon$, then $a_{2,t} = a_2 + \varepsilon$ in all subsequent periods. The system stays at a different fundamental equilibrium and does not return to the original one. For this reason, the local stability analysis below focuses on $0 \leq \omega < 1$.

To analyze local stability, we consider the first-order approximation of the deterministic system around the fundamental equilibrium. Define the deviations

$$\tilde{p}_t = p_t - \bar{p}, \quad \tilde{u}_t = u_t - \bar{p}, \quad \tilde{v}_t = v_t, \quad \tilde{a}_t = a_{2,t} - a_2. \quad (4.1)$$

At the fundamental equilibrium, the variance equation contains the second-order term $\delta(1 - \delta)(p_t - u_{t-1})^2$, which vanishes under the first-order approximation. The linearized variance dynamics therefore reduce to $\tilde{v}_{t+1} = \delta\tilde{v}_t$. Likewise, the realized profit of trend followers $\pi_{C,t-1} = z_{2,t-1}(p_t + D_t - Rp_{t-1})$ is of second order because both trading demand and the excess payoff vanish at the fundamental equilibrium. Hence, the profit-dependent component of the risk-aversion equation disappears, and the linearized risk-aversion dynamics reduce to $\tilde{a}_{t+1} = \omega\tilde{a}_t$. Perceived variance and time-varying risk aversion do not feed back into the first-order price-trend subsystem.

The resulting linearized price-trend subsystem is

$$\begin{pmatrix} \tilde{p}_{t+1} \\ \tilde{u}_{t+1} \end{pmatrix} = \begin{pmatrix} A & B \\ (1 - \delta)A & \delta + (1 - \delta)B \end{pmatrix} \begin{pmatrix} \tilde{p}_t \\ \tilde{u}_t \end{pmatrix}, \quad (4.2)$$

where the coefficients are given by

$$A = 1 + \frac{\mu}{Q} [(1 + m_q)\bar{a}(\alpha - R) + (1 - m_q)(1 + \gamma - R)], \text{ and } B = -\frac{\mu\gamma(1 - m_q)}{Q}, \quad (4.3)$$

with $\bar{a} = a_2/a_1$ and $Q = 2a_2(1 + r^2)\sigma_1^2$. At the fundamental equilibrium, both trader types earn zero trading profits, so the performance measure in the switching mechanism is zero, implying that $m_t^* = \tanh(0) = 0$. The market composition is determined only by the fixed predisposition of traders. The equilibrium fraction of fundamentalists is therefore

$$q_1^* = \frac{1 + m_0}{2}n_0 + \frac{1}{2}(1 - n_0) = \frac{1 + n_0m_0}{2}. \quad (4.4)$$

For notational convenience, let $m_q = n_0m_0$, so that $q_1^* = \frac{1+m_q}{2}$. The corresponding characteristic polynomial of the price-trend subsystem is

$$\Gamma(\lambda) = \lambda^2 - [\delta + A + (1 - \delta)B]\lambda + \delta A. \quad (4.5)$$

Proposition 2. *Suppose that $0 < \delta < 1$ and $0 \leq \omega < 1$. The fundamental equilibrium is locally asymptotically stable if and only if the following Jury conditions are simultaneously satisfied: $\Gamma(1) > 0$, $\Gamma(-1) > 0$, and $|\delta A| < 1$.*

These conditions keep the price-adjustment process stable around the fundamental equilibrium and ensure convergence after small disturbances. Since the profit-dependent term vanishes at the fundamental equilibrium, it does not affect the first-order local stability conditions. Its effect appears only when trend followers earn non-zero trading profits, which change their risk aversion and therefore their demand and future price dynamics.

5 Data and Calibration

This section introduces the data and the calibration procedure used in the empirical analysis. We first describe the DAX 30 data and summarize the main stylized facts that the model aims to reproduce. We then present the simulation-based calibration procedure used to estimate the model parameters.

5.1 Data and Stylized Facts

We use daily closing prices of the DAX index from 30 December 1987 to 31 July 2026, obtained from Yahoo Finance. The longest available sample is chosen to cover a wide range of market conditions and volatility periods. The index contained 30 constituents until September 2021,

when it was expanded to 40 constituents. For consistency with [He and Li \(2015, 2017\)](#), we refer to the empirical series as the DAX 30 throughout the paper. Daily log returns are computed as $r_t = \ln p_t - \ln p_{t-1}$, yielding a sample of $T = 9,756$ return observations. All three models are calibrated to the same dataset to ensure a fair comparison.

Table 1: Summary Statistics of DAX 30 Daily Log Returns (1987-2026)

Mean	Std	Skewness	Kurtosis	Min	Max	JB
0.000332	0.01366	-0.300	6.834	-0.1409	0.1080	19,110

Note: Std denotes the standard deviation, and JB denotes the Jarque-Bera test statistic for normality.

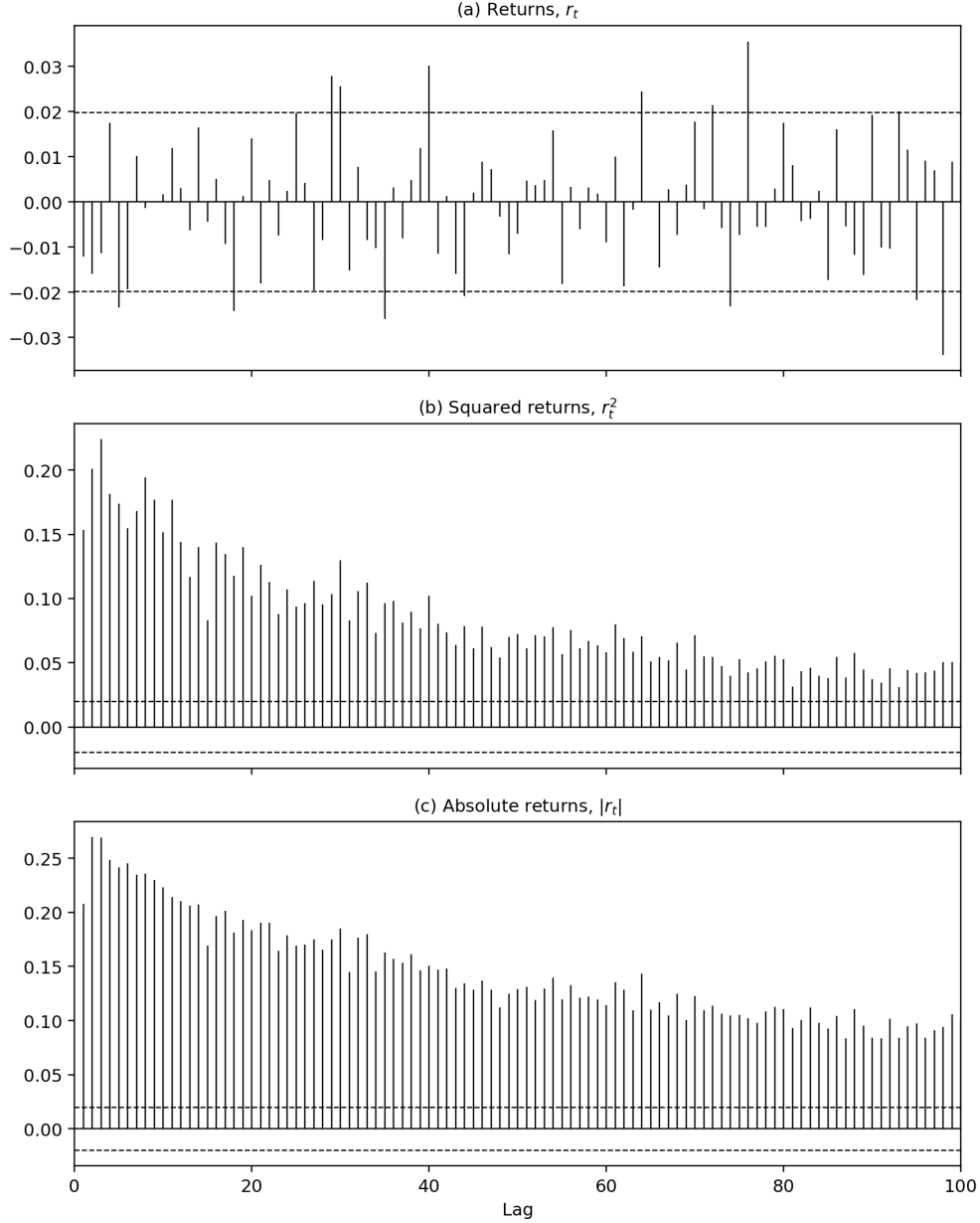


Figure 1: Sample autocorrelation functions of DAX 30 daily log returns, squared returns, and absolute returns. The dashed lines indicate the approximate 95% confidence bounds under the null hypothesis of zero autocorrelation.

Table 1 reports the summary statistics of the DAX 30 daily log returns. Kurtosis is reported as excess kurtosis. The distribution exhibits pronounced fat tails, with an excess kurtosis of 6.834, while the Jarque-Bera statistic of 19,110 strongly rejects the null hypothesis of normality. The return distribution is also negatively skewed, with a skewness coefficient of -0.300 , indicating greater asymmetry towards large negative returns. These features are commonly observed in financial return data and are therefore useful for assessing how well

the models reproduce the empirical distribution of returns. Figure 1 reports the sample autocorrelation functions of daily returns, squared returns, and absolute returns for the DAX 30. The autocorrelations of raw returns fluctuate around zero and are generally insignificant, indicating little serial dependence in daily returns. Squared returns and absolute returns remain positively autocorrelated over many lags, showing persistent volatility clustering. Their slow decay is also consistent with the long-memory behavior examined later in the paper. These autocorrelation functions are used as the main dynamic moments in the calibration.

5.2 Calibration Procedure

The model is calibrated by matching simulated and empirical autocorrelation functions (ACFs). The target moments consist of the ACFs of log returns, squared returns, and absolute returns. Although the model is formulated in terms of price levels, each simulated price series is converted into log returns before computing the ACFs, so that the simulated and empirical moments are measured in the same way. The optimization is carried out over the parameter space Θ , summarized in Table 2.

