Heston-Nandi GARCH Parameter Estimation Using Market Index Data

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Abstract

This essay focuses on the Heston-Nandi GARCH (1,1) model from Heston and Nandi (2000) for parameter estimation using simulation-based approach and real market returns. The primary objective is to compare the theoretical parameterization formulas proposed by Badescu et al. (2019) with the estimated relationships across various frequencies and parameter values. Three U.S. market indices are employed for our parameter estimation. The study aims to provide investors with insights into how data frequency can influence the accuracy of estimation results for GARCH (1,1) model.

Contents

1 Introduction

The Generalized Autoregressive Conditional Heteroskedasticity Model (GARCH) is widely used in financial analysis and modeling because of its ability to capture the dynamic nature of volatility in financial time series data (Bollerslev, 1986). The GARCH model effectively addresses key properties of financial data including volatility clustering, where periods of high or low volatility tend to cluster together. Extending the original GARCH model Heston and Nandi proposed closed-form formula for European option pricing under the assumption that the spot asset price follows a GARCH (p, q) process in discrete time setting (Heston & Nandi, 2000).

Accurate GARCH parameter estimation is essential for investors and traders to better understand portfolio performance and tailor investment strategies to suit their chosen frequency of portfolio rebalancing. Most financial participants in the market primarily focus on daily frequency for analyzing asset returns and assume that GARCH parameters remain constant across various frequencies. However, empirical studies suggest that the GARCH process at different frequencies may lack consistency, potentially affecting the accuracy of estimated parameters. In both Heston and Nandi's paper as well as another paper by Badescu, Cui, and Ortega (Badescu et al., 2019), they proposed parameterization formulas for the relationship between parameter values and various frequencies. Nevertheless, it's worth noting that further validation of the proposed parameterization formulas for either paper has not yet been carried out. Our objective is to bridge the existing literature gap by investigating the relationship between different frequencies and estimated parameters using the Heston and Nandi's GARCH model with real returns (Heston & Nandi, 2000).

This essay will specifically focus on the GARCH (1,1) model, which denotes that GARCH model includes one autoregressive lag (ARCH lag) and one moving average lag (GARCH lag). Our empirical exploration will provide a practical insight of how various frequencies are linked to parameter estimation. First, we will analyze the parameter estimation results by simulating hourly, daily and weekly returns for 10 years. As the true parameter values are known for simulated data, we can test the reliability of our estimation by checking the convergence of different initial value sets to the same optimal parameter value. After confirmation on the reliability of our estimator from simulation, we will employ the same methodology to perform parameter estimation on selected market index data. While previous studies predominately focused on single market instruments, such as the S&P 500, to validate their theoretical findings, our research expands to encompass three commonly used US stock market indices spanning from 2008 to 2022: the Russell 2000 (RUT), S&P500 (SPX) and Dow Jones Industrial Average (DOW).

This paper is structured as follows: in Section 2, we discuss the existing literature on the original GARCH model and its extensions. Section 3 presents the fundamental properties and setup of the HN-GARCH (1,1) model, along with the proposed parameterization formulas. Section 4 discusses the interpretation and constraints of HN-GARCH model parameters, the application of the maximum likelihood function, and our simulation methodology. In Section 5, we test our estimation method using simulated returns, followed by a discussion of our results. Section

6 describes the index data used for estimation and the estimation methodology. We present the estimation results and plots illustrating the relationships between frequencies and estimated parameters, with a focus on the Russell 2000. Section 6 presents the key findings of this essay, suggests areas for improvement, and outlines potential directions for future study.

2 Literature Review

The Autoregressive Conditional Heteroskedasticity (ARCH) model was first introduced in 1982 by Robert F. Engle, laying the foundation for incorporating time-varying volatility into financial modeling (Engle, 1982).

In the late 1980s, the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model extended the original ARCH model by incorporating additional lagged conditional terms for a more flexible lag structure (Bollerslev, 1986). GARCH is widely employed in various areas of volatility analysis and forecasting, including portfolio management, option pricing and risk management, enabling the capture of the dynamic nature of financial data volatility.

The introduction of the continuous-time stochastic volatility model by Heston marked a significant advancement over the original Black-Scholes model (BS model) as it incorporates stochastic volatility into option pricing (Heston, 1993). It addresses a shortcoming of BS model by allowing the volatility of option price to be stochastic rather than constant. Nonetheless, continuous-time setting makes the testing and implementation of the model parameters in the Heston 1993 model challenging. This is due to volatilities of option returns from GARCH process are not directly observable. As a result, substantial computational work is required to estimate implied volatilities from historical option records (Heston, 1993). Inspired by Heston 1993 model, Heston and Nandi later introduced a discrete-time model, offering a closed-form formula for option pricing for the first time, which is known as the HN-GARCH model (Heston & Nandi, 2000). In the HN-GARCH model, the spot option price follows a new GARCH (p, q) process, which is described as a function of the current spot price and the observed historical spot prices. This model and most of its extensions rely on daily asset log returns for parameter estimation. The HN-GARCH model inspired several extensions in the area of asset pricing. Christoffersen, Heston, and Jacobs (2013) presented a more general HN-GARCH model by introducing a U-shaped pricing kernel for the valuation of index returns (Christoffersen et al., 2013). Another paper by Budescu et al. builds upon the original HN-GARCH model and introduces a closed-form pricing formula for volatility derivatives, particularly variance swap pricing (Badescu et al., 2019). Their study explores the weak diffusion limit within the framework of a variance-dependent pricing kernel and shows that the discretely sampled GARCH swap rate converges to its continuous-time counterpart. To determine the affine GARCH diffusion limit, the authors propose theoretical formulas that establish the connection between GARCH parameters and different frequencies.

Our paper contributes to the current literature through the following ways. First, we establish connections between the model set up in Heston and Nandi's (2000) as well as Badescu's (2019), comparing the suggested parameterization formulas from both papers. We also discuss some of the criticisms about the proposed formulas. Secondly, we test an estimation methodology for the HN-GARCH (1,1) model across various frequencies using simulated data. Additionally, we apply real financial market data to compare the relationship between the proposed and estimated values of parameters across different frequencies. Our primary contribution lies in bridging the gap by using real returns to demonstrate the differences between theoretical parameterization formulas and their real-world relationships.

3 Model Set Up

3.1 Heston-Nandi's GARCH (1,1)

We assume that our spot index price follows the Heston-Nandi's GARCH (1,1) process. Heston-Nandi's GARCH (1,1) model for a frequency Δ can be specified as:

$$
\begin{aligned}
y_t &= \log(S_t) - \log(S_{t-\Delta}) = r(\Delta) + \lambda(\Delta)h_t + \sqrt{h_t}z_t, && (1)\\
h_t &= \omega(\Delta) + \beta(\Delta)h_{t-\Delta} + \alpha(\Delta)\left(z_t - \gamma(\Delta)\sqrt{h_{t-\Delta}}\right)^2, && (2)
\end{aligned}
$$

where $\{z_t\}_t$ is a sequence of independent standard normal random variables, the parameters are: $\lambda(\Delta)$, $\omega(\Delta)$, $\beta(\Delta)$, $\alpha(\Delta)$ and $\gamma(\Delta)$. The dependence on Δ is explicit in their representation. $r(\Delta)$ is the short-term interest free rate for the time interval: Δ.

We aim to understand how the GARCH (1,1) parameters depend on Δ with real data in a discrete-time setting (weekly, every-second-day, daily, half-day, two-hour, hourly). It will provide us relationship between estimated model parameter value and various frequencies.

We use $\Delta = 1$ to represent daily frequency to be consistent with Heston-Nandi's GARCH (1,1) model. This is special as most estimations choose daily to be the benchmark frequency, so we denote the parameters of this estimation with a subindex 1, e.g., generically $\theta(1) = \theta_1$:

$$
\begin{aligned}
y_t &= y_{t-1} + r_1 + \lambda_1 h_t + \sqrt{h_t}z_t, && (3)\\
h_t &= \omega_1 + \beta_1 h_{t-1} + \alpha_1\left(z_t - \gamma_1\sqrt{h_{t-1}}\right)^2, && (4)
\end{aligned}
$$

Define the variance per unit of time as $v_t = \frac{h_t}{\Delta}$, this leads to:

$$\begin{aligned}
y_t - y_{t-\Delta} &= r(\Delta) + \lambda(\Delta) v_t \Delta + \sqrt{\Delta}\sqrt{v_t} z_t && (5)\\
v_t &= \frac{\omega(\Delta)}{\Delta} + \beta(\Delta) v_{t-\Delta} + \frac{\alpha(\Delta)}{\Delta}\left(z_t - \gamma(\Delta)\sqrt{\Delta}\sqrt{v_{t-\Delta}}\right)^2 && (6)\\
&= \omega_v + \beta_v v_{t-\Delta} + \alpha_v \left(z_t - \gamma_v \sqrt{v_{t-\Delta}}\right)^2 && (7)
\end{aligned}$$

where,

$$\omega_v = \frac{\omega(\Delta)}{\Delta}, \quad \beta_v = \beta(\Delta), \quad \alpha_v = \frac{\alpha(\Delta)}{\Delta}, \quad \gamma_v = \gamma(\Delta)\sqrt{\Delta}$$

3.2 Heston and Nandi's Parameterization

The relationships between parameters for variance depending kernel are introduced by Nelson as (Nelson, 1990):

$$\theta = \omega + \frac{\alpha}{\kappa}, \sigma = 2\alpha\gamma, \kappa = 1 - \beta - \alpha\gamma^2$$

where:

- θ is related to the average volatility in the model consists of a constant term (ω) and a term proportional to the inverse of the mean-reverting speed (κ).
- κ is involved in the mean-reverting behavior of the volatility. The term $1-\beta$ is often related to the long-term variance, and the term $\alpha\gamma^2$ represents the volatility risk premium.
- σ is directly proportional to the product of the volatility of the variance process (α) and the correlation between the asset price process and the volatility process (γ).