Table 2: Parameter Space Used in the Calibration

Parameter	Description	Search interval
α	Fundamentalists' expectation parameter	$[0, 1]$
γ	Trend followers' extrapolation coefficient	$[0.05, 5.5]$
a_1	Fundamentalists' risk-aversion coefficient	$[0.001, 9]$
a_2	Baseline risk aversion of trend followers	$[0.001, 9]$
μ	Market maker's price-adjustment coefficient	$[0.1, 5]$
n_0	Fraction of fixed-strategy investors	$[0.05, 0.995]$
m_0	Composition of the fixed-strategy group	$[-1, 1]$
δ	Trend followers' memory parameter	$[0, 1]$
b	Trend followers' volatility-sensitivity coefficient	$[0.05, 8.5]$
β	Strategy-switching intensity parameter	$[0.5, 1.5]$
σ_ϵ	Volatility parameter of the stochastic fundamental process	$[0.005, 0.05]$
σ_δ	Standard deviation of liquidity shocks	$[0.05, 8.5]$
ω	Persistence parameter of trend followers' risk aversion (PDRA models only)	$[0, 1]$
s	Profit-sensitivity parameter (symmetric PDRA model only)	$[0, 1]$
s_{gain}	Gain-sensitivity parameter (asymmetric PDRA model only)	$[0, 1]$
s_{loss}	Loss-sensitivity parameter (asymmetric PDRA model only)	$[0, 1]$

Note: The search intervals for α , m_0 , n_0 , δ , ω , s , s_{gain} , and s_{loss} follow the model specification. The remaining parameter ranges follow the implementation of [He and Li \(2017\)](#) and are retained to ensure comparability with the benchmark model. The symmetric PDRA model introduces a single profit-sensitivity parameter s , whereas the proposed asymmetric PDRA model replaces it with two parameters, s_{gain} and s_{loss} , allowing realized gains and realized losses to affect the risk aversion of trend followers differently.

For each candidate parameter vector, the model is simulated $N = 1000$ times, each over 9,000 periods. The first 1,000 observations of each simulation are discarded as burn-in before computing the sample autocorrelations. For each simulated series, the autocorrelation functions of returns, squared returns, and absolute returns are computed at lags 1-50 and every fifth lag from 55 to 100. This lag structure uses all short lags while retaining selected longer lags to capture persistence. It yields 60 autocorrelations for each series and therefore 180 moment conditions in total. Let $\hat{\beta}^n(\theta) \in \mathbb{R}^{180}$ denote the autocorrelation vector obtained from the n th simulation, where $n = 1, \dots, N$. For each candidate parameter vector, the simulated moment vectors are averaged across the N simulations as $\bar{\beta}(\theta) = N^{-1} \sum_{n=1}^N \hat{\beta}^n(\theta)$. The empirical target vector $\hat{\beta}^{\text{DAX}} \in \mathbb{R}^{180}$ is computed from the observed DAX 30 returns using the same series and lag structure.

The parameter vector is estimated by matching the simulated moments to the empirical moments. While [He and Li \(2017\)](#) use an unweighted Euclidean distance, this paper weights each moment by the inverse of its estimated variance. The optimization problem is

$$\hat{\theta} \in \arg \min_{\theta \in \Theta} \left(\bar{\beta}(\theta) - \hat{\beta}^{\text{DAX}} \right)^\top W \left(\bar{\beta}(\theta) - \hat{\beta}^{\text{DAX}} \right), \quad (5.1)$$

where $W = \text{diag}(w_1, \dots, w_{180})$ is a diagonal weighting matrix with $w_i = 1/\widehat{\text{Var}}(\hat{\beta}_i)$. The sampling variances are estimated using 1,000 circular moving-block bootstrap replications with a block length of 20 trading days. The inverse-variance weights therefore assign greater weight to moments with lower estimated sampling variance and less weight to moments with higher sampling variance. The original calibration procedure of [He and Li \(2017\)](#) uses the derivative-free pattern-search algorithm APPSPACK. This paper instead uses the Differential Evolution (DE) algorithm of [Storn and Price \(1997\)](#). DE is a population-based optimization method that does not require derivatives and is suitable for nonlinear objective functions with multiple local optima ([Storn and Price, 1997](#); [Price et al., 2005](#)). The DE implementation uses a population of 15 candidate parameter vectors, a mutation factor drawn from $[0.5, 1.0]$, a crossover probability of 0.7, and a convergence tolerance of 10^{-6} .

Parameter uncertainty is assessed using a circular moving-block bootstrap. Because daily stock returns exhibit serial dependence and volatility clustering, observations are resampled in consecutive blocks, not individually. The block length is set to 250 trading days, which exceeds the maximum autocorrelation lag used in the calibration and preserves dependence over the relevant lag range. For each bootstrap sample, the model is recalibrated using the same simulated method of moments procedure as in the original estimation. Given the computational cost of recalibrating the model for each bootstrap sample, the number of bootstrap replications is limited to 50. The reported 95% confidence intervals are given by the 2.5th and 97.5th percentiles of the bootstrap parameter estimates.

6 Empirical Results

This section presents the empirical results for the benchmark, PDRA, and asymmetric PDRA models. All three models are calibrated using the same DAX 30 dataset and calibration framework. The results cover the calibrated parameter estimates, volatility persistence, long-memory behavior, tail properties, volatility asymmetry, and within-trend volatility patterns.

6.1 Calibrated Parameters

The calibrated parameter estimates for the PDRA and asymmetric PDRA models are reported in [Table 3](#), together with their bootstrap 95% confidence intervals. The annualized fundamental-value volatility is defined as $\sigma = \sqrt{K}\sigma_\epsilon$. The calibration uses the one-period

volatility parameter σ_ϵ , while Table 3 reports the corresponding annualized value σ . The asymmetric PDRA model has a lower trend-extrapolation parameter, with $\hat{\gamma} = 0.940$ compared with $\hat{\gamma} = 4.532$ in the PDRA model. Its trend followers also have higher baseline risk aversion, with $\hat{a}_2 = 8.124$ compared with 1.222 under PDRA. The estimated trend-memory parameter is similar across the two models, $\hat{\delta} = 0.675$ for PDRA and 0.659 for asymmetric PDRA. The asymmetric PDRA model also has a smaller fraction of non-switching traders and a lower estimated volatility of the fundamental-value process.

The main parameters of interest for the PDRA mechanism are the profit-sensitivity and persistence parameters. In the PDRA model, the estimated common profit-sensitivity parameter is $\hat{s} = 0.479$, while the persistence parameter is $\hat{\omega} = 0.475$, indicating moderate persistence in profit-dependent risk aversion. In the asymmetric PDRA model, the estimated sensitivity parameters are $\hat{s}_{\text{gain}} = 0.489$ and $\hat{s}_{\text{loss}} = 0.337$. The gain sensitivity is close to the common sensitivity estimated under the symmetric PDRA model, while the loss sensitivity is lower. The point estimates therefore suggest a stronger response of risk aversion to gains than to losses. However, the marginal confidence intervals for the two sensitivity parameters show a large degree of overlap, so these estimates alone do not establish a statistically significant difference between s_{gain} and s_{loss} . A formal test would require inference on $s_{\text{gain}} - s_{\text{loss}}$, which would involve additional bootstrap recalibration and is left for future work. The persistence parameter is higher in the asymmetric PDRA model, with $\hat{\omega} = 0.829$ compared with $\hat{\omega} = 0.475$ in the PDRA model.

Table 3: Calibrated Parameters of the PDRA and Asymmetric PDRA Models

Parameter	PDRA	95% CI	Asym. PDRA	95% CI	Interpretation
α	0.711	[0.507, 0.951]	0.846	[0.341, 0.972]	Asymmetric PDRA implies slower expected adjustment towards the fundamental value.
γ	4.532	[0.251, 5.157]	0.940	[0.466, 2.297]	Trend extrapolation is much weaker under asymmetric PDRA.
a_1	8.719	[5.804, 8.972]	7.304	[5.306, 8.834]	Fundamentalists are slightly less risk averse in asymmetric PDRA.
a_2	1.222	[0.174, 3.968]	8.124	[4.231, 8.179]	Trend followers exhibit a much higher baseline risk aversion in asymmetric PDRA.
μ	0.679	[0.180, 2.421]	0.569	[0.227, 2.465]	Both models imply similar market-maker adjustment speeds.
n_0	0.911	[0.207, 0.993]	0.277	[0.132, 0.983]	Asymmetric PDRA requires a much smaller fraction of non-switching traders.
m_0	0.893	[-0.143, 0.903]	0.701	[-0.106, 0.778]	In both models, fundamentalists make up the majority of the fixed-strategy group, although their share is smaller in the asymmetric PDRA model.
δ	0.675	[0.177, 0.893]	0.659	[0.155, 0.882]	Trend-memory persistence is very similar across the two models.
b	4.727	[1.152, 7.120]	5.649	[1.134, 7.995]	Perceived risk reacts slightly more strongly to volatility in asymmetric PDRA.
β	0.724	[0.629, 1.393]	0.682	[0.529, 1.453]	Strategy-switching intensity is similar across models.
σ	0.150	[0.079, 0.187]	0.079	[0.029, 0.123]	Asymmetric PDRA implies a lower volatility of the fundamental-value process.
σ_δ	6.247	[3.052, 7.444]	4.413	[3.636, 7.322]	Market-noise shocks are smaller in asymmetric PDRA.
ω	0.475	[0.098, 0.862]	0.829	[0.178, 0.859]	Risk aversion is more persistent in asymmetric PDRA.
s	0.479	[0.115, 0.817]	—	—	PDRA assumes a common response to gains and losses.
s_{gain}	—	—	0.489	[0.054, 0.842]	Sensitivity of risk aversion to realized gains.
s_{loss}	—	—	0.337	[0.100, 0.719]	Sensitivity of risk aversion to realized losses.