Note: More details about the convergence of the discrete-time process of the GARCH (1,1) model can be found in Nelson (1990, p. 15). Other parameter nterpretation (α, λ, ω, β, γ) is illustrated in Section 4.1.

By imposing the same diffusion limit (where a discrete-time model is transformed to a continuous-time model), Heston and Nandi (2000) compute the convergence of v_t in order to propose the following parameterization (Heston & Nandi, 2000):

$$\begin{aligned}
\gamma(\Delta) &= \frac{2}{\sigma\Delta} - \frac{\kappa}{\sigma} && (8)\\
\omega(\Delta) &= \left(\kappa\theta - \frac{1}{4}\sigma^2\right)\Delta^2\\
\beta(\Delta) &= 0\\
\alpha(\Delta) &= \frac{1}{4}\sigma^2\Delta^2\\
\lambda(\Delta) &= \lambda
\end{aligned}$$

As v_t is observable and they compute the following equation by taking the conditional expectation at time $t-\Delta$:

$$\frac{E_{t-\Delta}\left[v_{t+\Delta}-v_t\right]}{\Delta} \underset{\Delta\to 0}{\to} \kappa\left(\theta-v_t\right)$$
$$\frac{Var_{t-\Delta}\left[v_{t+\Delta}\right]}{\Delta} \underset{\Delta\to 0}{\to} \sigma^2 v_t$$
$$Corr_{t-\Delta}\left[v_{t+\Delta}, y_t\right] \underset{\Delta\to 0}{\to} 1 \text{ or } -1$$

Note: The above computation is used for deriving the diffusion limit for our GARCH (1,1) model. For detailed proof and explanation, refer to Heston and Nandi (2000) Appendix B.

Hence the discrete-time model, Heston and Nandi's (2000) GARCH, converges to continuous time limit (Heston & Nandi, 2000):

$$\begin{aligned} dy_t &= \left(r+\lambda v_t\right) dt+\sqrt{v_t} dW_t \\ dv_t &= \kappa\left(\theta-v_t\right) dt+\sigma \sqrt{v_t} dW_t \end{aligned} \tag{9}$$

The parameterization in equation (8) is written in terms of the continuous time parameters, which are unknown, we can better write it in terms of the one-day parameters, as they are the most commonly estimated. First note:

$$\gamma_1=\frac{2}{\sigma}-\frac{\kappa}{\sigma}, \quad \omega_1=\left(\kappa\theta-\frac{1}{4}\sigma^2\right), \quad \alpha_1=\frac{1}{4}\sigma^2 \tag{10}$$

This leads to the following important relationship:

$$\begin{aligned} \gamma(\Delta) &= \frac{2}{\sigma\Delta}-\frac{\kappa}{\sigma}=\frac{1}{\sqrt{\alpha_1}}\left(\frac{1-\Delta}{\Delta}\right)+\gamma_1 \\ \omega(\Delta) &= \omega_1\Delta^2 \\ \beta(\Delta) &= 0 \\ \alpha(\Delta) &= \alpha_1\Delta^2 \\ \lambda(\Delta) &= \lambda \end{aligned} \tag{11}$$

We can see several drawbacks of HN-GARCH parameterization:

- when $\Delta=1$, we do not recover the corresponding parameters.
- $\beta(\Delta)$ is always 0, this does not coincide with the estimated value at $\Delta=1$.
- $\lambda(\Delta)$ is always constant.
- $\gamma(\Delta)$ has a strange behaviour when Δ is small.

- for $\Delta > 1$, $\gamma(\Delta)$ could be negative.
- the proposed relation is not supported by data.

3.3 Badescu's Parameterization.

The notation in Badescu et al.(2019) paper is different, the authors work with the representation (taking $t = k\Delta$ and $\epsilon_{k\Delta} = z_t$) (Badescu et al., 2019):

$$
\begin{aligned}
y_t &= \log(S_t) - \log(S_{t-\Delta}) = (r + \lambda h_t)\Delta + \sqrt{\Delta}\sqrt{h_t}z_t, \\
h_t &= \omega_B(\Delta) + \beta_B(\Delta)h_{t-\Delta} + \alpha_B(\Delta)\left(z_t - \gamma_B(\Delta)\sqrt{h_{t-\Delta}}\right)^2,
\end{aligned}
\tag{12}
$$

Note their h_t is actually the v_t in Heston and Nandi's notation, hence we have to be careful with the mapping (where we use different notation to represent the same parameter), this is why the parameters have the subindex B. Writting their representation in Heston and Nandi's notation which will be used in this paper, replacing their $h_t = v_t$ with $\frac{h_t}{\Delta}$:

$$
\begin{aligned}
y_t &= \log(S_t) - \log(S_{t-\Delta}) = \left(r + \lambda\frac{h_t}{\Delta}\right)\Delta + \sqrt{\Delta}\sqrt{\frac{h_t}{\Delta}}z_t \\
&= r\Delta + \lambda h_t + \sqrt{h_t}z_t
\end{aligned}
$$

$$
\begin{aligned}
\frac{h_t}{\Delta} &= \omega_B(\Delta) + \beta_B(\Delta)\frac{h_{t-\Delta}}{\Delta} + \alpha_B(\Delta)\left(z_t - \gamma_B(\Delta)\sqrt{\frac{h_{t-\Delta}}{\Delta}}\right)^2 \\
&\Leftrightarrow \\
h_t &= \omega_B(\Delta)\Delta + \beta_B(\Delta)h_{t-\Delta} + \alpha_B(\Delta)\Delta\left(z_t - \frac{\gamma_B(\Delta)}{\sqrt{\Delta}}\sqrt{h_{t-\Delta}}\right)^2
\end{aligned}
$$

We then have the following mapping between Badescu's parameters and Heston and Nandi's parameters:

$$\begin{aligned}
\omega(\Delta) &= \omega_B(\Delta)\Delta \\
\beta(\Delta) &= \beta_B(\Delta) \\
\alpha(\Delta) &= \alpha_B(\Delta)\Delta \\
\gamma(\Delta) &= \frac{\gamma_B(\Delta)}{\sqrt{\Delta}} \\
\lambda(\Delta) &= \lambda \\
r(\Delta) &= r\Delta.
\end{aligned}$$

[1].

Badescu et al. propose the following parameterization, again based on capturing the same limiting in Heston and Nandi's model (Badescu et al., 2019).

$$\begin{aligned}
\gamma_B(\Delta) &= \frac{\gamma_1}{\sqrt{\Delta}} \qquad (13)\\
\omega_B(\Delta) &= \omega_1\Delta \\
\alpha_B(\Delta) &= \alpha_1\Delta \\
\beta_B(\Delta) &= 1-\left(1-\beta_1-\alpha_1\gamma_1^2\right)\Delta-\alpha_B(\Delta)\gamma_B^2(\Delta) \\
&= 1-\left(1-\beta_1-\alpha_1\gamma_1^2\right)\Delta-\alpha_1\gamma_1^2
\end{aligned}$$

In particular, the authors show the following connection between continuous time parameters and one-day parameters:

$$\theta=\frac{\omega_1+\alpha_1}{1-\beta_1-\alpha_1\gamma_1^2}, \quad \sigma=2\alpha_1\gamma_1, \quad \kappa=1-\beta_1-\alpha_1\gamma_1^2 \tag{14}$$

We can rewrite Badescu's parameterization formulas in terms of our notation as:

$$\begin{aligned}
\gamma(\Delta) &= \frac{\gamma_B(\Delta)}{\sqrt{\Delta}}=\frac{\gamma_1}{\Delta} \qquad (15)\\
\omega(\Delta) &= \omega_B(\Delta)\Delta=\omega_1\Delta^2 \\
\beta(\Delta) &= \beta_B(\Delta)=1-\left(1-\beta_1-\alpha_1\gamma_1^2\right)\Delta-\alpha_1\gamma_1^2 \\
\alpha(\Delta) &= \alpha_B(\Delta)\Delta=\alpha_1\Delta^2 \\
\lambda(\Delta) &= \lambda \qquad (16)
\end{aligned}$$

A few limitations of Badescu's parametrization:

[1]Badescu's (2019) paper assumes λ is constant to ensure average spot return only depends on the level of risk.