6.2 Conditional Volatility Persistence: GARCH Estimates

To compare volatility persistence in the DAX 30 and the simulated series, we estimate an AR(1)-GARCH(1,1) model for each series. For the three calibrated models, the reported GARCH parameter estimates are the averages obtained from 200 simulated series. Returns are multiplied by 100 before estimation to improve numerical stability, since daily returns are typically very small in magnitude. The estimated α_0 is then converted back to the original scale for comparison with the empirical series. Following standard practice in the GARCH literature (Tsay, 2010), the estimated specification is

$$r_t = a + b r_{t-1} + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2. \quad (6.1)$$

In the mean equation $r_t = a + b r_{t-1} + \varepsilon_t$, a is the constant term and b captures first-order serial dependence in returns. The innovation $\varepsilon_t = \sigma_t z_t$ is scaled by the time-varying conditional volatility σ_t . In the variance equation $\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$, α_0 is the baseline level of

conditional variance, α_1 captures the response of current volatility to the previous shock, and β_1 captures persistence from past volatility. The sum $\alpha_1 + \beta_1$ is used as the measure of volatility persistence, with values closer to one indicating slower decay of volatility shocks.

Table 4: GARCH(1,1) Parameter Estimates for the Empirical DAX and the Calibrated Models

	$a \times 10^3$	b	$\alpha_0 \times 10^4$	α_1	β_1	$\alpha_1 + \beta_1$
DAX 30	7.29	-0.0118	0.040	0.1006	0.8785	0.9792
Benchmark	-0.30	0.1113	2.47	0.0657	0.7425	0.8082
PDRA	0.04	-0.0080	0.21	0.0548	0.9393	0.9941
Asymmetric PDRA	0.02	0.0128	0.13	0.0582	0.9355	0.9938

Table 4 reports the estimated GARCH(1,1) parameters for the DAX 30 and the three calibrated models. The main statistic of interest is the persistence measure $\alpha_1 + \beta_1$, which is estimated at 0.9792 for the DAX 30. The corresponding estimates are 0.8082 for the benchmark model, 0.9941 for the PDRA model, and 0.9938 for the asymmetric PDRA model. The benchmark model therefore underestimates volatility persistence, while both PDRA specifications slightly overestimate it. Their deviations from the empirical estimate are much smaller than those of the benchmark model, with the asymmetric PDRA model marginally closer to the DAX 30. The estimated GARCH coefficient β_1 is 0.7425 for the benchmark model, 0.9393 for the PDRA model, and 0.9355 for the asymmetric PDRA model, compared with 0.8785 for the DAX 30. This suggests that past conditional variance has a weaker effect on current volatility in the benchmark model and a somewhat stronger effect in the two PDRA models than in the empirical data.

6.3 Long Memory in Conditional Volatility: FIGARCH Estimates

While the GARCH model captures volatility persistence, it assumes that the effects of volatility shocks decay exponentially over time. We also estimate an AR(1)-FIGARCH(1, d ,1) model (Baillie et al., 1996) to examine long memory in conditional volatility. In the FIGARCH specification, volatility shocks decay hyperbolically, not exponentially. The conditional mean equation remains the same as in the AR(1)-GARCH(1,1) specification, while the conditional variance follows

$$\sigma_t^2 = \alpha_0 + \left[1 - \beta_1 L - \phi_1 L(1 - L)^d\right] \varepsilon_t^2 + \beta_1 \sigma_{t-1}^2, \quad (6.2)$$

where L denotes the lag operator. Compared with the GARCH(1,1) model, the FIGARCH specification allows the current conditional variance to depend on shocks from a longer history rather than only the most recent period. The effects of past shocks therefore decay more

slowly over time. The fractional integration parameter d determines the rate of this decay. A larger value of d indicates greater persistence in conditional volatility, while a positive value indicates the presence of long memory. Table 5 reports the estimated FIGARCH(1, d ,1) parameters for the DAX 30 and the three calibrated models. For the DAX 30, the estimate is $d = 0.4814$. The benchmark model gives $d = 0.2309$, while the PDRA and asymmetric PDRA models give $d = 0.4675$ and $d = 0.6609$, respectively. Both PDRA specifications provide a closer match to the empirical DAX 30 long-memory estimate than the benchmark model, which underestimates the degree of long memory. Of the two PDRA models, the symmetric specification provides the closer match, while the asymmetric PDRA model overestimates long-memory persistence.

Table 5: FIGARCH(1, d ,1) Parameter Estimates for the Empirical DAX and the Calibrated Models

	$a \times 10^3$	b	$\alpha_0 \times 10^4$	ϕ_1	d	β_1
DAX 30	0.741	-0.0080	0.060	0.0885	0.4814	0.5032
Benchmark	-0.310	0.1147	3.040	0.2676	0.2309	0.3600
PDRA	0.035	-0.0081	0.510	0.2562	0.4675	0.6651
Asymmetric PDRA	0.018	0.0128	0.230	0.1654	0.6609	0.7689

6.4 Long-Memory Properties: Wald Tests

Although the GARCH and FIGARCH estimates allow the models to be compared descriptively, they do not show whether the differences from the DAX 30 are statistically significant. We therefore use Wald tests to test whether the estimated long-memory parameters of the models are equal to those of the DAX 30. Following He and Li (2017), we use both the Geweke-Porter-Hudak (GPH) estimator (Geweke and Porter-Hudak, 1983) and the Robinson-Henry (RH) estimator (Robinson and Henry, 1999). Both estimators use the behavior of the series over long horizons to estimate the long-memory parameter. The GPH estimator uses a log-periodogram regression, while the RH estimator applies a different method in the frequency domain.

The GPH and RH estimators are calculated using bandwidths $m \in \{50, 100, 150, 200, 250\}$ to check the sensitivity of the results to the bandwidth choice. Following He and Li (2015), 1,000 independent simulations are performed at the calibrated parameter vector. For each bandwidth m , the long-memory estimates from the simulated series are averaged and compared with the corresponding DAX 30 estimate. Averaging across the 1,000 simulations reduces the random variation arising from the simulation procedure. For each bandwidth m , the null hypothesis is $H_0 : d_m^{\text{model}} = d_m^{\text{DAX}}$, against the two-sided alternative that the two long-memory

parameters differ. Let $\bar{d}_m^{\text{model}}$ denote the average estimate from the 1,000 simulations and $\hat{d}_m^{\text{DAX}}$ the corresponding estimate from the DAX 30. The asymptotic justification of the Wald test assumes that both the number of independent simulations and the length of each simulated series tend to infinity. In the numerical analysis, each simulated series contains 8,000 observations after the burn-in period. The test statistic is

$$W_m = \frac{(\hat{d}_m^{\text{DAX}} - \bar{d}_m^{\text{model}})^2}{\widehat{\text{Var}}(\hat{d}_m^{\text{DAX}})}, \quad (6.3)$$

where $\bar{d}_m^{\text{model}}$ denotes the average long-memory estimate obtained from the 1,000 simulated series, and $\widehat{\text{Var}}(\hat{d}_m^{\text{DAX}})$ denotes the estimated sampling variance of the corresponding DAX 30 estimate. The statistic compares the difference between the model and DAX 30 estimates with the sampling uncertainty of the empirical estimate. Under the null hypothesis, W_m follows a $\chi^2(1)$ distribution. The critical values are 3.842 at the 5% significance level and 6.635 at the 1% significance level. The null hypothesis is rejected when W_m exceeds the corresponding critical value.

Table 6: Wald Statistics for the Benchmark, PDRA, and Asymmetric PDRA Models

Series	Estimator	m	Benchmark	PDRA	Asym. PDRA
Absolute returns	GPH	50	1.0155	0.0022	0.0046
		100	0.0021	0.0352	0.1652
		150	2.4016	0.6648	0.0013
		200	7.8691	1.3670	0.0485
		250	14.5172	2.0355	0.1444
	RH	50	3.2076	0.5967	0.5258
		100	0.0321	0.0757	0.3798
		150	2.9713	0.5202	0.1642
		200	11.0630	1.8106	0.0271
		250	24.0437	5.1771	1.1711
Returns	GPH	50	0.0006	18.4075	15.3336
		100	0.0273	29.6643	16.0846
		150	0.0100	32.5821	15.0363
		200	0.0861	31.5632	12.7238
		250	0.6680	43.3013	19.9032
	RH	50	0.0225	25.4198	21.8203
		100	0.2319	52.6252	25.8418
		150	0.0011	50.8298	17.9548
		200	0.0000	51.4724	16.2181
		250	0.9219	63.5311	22.5468
Squared returns	GPH	50	1.9964	0.0156	0.0396
		100	0.3002	0.2053	0.0434
		150	0.0063	0.0844	0.0448
		200	0.7244	0.0748	0.1335
		250	4.8641	0.8739	0.0582
	RH	50	6.0935	0.9032	0.3635
		100	1.5011	0.0309	0.3337
		150	0.1195	0.2238	0.0498
		200	1.5648	0.2069	0.1295
		250	7.8261	1.6539	0.1800

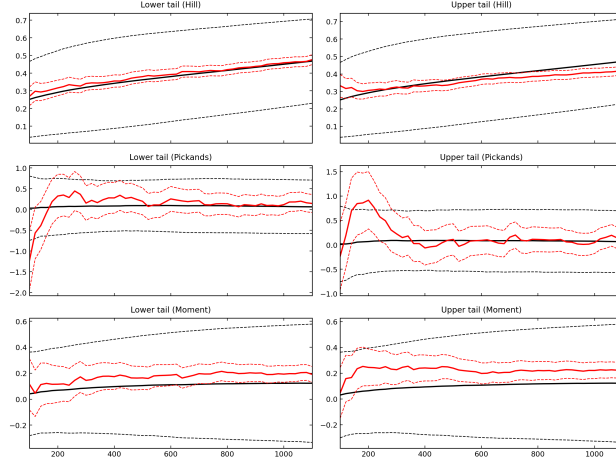
Notes: The 5% critical value of the Wald statistic is 3.842. Bold entries mark cases where the test rejects equality between the model and DAX long-memory estimates.