- when $\Delta > 1$, then $\beta(\Delta)$ could be negative.
- $\lambda(\Delta)$ is always the constant.
- the proposed relation is not supported by data.

3.4 Summary and Comments

The parametrizations in equations (11) for HN, and equation (15) for Badescu, are the only two examples available on how the parameters depend on frequency. We want to check how real data differs from those proposals.

We also want to propose a more realistic parametrization. For this task, we first need to compute the following objects generically (i.e., using the generic representation from equation (5))

$$
\begin{aligned}
& \frac{E_{t-\Delta}\left[v_{t+\Delta} - v_t\right]}{\Delta} = \frac{\omega_v + \left(\beta_v + \alpha_v \gamma_v^2 - 1\right) v_t + \alpha_v}{\Delta} \\
= \quad & \frac{\frac{\omega(\Delta)}{\Delta} + \left(\beta(\Delta) + \alpha(\Delta)\gamma^2(\Delta) - 1\right) v_t + \frac{\alpha(\Delta)}{\Delta}}{\Delta} \\
& \frac{Var_{t-\Delta}\left[v_{t+\Delta}\right]}{\Delta} = \frac{2\alpha_v^2 + 4\alpha_v^2\gamma_v^2 v_t}{\Delta} \\
= \quad & \frac{2\frac{\alpha^2(\Delta)}{\Delta^2} + 4\frac{\alpha^2(\Delta)}{\Delta}\gamma^2(\Delta) v_t}{\Delta} \\
& Corr_{t-\Delta}\left[v_{t+\Delta}, y_t\right] = \frac{-sign\left(\gamma_v\right)\sqrt{2\gamma_v^2 v_t}}{\sqrt{1 + 2\gamma_v^2 v_t}}
\end{aligned}
$$

Here, we want to compute the continious-time limit for the varaince per unit term, $v(t)$ as Δ goes to zero. Then we can propose specific choices of $\omega(\Delta),\ \ \beta(\Delta),\ \ \alpha(\Delta),\ \ \gamma(\Delta)$ in terms of κ, σ and θ such that:

$$
\begin{aligned}
\frac{E_{t-\Delta}\left[v_{t+\Delta} - v_t\right]}{\Delta} & \underset{\Delta \to 0}{\rightarrow} \kappa\left(\theta - v_t\right) \\
\frac{Var_{t-\Delta}\left[v_{t+\Delta}\right]}{\Delta} & \underset{\Delta \to 0}{\rightarrow} \sigma^2 v_t \\
Corr_{t-\Delta}\left[v_{t+\Delta}, y_t\right] & \underset{\Delta \to 0}{\rightarrow} 1 \text{ or } -1
\end{aligned}
$$

Lastly we use the previous connection from (13) to write $\omega(\Delta),\ \ \beta(\Delta),\ \ \alpha(\Delta),\ \ \gamma(\Delta)$ in terms of $\omega_1,\ \ \beta_1,\ \ \alpha_1,$ γ_1.

Note: The goal of above computation is to connect parameters at different frequency with daily parameter values, we will focus on equation (15) for parameter estimation in this paper. Heston and Nandi (2000) provides the proof of convergence of single lag GARCH model (HN-GARCH (1,1)) in Appendix B (Heston & Nandi, 2000).

4 Simulation

4.1 Parameter Interpretation

The parameter set, denoted as θ, in the HN-GARCH model comprises five parameters, which can be represented as follows:

$$\theta = (\lambda, \omega, \alpha, \beta, \gamma) \tag{17}$$

- λ: the risk premium term serves as an indicator of asset volatility. A higher value of λ means that the underlying assets exhibit greater volatility. Those assets tends to be priced higher than risk-natural assets to due to higher expected extreme gain.

- ω and β: these are parts of the volatility process, where $h(t+1)$, the next-day volatility, is equal to ω plus a percentage (β) of the previous day's volatility, $h(t)$. The value of β reflects the extent to which past volatility persists in affecting current volatility. A β value closer to 1 indicates that a significant portion of the current day's volatility is linked to the volatility observed on the previous day.

- α: governs the shape of the distribution of $h(t)$. A smaller α means the distribution of $h(t)$ has a narrower tail. When α equals zero, it suggests that the volatility process $h(t)$ is not random.

- γ: controls the asymmetry of the $h(t)$ distribution. When γ equals zero, $h(t)$ is symmetric, meaning that extreme noise values on both sides of the distribution have equal effects on $h(t)$. In other words, a large positive return shock has the same impact on the variance as a large negative shock.

4.2 Parameter Constraints

To ensure stationary condition is met for HN-GARCH process and $h(t) > 0$ for all time period, we will impose following constraint on all parameters: $\beta + \alpha \cdot \gamma^2 < 1$.

Furthermore, additional constraints will be imposed on each parameter before we begin constructing our initial parameter sets. We will provide a brief explanation of the rationale about each constraint. More detailed information about the proof, can be referred to Ye's paper (Ye, 2021) and Bollerslev's GARCH paper from 1986 (Bollerslev, 1986).

- λ constraint: as empirical findings suggest that log asset returns experience a positive risk premium, we assume that $\lambda > 0$.

- α, and β constraints: to ensure the positivity of conditional variance $h(t)$, we impose the following constraints: $\alpha > 0$, and $\beta > 0$.

- ω constraint: to avoid negative value of ω we keep $\omega = 0$ for all of our estimation.

4.3 Application of Maximum Likelihood Function (MLE)

Maximum likelihood estimator is widely used in finding the parameter values that best fit the underlying objective function. We want to derive the likelihood function in order to find the optimal initial parameter values based on asset return. As we use estimator **fminunc** in Matlab, the parameter set with the smallest function value where $\hat{\theta}(\lambda, \omega, \beta, \gamma)$ will be the optimal parameter set.

In a discrete time setting, we can observe th e initial asset price and the next N-day of asset price $S(0), S(1), ...S(0)$ with true parameter value θ_0 (values of θ_0 are reported in Appendix A). Then, we take the log for each day asset prices and denoted as $X_1, ...X_N$, we can derive the log likelihood function as:

$$\log L(\theta) = -\frac{1}{2}\sum_{t=1}^{T}\left(\log(2\pi) + \log(h_t) + z(t)^2\right) \tag{18}$$

where $z(t)$ is implied daily noise for log asset price returns. Now we can obtain $\hat{\theta}$ which maximize the joint density function for log asset price through

$$\hat{\theta} = \arg\max \log L(\theta) \tag{19}$$

For a random sample of n independent and identically distributed (i.i.d.) random variables, denoted as $X_1, X_2, \ldots, X_n$, representing the log prices of an asset, we can construct the Fisher's information matrix to quantify how the log-likelihood function varies with changes in our parametric space θ:

$$I(\theta) = -E\left[\frac{\partial^2 \log l(X|\theta)}{\partial\theta\partial\theta^T}\right] \tag{20}$$

The estimator $\hat{\theta}$ obtained in Equation 19 has nice statistical properties that we can leverage. It is strongly consistent and asymptotically normally distributed. We can rewrite the Fisher information matrix as:

$$I(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left[\frac{\partial^2 \log l(x_i|\theta)}{\partial\theta\partial\theta^T}\right] \tag{21}$$

By averaging the second partial derivative with respect to the gradient matrix, the standard error can be expressed in terms of the gradient, denoted as D:

$$SE = \sqrt{\frac{1}{D^2}} \tag{22}$$

An alternative method for computing standard errors is through simulation by applying the law of large number.

4.4 Simulation Methodology

Before performing the estimation on real data, we need to investigate whether various initial value sets all converge towards a single global minimum, which represents the optimal value for our objective function. Since the true value for each parameter is known in the simulation, we can verify whether the confidence intervals of the estimated parameters encompass the true parameter values. Consequently, this serves as an indicator that our code successfully recovers the true parameter values.

4.5 Parameter Set up

As discussed by Ye, the selection of initial parameter values can impact the accuracy of our estimation because these parameters can determine the moments and correlations of stock returns (Ye, 2021). The author compared three sets of parameter values: Heston-Nandi 2000 (H-N 2000) parameters, Christoffersen-Jacobs 2006 (C-J 2006) parameters, and Christoffersen-Jacobs 2013 (C-J 2013) parameters and suggested using the C-J 2013 parameters. This set encompasses a longer historical period of 21 years, in contrast to H-N 2000 parameter set, which only spans 3 years. Furthermore, it aligns more closely with recent financial market data. Therefore, we also selected the C-J 2013 set as the reference parameters to configure our daily parameters for simulating log price returns.

Christopersen-Heston-Jacobs 2013 parameters

$$\begin{cases} \lambda & = 1.094, \\ \omega_1 & = 0, \\ \alpha_1 & = 3.364 \times 10^{-6}, \\ \beta_1 & = 0.838, \\ \gamma_1 & = 196.82, \end{cases} \tag{23}$$

We will keep the same initial values and interest rate where:

$$r_0 = 0, \qquad s_0 = 0, \qquad h_1 = 1.0617e^{-4}.$$

For the 10-year daily return simulation, we chose the true parameter values as outlined in Equation 23. We created 10 different sets of initial conditions for daily frequency ($\Delta = 1$) to assess whether most of these sets would converge to the same optimal function value. The estimation results for the simulated returns are available in Appendix A. We applied the same methodology to the simulate 10-year hourly and weekly returns. The true parameters for weekly and hourly log returns were computed using Badescu's parameterization formulas based on daily parameter values. We employed $\Delta = 5$ for weekly data and $\Delta = 1/8$ for hourly data because our weekly index data comprises 5 trading days in a week, and hourly data includes 8 hourly observations for each trading day.

For weekly frequency ($\Delta = 5$):

$$\begin{cases} \lambda(5) & = \lambda = 1.094, \\ \omega(5) & = 5\omega_1 = 0, \\ \alpha(5) & = 5\alpha_1^2 = 1.68e \times 10^{-5}, \\ \beta(5) & = 1 - 5(1 - \beta_1 - \alpha_1\gamma_1^2) - \alpha_1\gamma_1^2 = 0.711, \\ \gamma & = \frac{\gamma_1}{\sqrt{5}} = 88.021. \end{cases} \tag{24}$$

For hourly frequency ($\Delta = 1/8$):

$$\begin{cases} \lambda(1/8) & = \lambda = 1.094, \\ \omega(1/8) & = \frac{1}{8}\omega_1 = 0, \\ \alpha(1/8) & = \frac{1}{16}\alpha_1 = 2.10 \times 10^{-7}, \\ \beta(1/8) & = 1 - \frac{1}{8}(1 - \beta_1 - \alpha_1\gamma_1^2) - \alpha_1\gamma_1^2 = 0.866, \\ \gamma & = \frac{\gamma_1}{\sqrt{1/8}} = 556.691. \end{cases} \tag{25}$$

4.6 Discussion

Most of our initial parameter sets converge to the optimal function value. We reported the estimation results for 2 out of the 10 initial parameter conditions in Tables 9 for our benchmark frequency, daily, where $\Delta = 1$. We also reported some parameter sets that do not converge to the optimal function value. Our analysis indicates that the estimation results are consistent with different initial value sets. In Table 10, we want to learn if our estimation method is still valid when the true parameter value β=0 while fix all other parameter values. Zero β value indicates that the next day variance, $h(t+1)$ is not correlated with previous day variance, $h(t)$.

Now, we can simulate weekly returns ($\Delta = 5$) and hourly ($\Delta = 1/8$) returns and perform estimation by employing the methodology discussed above. We reported the estimation results in Table 11 and Table 12. The three tables for hourly, daily and weekly simulated data show that the confidence intervals decrease for higher frequencies. Furthermore, we compare the standard errors across the three frequencies for the same estimated parameters and observe that higher frequency results in lower standard errors. This observation confirms the consistency of our estimation methods which aligns with the application of the law of large numbers in the calculation of standard errors.

5 Estimation

5.1 Index Data

We selected three index funds to perform HN-GARCH (1,1) parameter estimation: Russell 2000 (RUX), Standard and Poor 500 (SPX), and Dow Jones Industrial Average (DOW), spanning from 2008 to 2022. Our selection aims to cover a wide spectrum of U.S. stocks, providing a comprehensive representation of the U.S. stock market.

The S&P 500, comprises the 500 largest publicly traded companies on U.S. stock exchanges and serves as a benchmark for the overall performance of the U.S. economy. Russell 2000, on the other hand, represents the small to medium-cap segment of U.S. companies, making it particularly interesting for observing potential differences in volatility patterns. Finally, the Dow Jones Index is one of the oldest indices, comprising the 30 largest U.S. companies, with a focus on the industrial sector that significantly influences U.S. economic growth.

In our analysis, we aimed to ensure a sufficiently large number of observations, N, particularly for the lowest frequency, which is weekly. A substantial large enough N is crucial for achieving a consistent estimator for the GARCH model. However, we must consider the trade-off between estimation precision and computational efficiency.

If we opt for a lager sample size, it is likely to yield smaller standard errors for parameter estimation, which results in a more precise parameter estimates. Nevertheless, when it comes to higher frequency, extensive large data sets can become burdensome in terms of computational time.

Therefore, in deciding on the sample size, we should consider the precision we require for parameter estimates while also taking into account the computational demands. We employed two separate data sources for our estimation: daily and weekly data is derived from Barchart and hourly data is derived from Firstratedata. The raw data comprises open, close, high, and low prices for each index. We use closing price to calculate log asset returns, denoted as y_t, using the following formula:

$$y_t = \ln\left(\frac{S_t}{S_{t-1}}\right) \tag{26}$$

5.2 Methodology

Section 4 demonstrates the success of our maximization method in recovering the true parameter values. We applied the same methodology used in the simulation to estimate parameters using real-world index data. However, we encountered the challenge of not having knowledge of the true parameter values for real index data. To address this challenge, we aimed to use a wide range of initial parameter sets to assess the consistency of estimation results, ensuring that most sets converge to the same minimum point. We formed our initial parameter

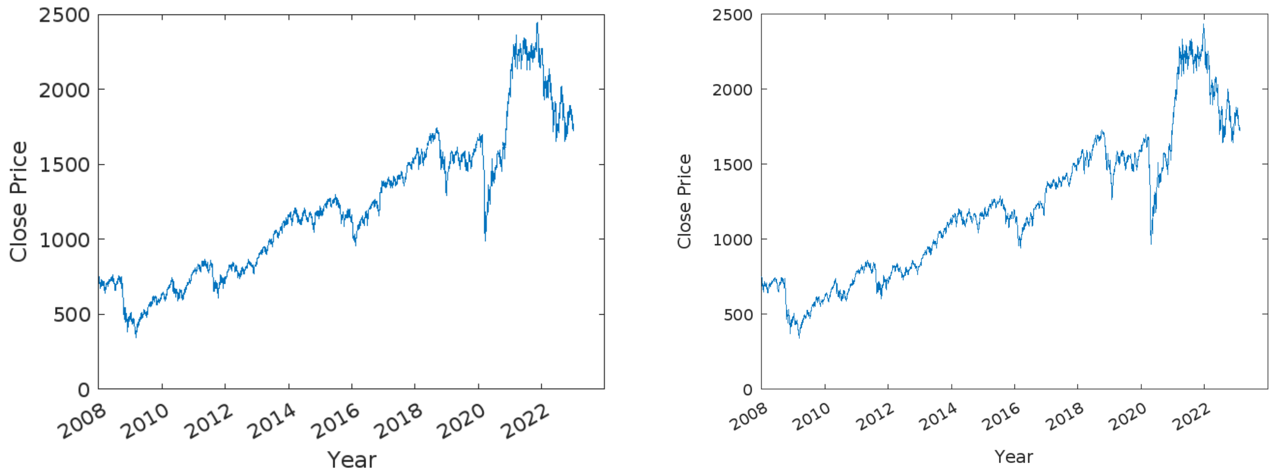


Figure 1: Close Price Plots from Hourly (left) and Daily (right) Data.
Notes: Both figures display the daily close price trend but with different data sources. Left figure is using hourly data and taking the close price at the end of each trading day to form daily data, while the right figure uses direct price information from daily data.

sets by combining the following values:

$$\begin{cases} \lambda = 1, 4, 9 \\ \omega_1 = 0 \\ \alpha_1 = 3 \times 10^{-6}, 5 \times 10^{-6}, 8 \times 10^{-6}, \\ \beta_1 = 0.6, 0.75, 0.9 \\ \gamma_1 = 150, 250, 350 \end{cases} \tag{27}$$

After combining the values in Equation 27, we will have a total of $3 \times 3 \times 3 \times 3 = 81$ initial sets to start with. In order to meet the stationary condition, we need to impose:

$$\beta_1 + \alpha_1 \gamma_1^2 < 1 \tag{28}$$

Following the exclusion of sets that do not meet the criteria outlined in Equation 28, we move on to conduct maximum likelihood estimation on the chosen index data. We employ the same MATLAB optimizer function **fminunc** and minimal value obtained in estimation corresponds to the optimal function value. Table 1 reports the initial set with the optimal function value (smallest value).