Table 6 reports the Wald statistics for the PDRA and asymmetric PDRA models, together with those for the benchmark model of He and Li (2017), recalibrated using the same DAX 30 dataset. For absolute and squared returns, the two PDRA models yield fewer rejections than the benchmark. The PDRA model is rejected only once, for absolute returns under the RH estimator with $m = 250$, while the asymmetric PDRA model is not rejected for either transformation. For raw returns, both PDRA models are rejected at all reported bandwidths, whereas the benchmark model is not rejected. The asymmetric PDRA model has one fewer rejection than the symmetric PDRA model. None of the three models matches the long-memory estimates for all three return measures simultaneously. A related pattern is reported by He and Li (2015). In their original application, the benchmark model is rejected for raw returns at every reported bandwidth, while no rejection is found for absolute or

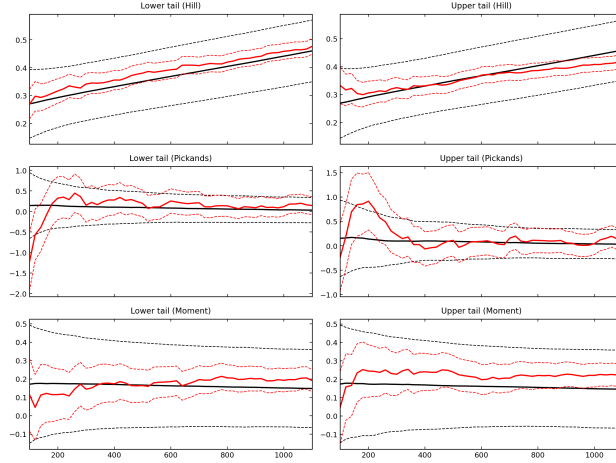
squared returns. With the updated DAX sample and recalibration used here, the pattern is partly reversed; the benchmark matches the raw-return estimates across all specifications, but is rejected in several cases for absolute and squared returns. This is in line with the finding in [He and Li \(2015\)](#) that the simple market fraction model does not match all empirical features simultaneously.

6.5 Tail Behavior

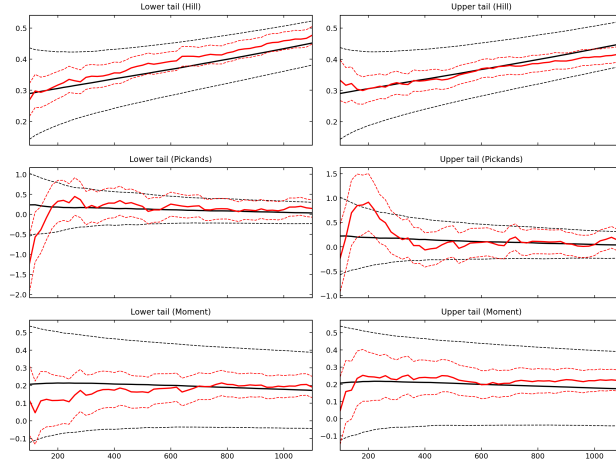
We examine tail behavior using three tail-index estimators: the Hill estimator ([Hill, 1975](#)), the Pickands estimator ([Pickands, 1975](#)), and the moment estimator ([Dekkers et al., 1989](#)). The three estimators differ in their sensitivity to the tail threshold and finite-sample properties, so results are reported for all three. For each estimator, the tail index is computed using the largest observations for the positive tail and the smallest observations for the negative tail, with k determining the number of tail observations included. We consider $k = 50, \dots, 1200$ to examine how the estimates vary with the tail threshold. [Figure 2](#) compares the tail-index estimates of the three calibrated models with those of the DAX 30. The three models produce similar tail-index estimates under the Hill and Pickands estimators. For the Hill estimator, the estimated curves of all three models track the DAX 30 closely over most values of k . The benchmark model has wider Monte Carlo confidence intervals, indicating greater variability in the estimated tail index. The Pickands estimates show a similar pattern, with little difference across the three models. For the moment estimator, the differences across models become more visible as k increases. For intermediate and larger values of k , the two PDRA models are generally closer to the DAX 30 estimates than the benchmark model. At smaller values of k , the differences across models are less clear. The benchmark model tends to underestimate the tail index for both the positive and negative tails over the higher range of k . Among the two PDRA models, the asymmetric specification generally provides the closer match.



(a) Benchmark model



(b) PDRA model



(c) Asymmetric PDRA model

Figure 2: Tail-index estimates for the positive and negative tails obtained using the Hill, Pickands, and moment estimators. The black solid line shows the model mean, and the black dashed lines its 95% simulation interval. The red solid line shows the DAX estimate, and the red dashed lines its asymptotic 95% confidence interval.

6.6 Volatility Asymmetry Diagnostics

The previous sections compare the three models in terms of several statistical properties of the DAX index. We now turn to volatility asymmetry to examine whether the different specifications of risk aversion lead to different volatility responses in the simulated data. Two diagnostics are used: the Engle-Ng Sign and Size Bias test (Engle and Ng, 1993) and the News Impact Curve based on the GJR-GARCH(1,1) model of Glosten et al. (1993). The Engle-Ng test examines whether asymmetric effects remain in the volatility residuals, while the News Impact Curve shows how conditional volatility responds to positive and negative return shocks.

Following Engle and Ng (1993), the Sign and Size Bias test is performed after estimating an AR(1)-GARCH(1,1) model for each return series. Letting $z_t = \varepsilon_t/\sigma_t$ denote the standardized residual, the auxiliary regression is

$$z_t^2 = c + b_1 S_{t-1}^- + b_2 S_{t-1}^- \varepsilon_{t-1} + b_3 S_{t-1}^+ \varepsilon_{t-1} + u_t, \quad (6.4)$$

where ε_t is the innovation from the estimated AR(1)-GARCH(1,1) model and $z_t = \varepsilon_t/\sigma_t$ is the standardized residual. The indicator S_{t-1}^- equals one when the previous innovation is negative and zero otherwise, while $S_{t-1}^+ = 1 - S_{t-1}^-$ identifies positive innovations. The coefficient b_1 captures sign bias, while b_2 and b_3 capture negative and positive size bias, respectively. A significant b_1 indicates different responses to negative and positive innovations. The coefficients b_2 and b_3 indicate whether these effects vary with the magnitude of negative and positive innovations. Under the null hypothesis, $b_1 = b_2 = b_3 = 0$. Rejection of the joint null indicates asymmetric effects that are not captured by the symmetric GARCH specification. The second diagnostic estimates an AR(1)-GJR-GARCH(1,1) model,

$$h_t = \omega + \alpha \varepsilon_{t-1}^2 + \gamma \varepsilon_{t-1}^2 I(\varepsilon_{t-1} < 0) + \beta h_{t-1}, \quad (6.5)$$

where h_t denotes the conditional variance of returns and ε_{t-1} is the previous return innovation. The constant ω gives the baseline level of conditional variance. The coefficient α measures the response to the previous innovation, while β reflects persistence through the lagged conditional variance h_{t-1} . The indicator $I(\varepsilon_{t-1} < 0)$ equals one for negative innovations and zero otherwise. The leverage parameter γ measures the additional effect of a negative innovation. A positive γ implies that negative innovations have a larger effect on future volatility than positive innovations of the same magnitude. Following Engle and Ng (1993), the News Impact Curve is constructed from the estimated GJR model by fixing lagged conditional variance at its unconditional level and varying the size and sign of the previous

innovation. The curve therefore shows how positive and negative innovations of the same magnitude affect future conditional volatility. For both diagnostics, the empirical statistics are first estimated from the full DAX sample. For each calibrated model, 1,000 valid Monte Carlo return series are generated. Each simulation contains 9,000 periods, with the first 1,000 observations discarded as burn-in, leaving 8,000 observations for the volatility-asymmetry analysis. Bootstrap replications of the DAX series are used to estimate the covariance matrix of the Engle-Ng statistics and the pointwise standard deviations of the News Impact Curve. These quantities are then used to standardize the differences between the simulated and empirical diagnostics. The three models are simulated using common random seeds, so that corresponding simulations are generated from the same sequence of shocks across the benchmark, PDRA, and asymmetric PDRA models. For each simulated return series, the Engle-Ng and GJR diagnostics are estimated using the same procedure as for the DAX 30. Each simulated diagnostic is then compared with the corresponding empirical diagnostic, giving 1,000 distance measures for each model. For the Engle-Ng diagnostic, the four-dimensional vector consisting of the Sign Bias statistic, Negative Size Bias statistic, Positive Size Bias statistic, and the square root of the Joint χ^2 statistic is compared with the DAX 30 vector using a Mahalanobis-type distance,

$$D_{\text{EN}} = \sqrt{\frac{(\mathbf{v} - \mathbf{v}_{\text{DAX}})' \Sigma_{\text{DAX}}^{-1} (\mathbf{v} - \mathbf{v}_{\text{DAX}})}{4}}, \quad (6.6)$$

where $\mathbf{v}$ denotes the simulated Engle-Ng statistic vector and Σ_{DAX} is the covariance matrix estimated from the DAX bootstrap distribution. The distance accounts for both the sampling variation and the correlation among the four Engle-Ng statistics. For the GJR News Impact Curve, shocks and conditional variance are normalized by a reference volatility to make the curves comparable across return series. Shocks are expressed as $\epsilon/\sqrt{h_{\text{ref}}}$, while conditional variance is expressed as h_t/h_{ref} . The normalized simulated and empirical News Impact Curves are compared using a bootstrap-standardized root mean squared error,

$$D_{\text{NIC}} = \sqrt{\frac{1}{K} \sum_{i=1}^K \left(\frac{\text{NIC}_i - \text{NIC}_i^{\text{DAX}}}{s_i} \right)^2}, \quad (6.7)$$

where NIC_i denotes the model News Impact Curve at shock level i , $\text{NIC}_i^{\text{DAX}}$ is the corresponding DAX 30 value, and s_i is the bootstrap standard deviation of the DAX News Impact Curve at that shock level. Each pointwise difference is divided by s_i before the root mean squared distance is calculated, so points with lower bootstrap variability receive greater weight. For each common simulation seed, the diagnostic distances of models A and B from the DAX 30 are computed and the paired difference is defined as $\Delta_i = D_{A,i} - D_{B,i}$. The mean

paired difference is $\bar{\Delta} = N^{-1} \sum_{i=1}^N \Delta_i$, where $N = 1,000$. A negative value of $\bar{\Delta}$ indicates that model A is closer to the DAX 30 than model B on average. The 95% bootstrap confidence interval is reported for the mean paired difference. One-sided paired t -tests are also used, with Holm-adjusted p -values to account for the three pairwise comparisons within each diagnostic.

Table 7: Pairwise Comparisons of Volatility Asymmetry Diagnostics Relative to the Empirical DAX

Diagnostic	Comparison	Mean Dist. (A)	Mean Dist. (B)	Mean Δ	95% Bootstrap CI	Holm p
Engle-Ng Sign/Size Bias	PDRA vs. Benchmark	7.167	32.193	-25.026	[-26.409, -23.690]	< 0.001
	Asym. PDRA vs. Benchmark	6.911	32.193	-25.282	[-26.680, -23.924]	< 0.001
	Asym. PDRA vs. PDRA	6.911	7.167	-0.256	[-0.537, 0.032]	0.039
GJR News Impact Curve	PDRA vs. Benchmark	2.128	4.239	-2.111	[-2.262, -1.966]	< 0.001
	Asym. PDRA vs. Benchmark	2.141	4.239	-2.099	[-2.251, -1.952]	< 0.001
	Asym. PDRA vs. PDRA	2.141	2.128	+0.013	[-0.009, 0.034]	0.885

Table 7 reports the pairwise comparisons of the three calibrated models under the two volatility asymmetry diagnostics. For the Engle-Ng Sign and Size Bias diagnostic, both the PDRA and asymmetric PDRA models have smaller diagnostic distances than the He-Li benchmark. The differences relative to the benchmark are statistically significant, with bootstrap confidence intervals entirely below zero and Holm-adjusted p -values below 0.001. The asymmetric PDRA model has the smallest mean diagnostic distance of the three models. However, its advantage over the symmetric PDRA model is much smaller. The one-sided Holm-adjusted test is significant at the 5% level, while the two-sided bootstrap confidence interval includes zero. The evidence in favor of the asymmetric specification is therefore weaker than the evidence for the improvement of the two PDRA models over the benchmark. For the GJR News Impact Curve, both PDRA models again have smaller diagnostic distances than the benchmark, with reductions of roughly one-half. These differences are statistically significant. The two PDRA specifications give very similar results. The asymmetric PDRA model has a slightly larger mean diagnostic distance than the symmetric PDRA model, while the bootstrap confidence interval includes zero and the Holm-adjusted p -value does not indicate a statistically significant difference. The two PDRA specifications fit the DAX more closely than the benchmark model under both the Engle-Ng Sign and Size Bias diagnostic and the GJR News Impact Curve. No clear overall difference is found between the symmetric and asymmetric PDRA specifications.

6.7 Sensitivity Analysis

The baseline specification sets the annual risk-free rate to $r = 0.05$, following [He and Li \(2015, 2017\)](#). For the sensitivity analysis, the calibrated behavioral parameters are held

fixed and r is varied between 3% and 7%. For each value of r , the daily gross risk-free return $R = 1 + r/K$ and the dividend-variance parameter $q = r^2$ are updated. All other parameters and simulation settings remain unchanged. The models are simulated using the same procedure as in the baseline case.

Table 8: Sensitivity to the Annual Risk-Free Rate: PDRA Model

r	Average Moment Distance	Raw ACF RMSE	Squared ACF RMSE	Absolute ACF RMSE	GARCH $\alpha + \beta$	FIGARCH d	Positive Tail Index	Negative Tail Index	Engle-Ng Distance	GJR NIC Distance
0.03	0.709550	0.022228	0.056854	0.063792	0.994059	0.467693	0.349876	0.350393	7.165902	2.129418
0.05	0.710434	0.022253	0.056900	0.063897	0.994058	0.467529	0.349830	0.350415	7.167240	2.127944
0.07	0.710457	0.022247	0.056926	0.063875	0.994056	0.467294	0.349959	0.350566	7.165088	2.125216

Table 9: Sensitivity to the Annual Risk-Free Rate: Asymmetric PDRA Model

r	Average Moment Distance	Raw ACF RMSE	Squared ACF RMSE	Absolute ACF RMSE	GARCH $\alpha + \beta$	FIGARCH d	Positive Tail Index	Negative Tail Index	Engle-Ng Distance	GJR NIC Distance
0.03	0.646073	0.023186	0.056554	0.052118	0.993772	0.660910	0.351270	0.351787	6.903347	2.141969
0.05	0.645452	0.023178	0.056511	0.052050	0.993774	0.660935	0.351340	0.351818	6.911226	2.140834
0.07	0.645934	0.023204	0.056530	0.052127	0.993774	0.661191	0.351426	0.351910	6.929026	2.139054

Tables 8 and 9 report the sensitivity results for the PDRA and asymmetric PDRA models. In both cases, changing the annual risk-free rate from 3% to 7% has little effect on the reported empirical measures. For the PDRA model, the largest percentage change relative to the baseline occurs in the absolute return ACF RMSE and is approximately 0.16%. The other ACF measures and the average moment distance also change only slightly. The GARCH persistence parameter ($\alpha + \beta$) is essentially unchanged, while the FIGARCH parameter d and the two tail indices vary by less than 0.05%. The Engle-Ng distance changes by about 0.03%, and the GJR News Impact Curve distance changes by at most approximately 0.13%. The asymmetric PDRA model shows a similar pattern. The largest change is observed for the Engle-Ng distance, at approximately 0.26% relative to the baseline. The ACF measures and the average moment distance change only slightly, while GARCH persistence, the FIGARCH parameter, and the two tail indices remain nearly unchanged. The GJR News Impact Curve distance also varies by less than 0.1%. These empirical results are insensitive to the choice of the annual risk-free rate over the range considered.

6.8 Observable Validation

The paper also examines whether the three models can reproduce patterns that are directly observed in the DAX 30 price series. This provides another way to compare the models using

only observable market data. Differences in subsequent volatility across trend states are first measured in the DAX 30 and used as empirical targets. The same definitions are then applied to the benchmark, PDRA, and asymmetric PDRA models to see how closely the simulated price series match these patterns. Robustness checks consider alternative target definitions and matching specifications.

6.8.1 Observable Market-State Evidence

Let P_t denote the DAX 30 closing price on trading day t , and define the daily log return as $r_t = \log P_t - \log P_{t-1}$. Each trading day is first classified by the direction of the preceding price trend. The trend is measured using the simple moving average $MA_{J,t-1} = \frac{1}{J} \sum_{j=0}^{J-1} P_{t-1-j}$, where J is the moving-average horizon. The baseline specification uses $J = 20$, approximately one trading month, while $J = 5$ and $J = 60$ are considered in the robustness checks. The trend measure is evaluated at $t - 1$, so the classification uses only information available before r_t is realized. An observation is classified as an uptrend when $P_{t-1} > MA_{J,t-1}$ and as a downtrend otherwise. The sign of the realized return is then combined with the preceding trend. This gives four market states: positive and negative returns following either an uptrend or a downtrend. These states are constructed entirely from observed prices and returns and can therefore be defined in the same way for both the DAX 30 data and the simulated price series. After the market state at date t is identified, subsequent volatility is measured by $V_{t+1,h} = \sum_{k=1}^h r_{t+k}^2$. The analysis considers $h \in \{1, 5, 20\}$. $h = 1$ measures next-day volatility, $h = 5$ approximately one trading week, and $h = 20$ approximately one trading month. The baseline specification uses $h = 5$, with the other two horizons used for robustness checks. Differences in $V_{t+1,h}$ across the four market states provide the empirical targets used in the model comparison.

Table 10: Observable DAX 30 Market States and Subsequent Volatility ($J = 20$, $h = 5$)

Prior trend	Return	N	Mean $V_{t+1,5}$	Median $V_{t+1,5}$
$P_{t-1} > MA_{J,t-1}$	$r_t > 0$	3117	0.000617	0.000350
$P_{t-1} > MA_{J,t-1}$	$r_t < 0$	2724	0.000720	0.000416
$P_{t-1} \leq MA_{J,t-1}$	$r_t > 0$	2065	0.001081	0.000602
$P_{t-1} \leq MA_{J,t-1}$	$r_t < 0$	1826	0.001607	0.000782

Table 10 reports the four observable market states for the baseline specification ($J = 20$, $h = 5$). Following an uptrend, mean five-day future volatility is 0.000617 after a positive return and 0.000720 after a negative return. Following a downtrend, the corresponding values are 0.001081 and 0.001607. Negative returns are therefore followed by higher future volatility

in both trend regimes, but the difference is much larger after a downtrend. To measure this difference while keeping the preceding trend fixed, two within-trend contrasts are used. For observations following an uptrend,

$$\Delta_{\text{Up}} = E[V_{t+1,h} \mid P_{t-1} > MA_{J,t-1}, r_t < 0] - E[V_{t+1,h} \mid P_{t-1} > MA_{J,t-1}, r_t > 0]. \quad (6.8)$$

A positive value of Δ_{Up} indicates that, conditional on the same preceding uptrend, future volatility is higher following a negative return than following a positive return. Similarly, for observations following a preceding downtrend,

$$\Delta_{\text{Down}} = E[V_{t+1,h} \mid P_{t-1} \leq MA_{J,t-1}, r_t < 0] - E[V_{t+1,h} \mid P_{t-1} \leq MA_{J,t-1}, r_t > 0]. \quad (6.9)$$

The two contrasts measure the difference in future volatility between negative and positive returns within the same preceding trend. These contrasts are used as empirical targets in the model comparison. For the baseline specification, $\Delta_{\text{Up}} = 0.000103$ and $\Delta_{\text{Down}} = 0.000526$. The 95% confidence intervals, based on 1,000 circular moving-block bootstrap replications with a block length of 20 trading days, are $[0.000061, 0.000143]$ for Δ_{Up} and $[0.000339, 0.000732]$ for Δ_{Down} . Both intervals exclude zero, showing that the within-trend differences are positive in both cases. Table 11 reports the two within-trend indicators for all combinations of $J \in \{5, 20, 60\}$ and $h \in \{1, 5, 20\}$. In all nine cases, both Δ_{Up} and Δ_{Down} are positive, and their bootstrap confidence intervals exclude zero. The same pattern therefore holds across different choices of the trend horizon J and the future-volatility horizon h .