The optimal estimated parameter values for the Russell 2000 daily data are used as the initial values for the closest next frequencies: half-day ($\Delta = 1/2$) and other-second day ($\Delta = 2$). We adjust the estimated parameter values as reported in Table 1 to establish a range until we have 10 initial parameter sets for each frequency. We always ensure that the stationary condition is satisfied at any stage. Tables 2 present the initial value sets along with the optimal function value for $\Delta = 1/2$ and $\Delta = 2$.

We then repeat these steps for the next closest frequencies: every two hours ($\Delta = 1/4$) and weekly ($\Delta = 5$).

Table 1: Russell 2000 Daily Results ($\Delta = 1$)

(a) Daily Estimation Result using Daily Data

	λ	α	β	γ
Initial value	1.000	3.000×10^{-6}	0.600	150.000
Estimate value	1.016	7.104×10^{-6}	0.814	147.083
Standard error	(1.005)	(5.783×10^{-7})	(0.012)	(10.714)
Confidence Interval	[-0.955, 2.986]	[5.970×10^{-6}, 8.237×10^{-6}]	[0.790, 0.838]	[126.083, 168.083]
Function value	-10995.177			
N	3780			

(b) Daily Estimation Result using Hourly Data

	λ	α	β	γ
Initial value	1.000	3.000×10^{-6}	0.600	150.000
Estimate value	1.029	7.231×10^{-6}	0.817	144.330
Standard error	(1.009)	(6.007×10^{-7})	(0.013)	(11.092)
Confidence Interval	[-0.957, 2.998]	[6.053×10^{-6}, 8.408×10^{-6}]	[0.792, 0.842]	[122.589, 166.071]
Function value	-10894.737			
N	3750			

Notes: Estimation in Panel (a) is performed using Barchart daily data. The parameter sets reported for daily data serve as the starting point to construct ranges for the next frequencies: half-day and every-second day. In Panel (b), we utilize hourly data from Firstrate to construct daily data and perform the estimation using the same methods as in Panel (a). Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

5.3 Plots of Log Close Price from Simulated Data

Table 2: Russell 2000 Half-day ($\Delta = 1/2$) and Every-second-day ($\Delta = 2$) Estimation Results

(a) Half-day Estimation Result using Hourly Data

	λ	α	β	γ
Initial value	0.500	5.000×10^{-6}	0.700	170.000
Estimate value	1.140	1.850×10^{-6}	0.840	276.707
Standard error	(1.057)	(9.589×10^{-8})	(0.007)	(11.982)
Confidence Interval	[-0.932, 3.212]	[1.663×10^{-6}, 2.038×10^{-6}]	[0.827, 0.852]	[253.223, 300.191]
Proposed value (Badescu)	1.016	3.552×10^{-6}	0.830	208.007
Function value	-24879.156			
N	7531			

(b) Every-second-day Estimation Result using Daily Data

	λ	α	β	γ
Initial value	1.500	6.000×10^{-6}	0.700	190.000
Estimate value	1.131	1.624×10^{-5}	0.817	93.657
Standard error	(1.118)	(1.608×10^{-6})	(0.017)	(8.964)
Confidence Interval	[-1.060, 3.322]	[1.309×10^{-5}, 1.939×10^{-5}]	[0.784, 0.850]	[76.087, 111.227]
Function value	-4912.116			
Proposed value (Badescu)	1.016	1.421×10^{-5}	0.782	104.003
N	1890			

Notes: In Panel (a), we employ hourly data from Firstrate to construct half-day data and perform the estimation using the same methods. Panel (b) employs daily data to construct 2-day data. We borrowed the optimal parameter values reported in Table 1 as starting initial sets and built 10 sets to test the consistency of the estimation results. Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

Table 3: Russell 2000 Weekly ($\Delta = 5$) and Every-second-day ($\Delta = 2$) Estimation Results

(a) Weekly Estimation Result

	λ	α	β	γ
Initial value	1.000	5×10^{-5}	0.800	50.000
Estimate value	1.414	7.591×10^{-5}	0.838	30.469
Standard error	(1.230)	(1.321×10^{-5})	(0.016)	(6.046)
Confidence Interval	[-0.998, 3.825]	[5.002×10^{-5}, 10.180×10^{-5}]	[0.807, 0.869]	[18.619, 42.318]
Proposed value (Badescu)	1.016	3.552×10^{-5}	0.685	65.778
Function value	-1744.903			
N	783			

(b) Every-second-day Estimation Result using Hourly Data

	λ	α	β	γ
Initial value	1.500	6.000×10^{-6}	0.700	190.000
Estimate value	1.131	1.624×10^{-5}	0.817	93.657
Standard error	(1.118)	(1.608×10^{-6})	(0.017)	(8.964)
Confidence Interval	[-1.060, 3.322]	[1.309×10^{-5}, 1.939×10^{-5}]	[0.784, 0.850]	[76.087, 111.227]
Function value	-4912.116			
Proposed value (Badescu)	1.016	1.421×10^{-5}	0.782	104.003
N	1890			

Notes: Panel (a) reports estimation results using weekly price data from Barchart. We borrowed the optimal parameter values reported from every second day as starting initial sets and built 10 sets to test the consistency of the estimation results. Panel (b) employs daily data to construct every-second-day data from Barchart and then performs the estimation. We borrowed the optimal parameter values reported in Panel (a) in Table 1. Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

Table 4: Russell 2000 Every-second-hour ($\Delta = 1/4$) and Every-second-day ($\Delta = 2$) Estimation Results

(a) Every-second-hour Estimation Result using Hourly Data

	λ	α	β	γ
Initial value	2.000	2×10^{-6}	0.800	270.000
Estimate value	1.204	6.816×10^{-7}	0.862	424.965
Standard error	(1.083)	(2.496×10^{-8})	(0.004)	(12.668)
Confidence Interval	[-0.919,3.326]	$[6.327 \times 10^{-7}$, $7.306 \times 10^{-7}]$	[0.854 ,0.870]	[400.136,449.794]
Proposed value (Badescu)	1.016	1.178×10^{-6}	0.838	294.166
Function value	-55730.079			
N	15102			

(b) Hourly Estimation Result

	λ	α	β	γ
Initial value	2.500	4×10^{-7}	0.750	500.000
Estimate value	1.226	1.666×10^{-7}	0.869	859.551
Standard error	(1.091)	(3.311×10^{-9})	(0.002)	(14.043)
Confidence Interval	[-0.912,3.364]	$[1.601 \times 10^{-7}$, $1.731 \times 10^{-7}]$	[0.865 ,0.874]	[832.028,887.075]
Proposed value (Badescu)	1.016	8.880×10^{-7}	0.842	416.014
Function value	-12392.616			
N	30540			

Notes: Panel (a) uses hourly data to construct every-second hour data and then performs the estimation. We borrowed the optimal parameter values reported in Table 3 as starting initial sets and built 10 sets to test the consistency of the estimation results. Panel (b) borrowed the optimal parameter values reported in Panel (a) as starting initial sets and built 10 sets to test the consistency of the estimation results.Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

We leverage the use of simulation to recreate daily price data for the Russell 2000 index. We generate six simulation data sets by altering one of the HN-GARCH parameter values each time based on the estimated parameters reported in daily estimation table where:

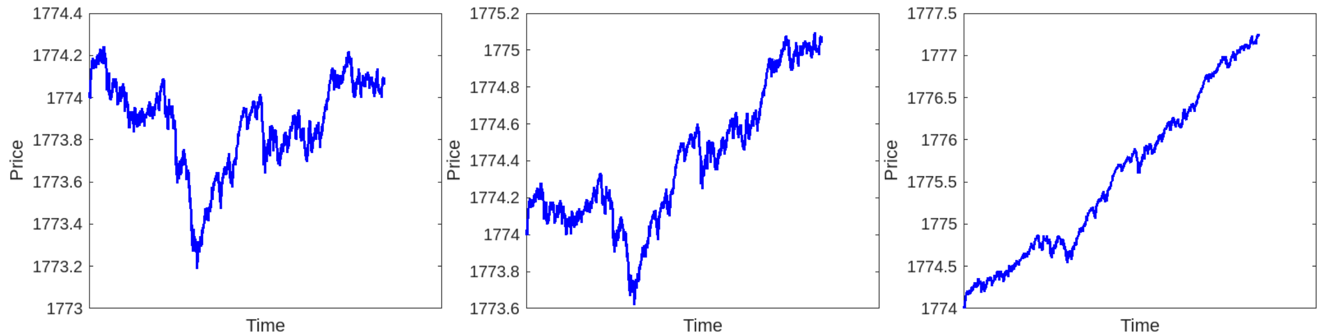
$$\begin{cases} \lambda & = 1.016, \\ \omega & = 0, \\ \alpha & = 7.104 \times 10^{-6}, \\ \beta & = 0.814, \\ \gamma & = 147.083, \end{cases} \tag{29}$$

The resulting plots (Figure 2 to Figure 5) are intended to offer a visual understanding of how the the underlying time series change as we manipulate parameter values.