Table 11: DAX 30 Within-Trend Indicators across Trend and Volatility Horizons

J	h	Δ_{Up}	95% CI	Δ_{Down}	95% CI
5	1	0.000057	[0.000038, 0.000079]	0.000090	[0.000048, 0.000139]
5	5	0.000154	[0.000089, 0.000218]	0.000459	[0.000302, 0.000635]
5	20	0.000447	[0.000219, 0.000710]	0.001212	[0.000771, 0.001711]
20	1	0.000040	[0.000025, 0.000057]	0.000111	[0.000064, 0.000171]
20	5	0.000103	[0.000061, 0.000143]	0.000526	[0.000339, 0.000732]
20	20	0.000250	[0.000100, 0.000425]	0.001464	[0.000898, 0.002052]
60	1	0.000045	[0.000028, 0.000066]	0.000111	[0.000058, 0.000167]
60	5	0.000125	[0.000082, 0.000174]	0.000543	[0.000331, 0.000800]
60	20	0.000336	[0.000199, 0.000489]	0.001475	[0.000829, 0.002185]

6.8.2 Cross-Model Comparison of Observable Within-Trend Indicators

The observable within-trend volatility differences in the DAX 30 are used as empirical targets for comparing the three models. The comparison asks whether the PDRA model matches these patterns more closely than the benchmark and whether allowing for asymmetric profit-dependent risk aversion leads to any further improvement. The same market-state classification is applied to the simulated price series from each model. For model $m \in \{\text{Benchmark, PDRA, Asymmetric PDRA}\}$ and simulation path $s = 1, \dots, 1000$, a price series of 9,000 trading periods is generated, with the first 1,000 observations discarded as burn-in. Common random seeds are used across the three models. The indicators Δ_{Up} and Δ_{Down} are then calculated from each simulated series in the same way as for the DAX 30, giving the simulated target vector $\mathbf{m}^{(m,s)} = (\Delta_{\text{Up}}^{(m,s)}, \Delta_{\text{Down}}^{(m,s)})'$. This is compared with the empirical DAX 30 target $\mathbf{m}_{\text{DAX}} = (\Delta_{\text{Up}}^{\text{DAX}}, \Delta_{\text{Down}}^{\text{DAX}})'$ using the Mahalanobis distance,

$$d^{(m,s)} = \left[\left(\mathbf{m}^{(m,s)} - \mathbf{m}_{\text{DAX}} \right)' \hat{\Sigma}_{\text{DAX}}^{-1} \left(\mathbf{m}^{(m,s)} - \mathbf{m}_{\text{DAX}} \right) \right]^{1/2}, \quad (6.10)$$

where $\hat{\Sigma}_{\text{DAX}}$ is the bootstrap covariance matrix of the empirical target vector. The Mahalanobis distance measures the joint difference between the two simulated indicators and their DAX 30 counterparts while accounting for their covariance. A smaller distance means that the simulated within-trend pattern is closer to the DAX 30 target. This procedure is repeated for 1,000 common-seed simulations, giving 1,000 paired distance observations for each model. The paired distances are then used to compare the three models.

Table 12: Observable Within-Trend Indicators and Cross-Model Comparison ($J = 20$, $h = 5$)

Panel A: DAX 30 targets and model means					
Indicator	DAX 30	Benchmark	PDRA	Asymmetric PDRA	Closest model
Δ_{Up}	0.000103	0.003142	0.000814	0.000575	Asymmetric PDRA
Δ_{Down}	0.000526	-0.001793	0.001894	0.001356	Asymmetric PDRA
Panel B: Standardized Mahalanobis distances					
Model	Mean distance	Median			
Benchmark	148.754	12.942			
PDRA	39.559	22.506			
Asymmetric PDRA	27.518	15.040			
Panel C: Paired common-seed comparisons					
Comparison	Mean D_s	Bootstrap 95% CI	Closer fraction	Holm-adjusted p	Favoured model
PDRA – Benchmark	-109.195	[−131.966, −88.344]	0.527	2.41×10^{-21}	PDRA
Asymmetric PDRA – Benchmark	-121.236	[−143.934, −99.323]	0.602	8.90×10^{-25}	Asymmetric PDRA
Asymmetric PDRA – PDRA	-12.040	[−16.127, −7.733]	0.718	1.47×10^{-8}	Asymmetric PDRA

Table 12 reports the baseline comparison for $J = 20$ and $h = 5$. Panel A compares

the average simulated values of Δ_{Up} and Δ_{Down} with the corresponding DAX 30 values. For both indicators, the asymmetric PDRA model is closest to the empirical target. The symmetric PDRA model is closer than the benchmark model, but remains farther from the DAX 30 values than the asymmetric PDRA model. Panel B reports the Mahalanobis distance between the simulated and empirical target vectors. The average distance is 148.754 for the benchmark model, 39.559 for the PDRA model, and 27.518 for the asymmetric PDRA model. Both PDRA models are closer to the DAX 30 target than the benchmark, and the asymmetric PDRA model has the smallest average distance. Panel C uses paired common-seed simulations to test whether the differences in Mahalanobis distance between the models are statistically significant. For each simulation path, the distance difference $D^{(s)} = d_A^{(s)} - d_B^{(s)}$ is calculated. The closer fraction gives the share of simulations in which one model has a smaller Mahalanobis distance than the other. Holm-adjusted p -values are reported for the three pairwise comparisons. Compared with the benchmark model, the average Mahalanobis distance is lower by 109.195 for the PDRA model and by 121.236 for the asymmetric PDRA model. The bootstrap confidence intervals exclude zero in both comparisons, and the Holm-adjusted p -values are very small. The asymmetric PDRA model has a lower average distance than the symmetric PDRA model by 12.040. Its confidence interval excludes zero, it is closer to the DAX 30 target in 71.8% of the common-seed simulations, and the Holm-adjusted p -value is 1.47×10^{-8} . The PDRA models match the observable within-trend volatility patterns more closely than the benchmark model. The asymmetric PDRA model is closer to the DAX 30 targets than the symmetric PDRA model.

6.8.3 Robustness across Alternative Empirical Targets

The comparison is also carried out for all combinations of the trend horizon $J \in \{5, 20, 60\}$ and the future-volatility horizon $h \in \{1, 5, 20\}$ considered in Subsection 6.8.1. For each (J, h) combination, the DAX 30 targets and the model indicators are recalculated, and the same Mahalanobis-distance and paired common-seed comparisons are used. Table 13 reports the model with the smallest average Mahalanobis distance for each specification and whether the pairwise differences remain statistically significant after the Holm adjustment.

Table 13: Cross-Model Robustness Across Alternative Observable Targets

J	h	Numerically closest model	Statistically supported best
5	1	Asymmetric PDRA	None
5	5	Asymmetric PDRA	Asymmetric PDRA
5	20	Asymmetric PDRA	Asymmetric PDRA
20	1	Asymmetric PDRA	Asymmetric PDRA
20	5	Asymmetric PDRA	Asymmetric PDRA
20	20	Asymmetric PDRA	Asymmetric PDRA
60	1	Asymmetric PDRA	Asymmetric PDRA
60	5	Asymmetric PDRA	Asymmetric PDRA
60	20	Asymmetric PDRA	Asymmetric PDRA

Table 13 shows that the model ranking changes little across the alternative target definitions. The asymmetric PDRA model has the smallest average Mahalanobis distance in all nine combinations of the trend horizon J and the future-volatility horizon h . The benchmark model is never the closest to the DAX 30 target. The paired common-seed tests show that the advantage of the asymmetric PDRA model is statistically significant in eight of the nine cases after Holm adjustment. The only exception is $(J, h) = (5, 1)$, where the asymmetric PDRA model still has the smallest average distance, but the difference is not statistically significant after the adjustment. The asymmetric PDRA model therefore remains the closest specification across all nine target definitions, with statistical support in eight of them.

A second robustness check considers whether the model comparison changes after controlling for recent volatility and the strength of the preceding trend. The matching is carried out separately for the DAX 30 data and for each simulated price path. The empirical and simulated observations are first divided into uptrend and downtrend groups using the same trend classification as in the baseline analysis. Within each trend group, positive- and negative-return observations are paired one-to-one based on the closest standardized covariate values, with each observation used only once. The matching covariates are the cumulative squared returns over the preceding five trading days, $V_{t,5}^{\text{past}} = \sum_{j=1}^5 r_{t-j}^2$, and prior trend strength,

$$\text{TrendStrength}_{t-1} = \frac{|P_{t-1} - MA_{J,t-1}|}{MA_{J,t-1}}. \quad (6.11)$$

A standardized caliper of 0.50 is used, so observations are matched only when the matching covariates are sufficiently similar. Observations without a suitable match are dropped. After matching, $\tilde{\Delta}_{\text{Up}}$ and $\tilde{\Delta}_{\text{Down}}$ are calculated using only the matched observations. This gives one matched target vector for the DAX 30, $(\tilde{\Delta}_{\text{Up}}^{\text{DAX}}, \tilde{\Delta}_{\text{Down}}^{\text{DAX}})'$, and one for each simulated path

from the benchmark, PDRA, and asymmetric PDRA models. The Mahalanobis distance is then calculated between each simulated matched target and the matched DAX 30 target. The covariance matrix is estimated using the circular moving-block bootstrap, with the full matching procedure repeated within each bootstrap replication.