5.4 Plots for Estimated Parameters against Frequency using Russel 2000 Data

In this section, we plot graphs of our estimated parameter values against different frequencies and compare them with values calculated using the Badescu's parameterization Equation 15.

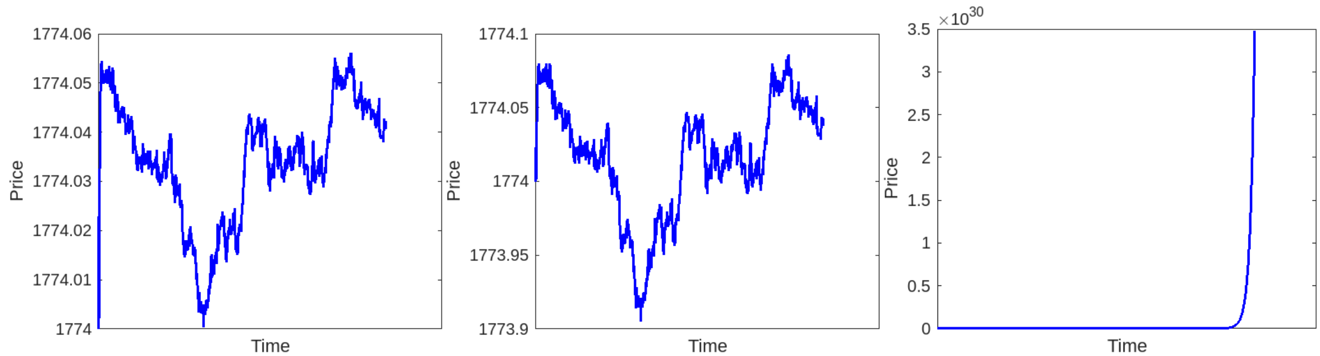
In Figure 6, the blue lines represent Badescu's proposed parameters, purple dots represent estimated parameter values using hourly data, black dots represent estimated parameter values using daily data, and red dots



(a) $\lambda = 0.2$ (b) $\lambda = 2$ (c) $\lambda = 6$

Figure 2: 15-year Close Price Plots From Simulated Data.

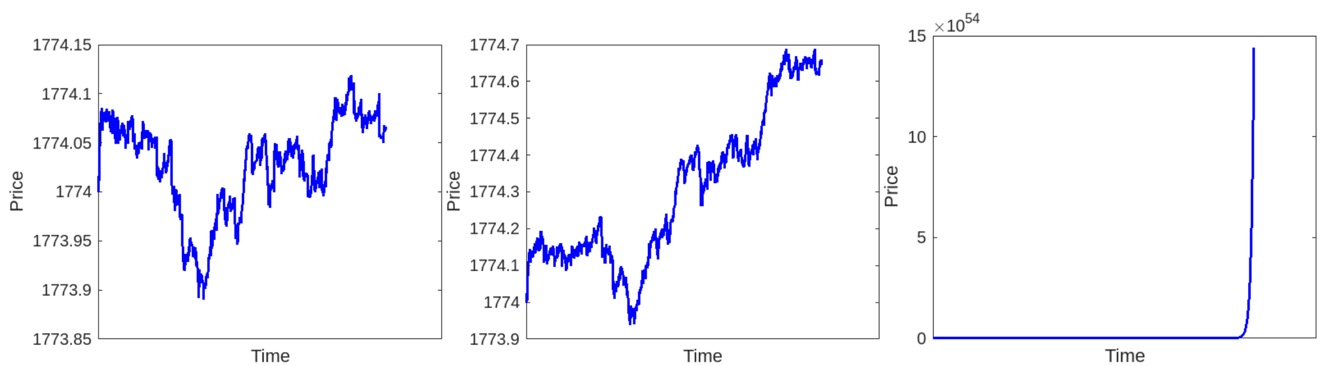
Notes: Plots (a) to (c) have parameter values: $\omega = 0$, $\alpha = 7.104 \times 10^{-6}$, $\beta = 0.814$, and $\gamma = 147.083$ with different λ values.



(a) $\alpha = 0.1 \times 10^6$ (b) $\alpha = 1 \times 10^6$ (c) $\alpha = 10 \times 10^6$

Figure 3: 15-year Close Price Plots From Simulated Data.

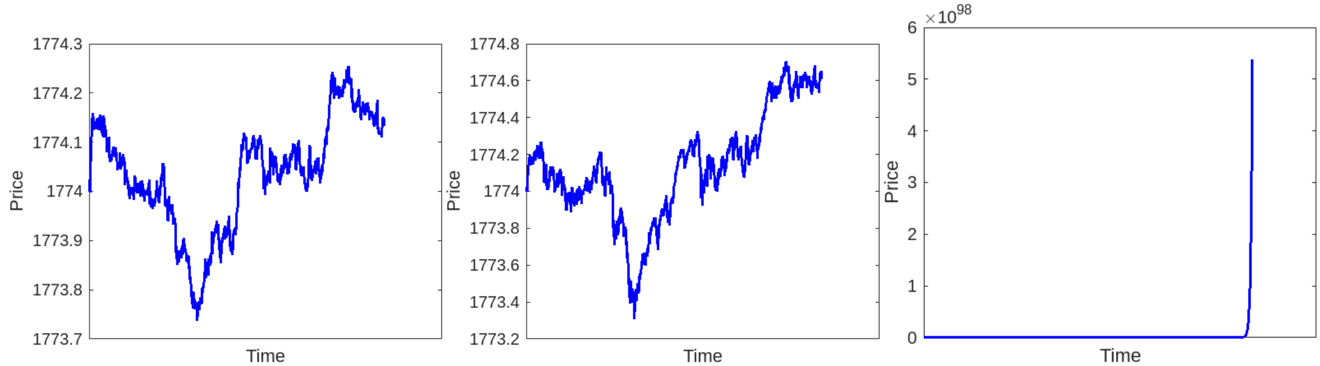
Notes: Plots (a) to (c) have parameter values: $\omega = 0$, $\lambda = 1.016$, $\beta = 0.814$, and $\gamma = 147.083$ with different α values.



(a) $\beta = 0.2$ (b) $\beta = 0.7$ (c) $\beta = 0.9$

Figure 4: 15-year Close Price Plots From Simulated Data.

Notes: Plots (a) to (c) have parameter values: $\omega = 0$, $\alpha = 7.104 \times 10^{-6}$, $\lambda = 1,016$, and $\gamma = 147.083$ with different β values.



(a) $\gamma = 10$ (b) $\gamma = 150$ (c) $\gamma = 200$

Figure 5: 15-year Close Price Plots From Simulated Data.

Notes: Plots (a) to (c) have parameter values: $\omega = 0$, $\alpha = 7.104 \times 10^{-6}$, $\beta = 0.814$, and $\lambda = 1.016$ with different γ values.

represent estimated parameter values using weekly data. These initial four plots in Figure 6 closely resemble how investors and traders typically conduct their analysis using various data frequencies. Additionally, a 95% confidence interval is constructed around each estimated point for each point.

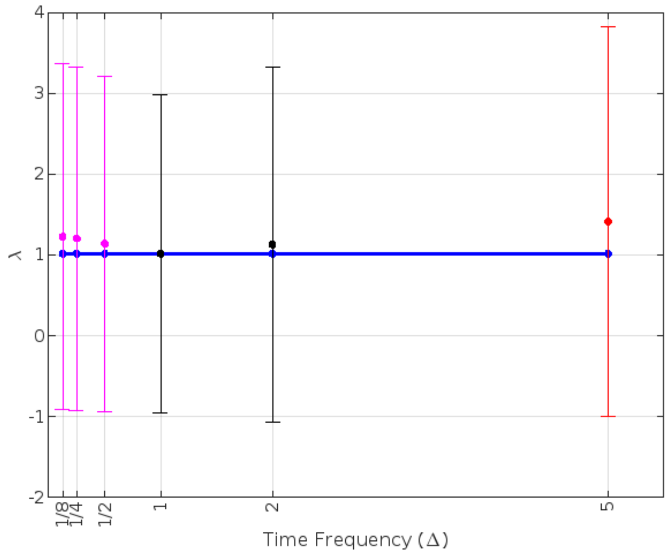
Notes: Interpretation of the x-axis: $\Delta = 1/8, 1/4, 1, 2, 5$ corresponds to data frequencies of hourly, every two hours, half-day, daily, every second day, and weekly.