Table 14: Matched Observable Within-Trend Indicators and Comparison ($J = 20$, $h = 5$)

Panel A: Matched DAX 30 targets and model means					
Indicator	DAX 30	Benchmark	PDRA	Asymmetric PDRA	Closest model
$\widetilde{\Delta}_{\text{Up}}$	0.000113	0.001792	0.001064	0.000718	Asymmetric PDRA
$\widetilde{\Delta}_{\text{Down}}$	0.000508	0.000981	0.002385	0.001607	Benchmark
Panel B: Standardized Mahalanobis distances					
Model	Mean distance	Median			
Benchmark	87.314	12.664			
PDRA	50.631	32.375			
Asymmetric PDRA	32.791	21.747			
Panel C: Paired common-seed comparisons					
Comparison	Mean D_s	Bootstrap 95% CI	Closer fraction	Holm-adjusted p	Favoured model
PDRA – Benchmark	-36.684	[−51.878, −22.703]	0.392	4.02×10^{-7}	PDRA
Asymmetric PDRA – Benchmark	-54.524	[−69.762, −40.277]	0.476	2.91×10^{-12}	Asymmetric PDRA
Asymmetric PDRA – PDRA	-17.840	[−22.686, −12.078]	0.724	4.90×10^{-11}	Asymmetric PDRA

Table 14 reports the matched comparison for the baseline specification. In Panel A, the asymmetric PDRA model is closest to the matched DAX 30 target for $\tilde{\Delta}_{Up}$, while the benchmark model is closest for $\tilde{\Delta}_{Down}$. Panel B gives average Mahalanobis distances of 87.314 for the benchmark model, 50.631 for the PDRA model, and 32.791 for the asymmetric PDRA model. Considering the two matched indicators together, the asymmetric PDRA model is closest to the matched DAX 30 target. The paired results in Panel C are broadly consistent with this ranking. Both PDRA models have significantly lower average Mahalanobis distances than the benchmark model. However, the closer fractions are below 50% in both comparisons with the benchmark, so this advantage is not present in a majority of individual common-seed simulations. The asymmetric PDRA model also has a significantly lower average distance than the symmetric PDRA model. Both robustness checks give the same general result. The PDRA models remain closer to the DAX 30 targets than the benchmark model. The asymmetric PDRA model also remains the closest overall specification.

7 Conclusion

This paper asks whether introducing profit-dependent risk aversion improves the empirical performance of the market fraction model relative to the constant-risk-aversion benchmark,

and whether allowing risk aversion to respond differently to gains and losses provides any additional improvement. Two PDRA specifications are considered. In the symmetric PDRA model, trend followers' risk aversion changes with realized trading profits through a common profit-sensitivity parameter. The asymmetric PDRA model allows gains and losses to affect risk aversion with different sensitivities. The benchmark, PDRA, and asymmetric PDRA models are calibrated to daily DAX 30 returns from December 1987 to July 2026 using differential evolution to match autocorrelation moments of raw returns, squared returns, and absolute returns. In the symmetric model, the estimated sensitivity is $\hat{s} = 0.479$. The asymmetric PDRA model gives $\hat{s}_{\text{gain}} = 0.489$ and $\hat{s}_{\text{loss}} = 0.337$. Under the model specification, these estimates imply that profitable trading outcomes increase trend followers' risk aversion, while losses reduce it. The larger estimate of s_{gain} suggests that risk aversion responds more strongly to gains than to losses in the asymmetric model. Both PDRA specifications improve the model fit along several dimensions, with some additional gains from the asymmetric extension. Both PDRA models reproduce the persistence of conditional volatility more closely than the benchmark model, as shown by the GARCH and FIGARCH estimates. The long-memory results are less uniform. The benchmark model is closer for raw returns, while the PDRA models perform better for squared and absolute returns in the Wald tests. The three models generate similar tail behavior under the Hill and Pickands estimators. Under the moment estimator, the PDRA models are closer to the empirical tails, with the asymmetric PDRA model giving the closest match. The two PDRA models also reproduce the volatility asymmetry in the DAX 30 more closely than the benchmark model, although the asymmetric PDRA model does not improve on the symmetric specification in both asymmetry diagnostics. The observable validation gives a clearer difference between the two PDRA models. Both match the within-trend volatility patterns more closely than the benchmark model, and the asymmetric PDRA model is closest overall across most of the robustness checks.

The paper can be extended in several directions. Direct inference on $s_{\text{gain}} - s_{\text{loss}}$ would allow the equality restriction $s_{\text{gain}} = s_{\text{loss}}$ to be tested explicitly. Separate confidence intervals for the two parameters do not provide a direct test of this restriction. The mixed Wald-test results suggest a need for further work, since none of the three specifications matches all long-memory features of the DAX 30 equally well. The empirical comparison could also be repeated for other financial markets and with out-of-sample data to examine whether the results are robust across different samples and market environments. In addition, the current specification represents exogenous influences through relatively simple stochastic shocks. Allowing for richer shock processes or additional sources of external variation would make it possible to study the profit-dependent risk-aversion mechanism under a wider range of market conditions.

Appendix A. Technical Proofs

A.1 Derivation of $\sigma_D^2 = r^2 \sigma_1^2$

Let σ_P^2 denote the annual variance of the fundamental price. The corresponding variance over one trading period is obtained by dividing the annual variance by the number of trading periods per year,

$$\sigma_1^2 = \frac{\sigma_P^2}{K}, \quad (\text{A-1})$$

where K is the number of trading periods per year. The annual dividend is assumed to satisfy $D_A = rp$, where D_A denotes the annual dividend. This assumes that annual dividends are proportional to the asset price. The proportionality factor is fixed at the annual risk-free rate r to avoid introducing an additional model parameter. Using $\text{Var}(cX) = c^2 \text{Var}(X)$ gives $\sigma_{D_A}^2 = r^2 \sigma_P^2$.

Following the same time-scaling assumption used for the price variance, the one-period dividend variance is assumed to satisfy

$$\sigma_D^2 = \frac{\sigma_{D_A}^2}{K}. \quad (\text{A-2})$$

Substituting $\sigma_{D_A}^2 = r^2 \sigma_P^2$ and (A-1) into (A-2) yields

$$\sigma_D^2 = \frac{\sigma_{D_A}^2}{K} = \frac{r^2 \sigma_P^2}{K} = r^2 \sigma_1^2. \quad (\text{A-3})$$

Hence, writing $\sigma_D^2 = q \sigma_1^2$ and comparing it with (A-3) immediately implies $q = r^2$.

A.2 Proof of Proposition 1

Proof. Consider the deterministic system obtained by setting $\varepsilon_t = \eta_t = 0$, $p_t^* = \bar{p}$, and $D_t = \bar{D} = (R - 1)\bar{p}$. The first condition removes the market-price and fundamental-price innovations, the second fixes the fundamental price at its long-run level, and the third fixes the dividend at its steady-state level. Since $\bar{p} + \bar{D} - R\bar{p} = \bar{p} + (R - 1)\bar{p} - R\bar{p} = 0$, the per-unit excess payoff of the risky asset is zero when the market price equals the fundamental value.

Let $\mathbf{x}^* = (p^*, u^*, v^*, a_2^*)^\top$ denote a fixed point of the resulting deterministic system. By definition, $p_{t+1} = p_t = p^*$, $u_t = u_{t-1} = u^*$, $v_t = v_{t-1} = v^*$, and $a_{2,t} = a_{2,t-1} = a_2^*$. The trend-reference equation $u_t = \delta u_{t-1} + (1 - \delta)p_t$ gives $u^* = \delta u^* + (1 - \delta)p^*$ at a fixed point, or equivalently $(1 - \delta)(u^* - p^*) = 0$. Since $\delta \in (0, 1)$, it follows that $u^* = p^*$. Similarly, the perceived-variance equation becomes $v^* = \delta v^* + \delta(1 - \delta)(p^* - u^*)^2$. Substituting $u^* = p^*$ gives $(1 - \delta)v^* = 0$, and hence, $u^* = p^*$, $v^* = 0$.

The fixed-point demands of the two trader types can now be evaluated. Since the fundamental price is fixed at $\bar{p}$, the demand of fundamentalists is

$$z_1^* = \frac{(1 - \alpha)(\bar{p} - p^*) - (R - 1)(p^* - \bar{p})}{a_1(1 + q)\sigma_1^2} = \frac{(R - \alpha)(\bar{p} - p^*)}{a_1(1 + q)\sigma_1^2}. \quad (\text{A-4})$$

Using $u^* = p^*$ and $v^* = 0$, the demand of trend followers becomes

$$z_2^* = \frac{\gamma(p^* - u^*) - (R - 1)(p^* - \bar{p})}{a_2^*(1 + q + bv^*)\sigma_1^2} = \frac{(R - 1)(\bar{p} - p^*)}{a_2^*(1 + q)\sigma_1^2}. \quad (\text{A-5})$$

Thus, at a fixed point, both demands are proportional to $\bar{p} - p^*$.

In the deterministic system, the price-adjustment equation is

$$p_{t+1} = p_t + \mu [q_{1,t}z_{1,t} + (1 - q_{1,t})z_{2,t}]. \quad (\text{A-6})$$

The fixed-point condition $p_{t+1} = p_t$, applied to Equation (A-6), together with $\mu > 0$, therefore requires $q_1^*z_1^* + (1 - q_1^*)z_2^* = 0$. Substituting Equations (A-4) and (A-5) and factoring out the common term $\bar{p} - p^*$ gives

$$(\bar{p} - p^*) \left[\frac{q_1^*(R - \alpha)}{a_1(1 + q)\sigma_1^2} + \frac{(1 - q_1^*)(R - 1)}{a_2^*(1 + q)\sigma_1^2} \right] = 0.$$

Under the parameter restrictions of the model, $q_1^* \in [0, 1]$, $\alpha \in (0, 1)$, $R > 1$, $a_1 > 0$, $a_2^* > 0$, $\sigma_1^2 > 0$, and $1 + q > 0$. The two terms inside the square brackets are therefore non-negative. If $q_1^* > 0$, the first term is strictly positive because $R - \alpha > 0$, while if $q_1^* < 1$, the second term is strictly positive because $R - 1 > 0$. At the boundary $q_1^* = 1$, the first term remains strictly positive, and at the boundary $q_1^* = 0$, the second term remains strictly positive. Consequently, the expression in square brackets is strictly positive for every $q_1^* \in [0, 1]$. It follows that $p^* = \bar{p}$.