In Figure 7, we repeated the same estimation steps and added the estimated parameters for daily ($\Delta = 1$), every-second day ($\Delta = 2$), and weekly ($\Delta = 5$) data by utilizing different frequencies for data construction. For instance, when $\Delta = 5$, there are three estimated values. The purple and black dots represent the estimated weekly parameter values derived from hourly and daily data used in constructing the weekly data. We use color coding to indicate the data source: purple for hourly, black for daily, and red for weekly.

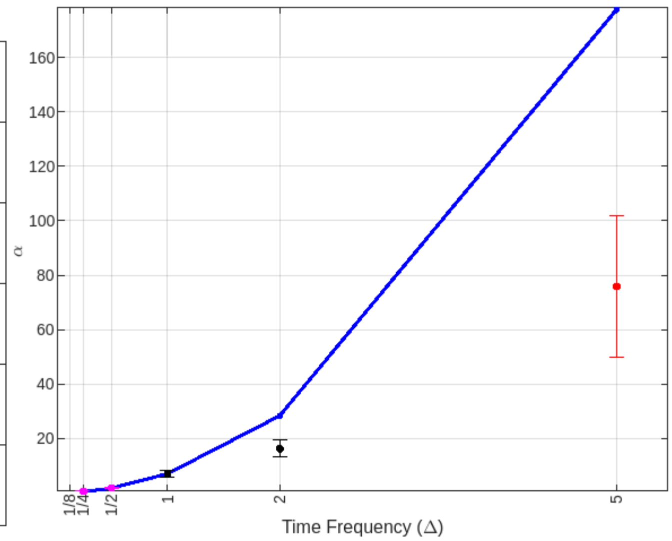
5.5 Discussion

In Figure 6 (a) and Figure 7 (a), we observe that the estimated parameter values for λ are relatively stable across different frequencies, with wide confidence intervals. This is in contrast to the constant value suggested by Badescu's paper. Figure 6 (c) and Figure 7 (c) illustrate that there is no discernible relationship between the value of β and the frequency of data, contradicting Badescu's parameterization formulas. The trends for α and γ align with our expectations, showing relatively small confidence intervals for each parameter. As we move up to a higher frequency, the value for estimated α decreases while γ tends to increases. Moreover, it is noteworthy that all confidence intervals overlap when employing data from different source frequencies (hourly, daily, weekly) for estimation.

Expanding on the observations from above, we are able to evaluate the consistency of trends in HN-GARCH parameters in comparison to other market indices. In pursuit of this objective, we have integrated the plots of S&P 500 and Dow Jones Industrial Average in Appendix B. We can draw the same conclusion as discussed earlier regarding the trends observed.

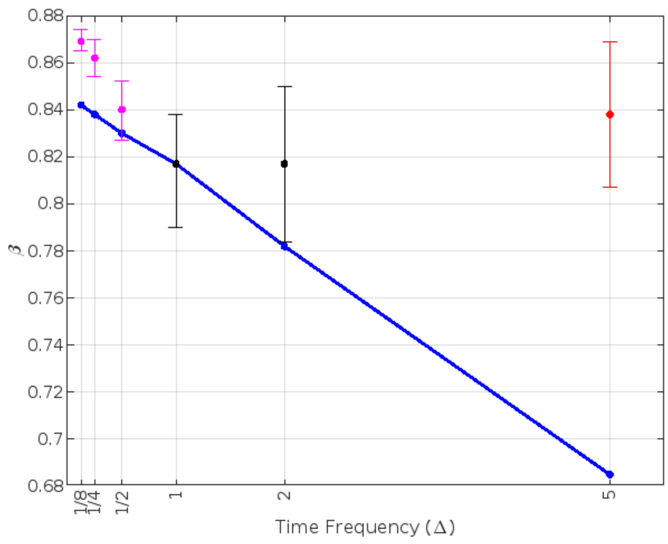


(a) Lambda Plot

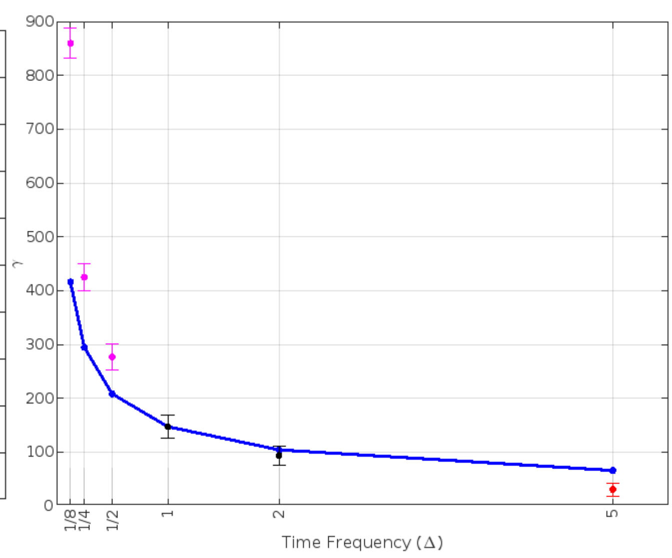


(b) Alpha Plot.[a]

[a]The values on the y-axis are scaled up by 10^6.

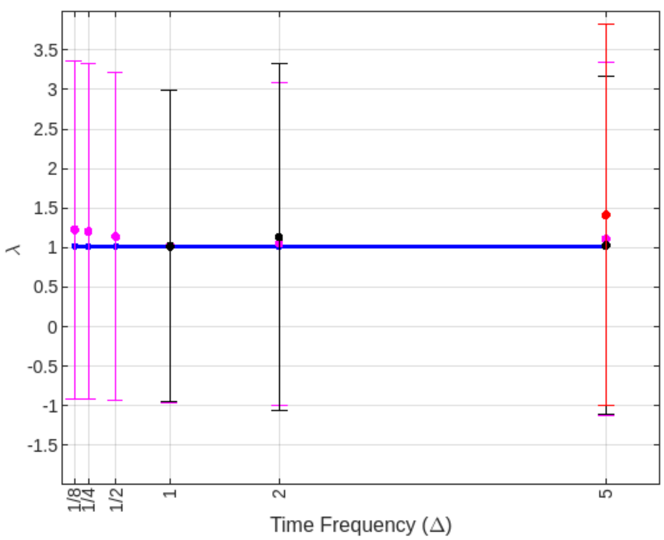


(c) Beta Plot

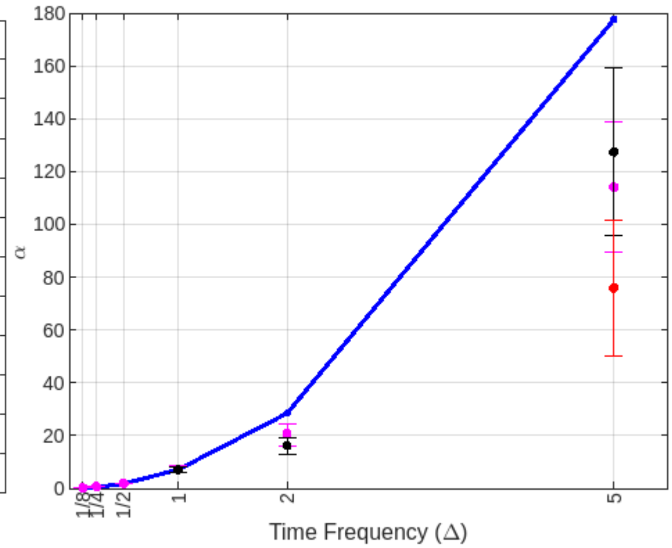


(d) Gamma Plot

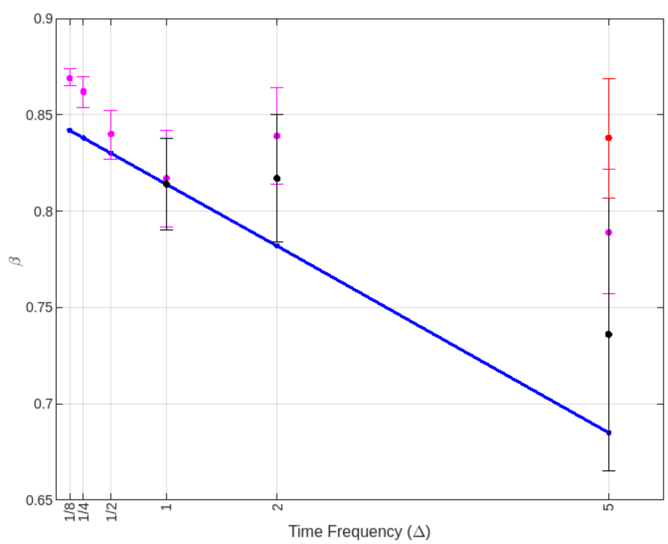
Figure 6: Plots for Proposed Relationship (blue line) and Estimated Relationship between Parameters Values and Frequencies