Combining this result with $u^* = p^*$ and $v^* = 0$ yields $p^* = u^* = \bar{p}$ and $v^* = 0$. Substitution into the demand functions then gives

$$z_1^* = \frac{(R - \alpha)(\bar{p} - \bar{p})}{a_1(1 + q)\sigma_1^2} = 0, \text{ and } z_2^* = \frac{(R - 1)(\bar{p} - \bar{p})}{a_2^*(1 + q)\sigma_1^2} = 0.$$

Hence, $z_1^* = z_2^* = 0$.

The realized profit of trend followers at the fixed point is $\pi_C^* = z_2^*(p^* + \bar{D} - Rp^*)$. Since $z_2^* = 0$, it follows immediately that $\pi_C^* = 0$. The same conclusion also follows from the per-unit excess payoff, because $p^* + \bar{D} - Rp^* = \bar{p} + (R - 1)\bar{p} - R\bar{p} = 0$. Thus, both the position held by trend followers and the excess payoff per unit of the risky asset are zero at

the fixed point.

The fixed-point value of trend followers' risk aversion is determined by

$$a_{2,t} = \begin{cases} (1 - \omega)a_2 \exp(s_{\text{gain}}\pi_{C,t-1}) + \omega a_{2,t-1}, & \pi_{C,t-1} \geq 0, \\ (1 - \omega)a_2 \exp(s_{\text{loss}}\pi_{C,t-1}) + \omega a_{2,t-1}, & \pi_{C,t-1} < 0. \end{cases}$$

At the fixed point, $\pi_C^* = 0$, so the gain branch applies and $a_2^* = (1 - \omega)a_2 \exp(s_{\text{gain}}\pi_C^*) + \omega a_2^*$. Since $\pi_C^* = 0$, $\exp(s_{\text{gain}}\pi_C^*) = \exp(0) = 1$, and therefore $a_2^* = (1 - \omega)a_2 + \omega a_2^*$. Rearranging gives $(1 - \omega)(a_2^* - a_2) = 0$.

If $0 \leq \omega < 1$, then $1 - \omega > 0$, so $a_2^* = a_2$. The fixed point is

$$\mathbf{x}^* = (\bar{p}, \bar{p}, 0, a_2)^\top. \quad (\text{A-7})$$

If $\omega = 1$, the risk-aversion updating equation reduces to $a_{2,t} = a_{2,t-1}$. In this boundary case, the equation imposes no restriction requiring the fixed-point value of risk aversion to equal its baseline level a_2 . Instead, every constant positive risk-aversion level is preserved over time. The deterministic system therefore admits a continuum of fixed points of the form $(\bar{p}, \bar{p}, 0, c)^\top, c > 0$. This feature requires some clarification. It does not imply that the benchmark model of [He and Li \(2015, 2017\)](#) admits multiple equilibria. It arises because the present extended system includes trend followers' risk aversion as an endogenous state variable, whereas the benchmark model treats a_2 as a fixed parameter. When $\omega = 1$, the endogenous updating mechanism becomes inactive and the initial value of the risk-aversion state is carried forward indefinitely. The subsequent analysis therefore focuses on the economically relevant case $0 \leq \omega < 1$, for which the equilibrium value of trend followers' risk aversion is uniquely determined. Hence, for every $\omega \in [0, 1)$, the deterministic system admits the fundamental equilibrium given by (A-7). $\square$

A.3 Proof of Proposition 2

Proof. To derive the local dynamics, we linearize the deterministic system around the fundamental equilibrium. The deviations from the fundamental equilibrium are defined as $\tilde{p}_t = p_t - \bar{p}$, $\tilde{u}_t = u_t - \bar{p}$, $\tilde{v}_t = v_t$, and $\tilde{a}_t = a_{2,t} - a_2$.

The demand of fundamentalists is $z_{1,t} = \frac{(1-\alpha)(p_{t+1}^* - p_t) - (R-1)(p_t - \bar{p})}{a_1(1+r^2)\sigma_1^2}$. Since $p_{t+1}^* = \bar{p}$, this becomes

$$z_{1,t} = \frac{(\alpha - R)\tilde{p}_t}{a_1(1 + r^2)\sigma_1^2}. \quad (\text{A-8})$$

Similarly, the demand of trend followers is $z_{2,t} = [\gamma(p_t - u_t) - (R-1)(p_t - \bar{p})] / [a_{2,t}(1 + r^2 + bv_t)\sigma_1^2]$,

which becomes

$$z_{2,t} = \frac{(1 + \gamma - R)\tilde{p}_t - \gamma\tilde{u}_t}{a_2(1 + r^2)\sigma_1^2}. \quad (\text{A-9})$$

The numerator is zero at the fundamental equilibrium. Therefore, variations in $a_{2,t}$ and v_t affect the demand only through second-order terms and do not enter the linear approximation.

At the fundamental equilibrium, both trader types have zero demand and therefore earn zero realized profits. The argument of the switching function is therefore zero, implying that $m_t^* = \tanh(0) = 0$. Substituting $m_t^* = 0$ into the market fraction equation yields the equilibrium fraction of fundamentalists,

$$q_1^* = \frac{1 + m_0}{2}n_0 + \frac{1}{2}(1 - n_0) = \frac{1 + n_0m_0}{2}. \quad (\text{A-10})$$

For notational convenience, let $m_q = n_0m_0$, so that $q_1^* = (1 + m_q)/2$ and $1 - q_1^* = (1 - m_q)/2$.

The deterministic price-adjustment equation is $p_{t+1} = p_t + \mu[q_{1,t}z_{1,t} + (1 - q_{1,t})z_{2,t}]$, where the demands of the two trader types are weighted by their respective market fractions. To linearize this equation, both the demands and the market fractions must be evaluated around the fundamental equilibrium. Since both demands are zero at the fundamental equilibrium ($z_1^* = z_2^* = 0$), changes in the market fractions enter the price equation only when multiplied by changes in demand. These products are second-order terms and are omitted from the linear approximation. The market fractions can therefore be evaluated at their equilibrium values, giving

$$\tilde{p}_{t+1} = \tilde{p}_t + \mu[q_1^*z_{1,t} + (1 - q_1^*)z_{2,t}].$$

Substituting the linearized demand functions (A-8)–(A-9) and the equilibrium market fractions (A-10) yields

$$\tilde{p}_{t+1} = \tilde{p}_t + \frac{\mu(1 + m_q)}{2} \frac{(\alpha - R)\tilde{p}_t}{a_1(1 + r^2)\sigma_1^2} + \frac{\mu(1 - m_q)}{2} \frac{(1 + \gamma - R)\tilde{p}_t - \gamma\tilde{u}_t}{a_2(1 + r^2)\sigma_1^2}. \quad (\text{A-11})$$

Define $\bar{a} = a_2/a_1$ and $Q = 2a_2(1 + r^2)\sigma_1^2$. Collecting the coefficients of $\tilde{p}_t$ and $\tilde{u}_t$ gives

$$\tilde{p}_{t+1} = A\tilde{p}_t + B\tilde{u}_t, \quad (\text{A-12})$$

where

$$A = 1 + \frac{\mu}{Q} [(1 + m_q)\bar{a}(\alpha - R) + (1 - m_q)(1 + \gamma - R)], \text{ and } B = -\frac{\mu\gamma(1 - m_q)}{Q}.$$

The trend reference evolves according to $u_{t+1} = \delta u_t + (1 - \delta)p_{t+1}$. Subtracting $\bar{p}$ from both sides gives $\tilde{u}_{t+1} = \delta\tilde{u}_t + (1 - \delta)\tilde{p}_{t+1}$. Substituting the linearized price equation (A-12)

obtained above yields

$$\tilde{u}_{t+1} = (1 - \delta)A\tilde{p}_t + [\delta + (1 - \delta)B]\tilde{u}_t. \quad (\text{A-13})$$

Combining the linearized price equation (A-12) with the linearized trend equation (A-13) gives

$$\begin{pmatrix} \tilde{p}_{t+1} \\ \tilde{u}_{t+1} \end{pmatrix} = J \begin{pmatrix} \tilde{p}_t \\ \tilde{u}_t \end{pmatrix}, \quad J = \begin{pmatrix} A & B \\ (1 - \delta)A & \delta + (1 - \delta)B \end{pmatrix}. \quad (\text{A-14})$$

The Jacobian matrix characterizes the local dynamics around the fundamental equilibrium. Its eigenvalues are determined by the characteristic polynomial, which for a 2×2 matrix depends only on its trace and determinant. The trace and determinant of the Jacobian matrix are $\text{tr}(J) = \delta + A + (1 - \delta)B$ and $\det(J) = \delta A$. Hence, the characteristic polynomial is

$$\Gamma(\lambda) = \lambda^2 - \text{tr}(J)\lambda + \det(J),$$

which is exactly (4.5).

For a second-order polynomial $\lambda^2 - T\lambda + D$, the Jury criterion requires

$$1 - T + D > 0, \quad 1 + T + D > 0, \quad |D| < 1. \quad (\text{A-15})$$

Since $T = \delta + A + (1 - \delta)B$ and $D = \delta A$, (A-15) becomes

$$\Gamma(1) > 0, \quad \Gamma(-1) > 0, \quad |\delta A| < 1.$$

This completes the proof. □

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