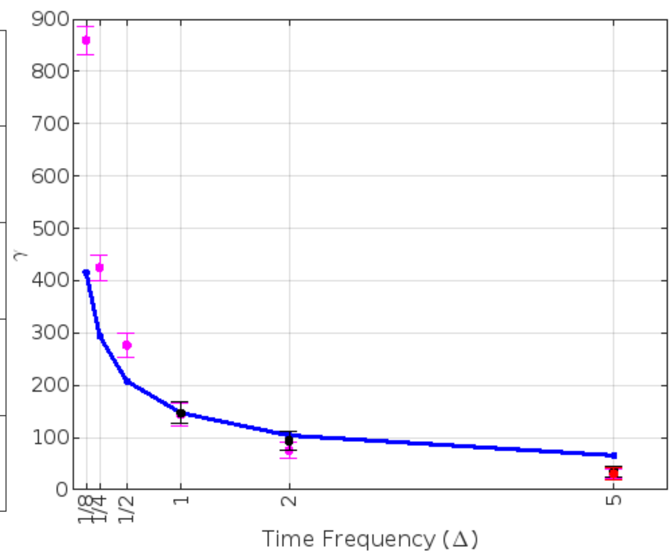
(a) Lambda Plot



(b) Alpha Plot



(c) Beta Plot



(d) Gamma Plot

Figure 7: Plots for Proposed Relationship (blue line) and Estimated Relationship between parameter values and various frequencies [2]

6 Conclusion and Further Research Directions

The main contribution of this essay can be viewed as twofolds. First, we examine the validity and consistency of our estimation methods by using simulated data for the HN-GARCH (1,1) model. The estimation results are promising using a simulation-based approach: as our sample size increases (when we move to higher frequency for the same time period), the standard errors decrease for most parameters (λ, α, β); different initial value sets, $\theta(\lambda, \omega, \alpha, \beta, \gamma)$, mostly converge to the same optimal function value; the 95% confidence intervals for estimated parameters cover the true parameter values. Secondly, we applied our consistent estimation methodology to real index returns. The plots in Section 5.5 and Appendix B demonstrate the relationships we discovered between estimated parameters and various frequencies, supported by U.S. index data from 2008 to 2022. While most research focuses on addressing the limitations of the HN-GARCH model by proposing various statistical approaches, this essay aims to shift the emphasis towards illustrating the disparity between the theoretical parameterization formulas and the estimated relationship by using market index data. Our findings emphasize the importance for investors to consider the frequency of their asset data in the context of their trading and investment strategies. This consideration enables them to make informed decisions that align with the dynamic nature of volatilities in the financial market.

Nevertheless, there are areas where further improvements could be made. Due to time constraints, we were unable to explore these areas within the scope of this essay. In terms of the choice of the log likelihood function, an interesting avenue for investigation involves using an alternative optimization tool to maximize the log likelihood function (MLE). Another Matlab solver, `fminsearch`, was used to estimate the HN-GARCH model parameters and perform option pricing based on simulated data (Ye, 2021). Additionally, we can also further explore the consistency of the estimated relationship in other markets. If we were to extend our analysis to include European and Asian markets, would the estimated relationship between parameters and frequencies be similar or differ from what we observed in the U.S. market?

Building upon the findings of this paper, we have several suggestions for the directions of future studies. Investors can utilize our findings to balance their portfolios at given frequencies, as the relationship between HN-GARCH parameters and frequencies has been illustrated using real returns. Escobar, Rastegari, and Stentoft (2019) provided an approximate closed-form solution for the dynamic portfolio rebalancing problem, where the asset spot price follows a standard GARCH(1,1) process (Escobar-Anel et al., 2022). By applying the same estimation methodology to higher frequency data, such as minutes, five minutes, and thirty minutes, one can investigate whether there are discontinuities in the relationship between frequencies and parameter values. This analysis may be particularly advantageous for investors, such as day traders, who primarily rely on high-frequency data. One can also apply alternative estimators and models when working with high-frequency data. Empirical studies have shown that the standard GARCH (1,1) model may not be the most appropriate choice for such data. Marcel

(2008) introduced a GARCH quasi-maximum likelihood estimator (QMLE) to enhance the efficiency of estimation when dealing with this specific data type (Visser, 2008).

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7 Appendix A

7.1 Simulation Table

Table 5: 10-year daily simulation results

True value	λ (1.094)	α (3.364×10^{-6})	β (0.838)	γ (196.82)
Initial value	1.105	3.398×10^{-5}	0.846	198.800
Estimate value	1.138	4.153×10^{-5}	0.848	163.427
	(1.979)	(5.504×10^{-7})	(0.014)	(16.692)
Confidence Interval	[-2.741,5.017]	[3.074×10^{-6} , 5.232×10^{-6}]	[0.821 , 0.875]	[130.712,196.143]
Log max value	-8145.995			
N	2510			
Initial value	0.000	4.000×10^{-5}	0.700	170.000
Estimate value	1.138	4.153×10^{-5}	0.848	163.429
	(1.979)	(5.503×10^{-7})	(0.013)	(16.692)
Confidence Interval	[-2.741,5.017]	[3.074×10^{-6} , 5.231×10^{-6}]	[0.822 , 0.875]	[130.712,196.143]
Log max value	-8145.995			
N	2510			
Initial Value	15.000	9.000×10^{-6}	0.000	100.000
Estimate value	146.362	8.997×10^{-6}	0.002	96.296
	(3.551×10^{-7})	(5.671×10^{-14})	(3.164×10^{-12})	(8.489×10^{-8})
Confidence Interval	[146.362,146.362]	[8.997×10^{-6},8.996×10^{-6}]	[0.002,0.002]	[96.296,96.296]
Log max value	70933.3			
N	2510			
Initial Value	2.000	7.000×10^{-6}	0.000	190.000
Estimate value	19.995	7.000×10^{-6}	-6.182×10^{-7}	189.994
Log max value	1013311.096			
N	2510			

Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

Table 6: 10-year daily simulation results

True value	λ (1.094)	α (3.364×10^{-6})	β (0.000)	γ (196.82)
Initial value	1.100	3.000×10^{-5}	0.800	200.000
Estimate value	-5.546	1.006×10^{-6}	0.774	85.326
	(9.840)	(3.704×10^{-8})	(0.003)	(85.326)
Confidence Interval	[-24.833,13.741]	[9.325×10^{-7} , 1.078×10^{-6}]	[0.768 , 0.780]	[55.186,115.467]
Log max value	-12208.074			
N	2510			
Initial value	1.100	3.000×10^{-6}	0.000	200.000
Estimate value	14.686	3.012×10^{-6}	0.004	201.808
	(1.107)	(3.665×10^{-14})	(2.933×10^{-10})	(3.244×10^{-6})
Log max value	21783.389			
N	2510			
Initial Value	1.100	2.000×10^{-6}	0.000	200.000
Estimate value	111.000	2.500×10^{-6}	0.1000	249.947
	(1.672)	(5.358×10^{-9})	(1.756×10^{-5})	(1.690)
Log max value	-8921.337			
N	2510			

Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

Table 7: 10-year weekly simulation results

True value	λ (1.094)	α (1.68×10^{-5})	β (0.711)	γ (88.021)
Initial value	1.203	1.852×10^{-5}	0.782	100.000
Estimate value	1.321	2.097×10^{-5}	0.679	77.115
	(4.252)	(3.883×10^{-6})	(0.042)	(13.457)
Confidence Interval	[-7.012,9.656]	[1.336×10^{-5} , 2.858×10^{-5}]	[0.598 , 0.761]	[50.738,103.490]
Log max value	-1635.514			
N	502			
Initial value	0.000	2.000×10^{-5}	0.640	96.000
Estimate value	1.321	2.097×10^{-5}	0.679	77.115
	(4.252)	(3.883×10^{-6})	(0.042)	(13.457)
Confidence Interval	[-7.012,9.656]	[1.335×10^{-5}, 2.858×10^{-5}]	[0.598, 0.761]	[50.740, 103.490]
Log max value	-1635.514			
N	502			
Initial value	0.800	0.000	0.6000	90.000
Estimate value	80.00	1.000×10^{-6}	0.600	90.00
	(4.273)	(2.566×10^{-8})	(0.004)	(7.107)
Confidence Interval	[71.624,88.376]	[9.497×10^{-7}, 1.050×10^{-6}]	[0.592, 0.608]	[76.070,103.930]
Log max value	-650.38			
N	502			
Initial value	1.000	0.900×10^{-5}	0.000	80.000
Estimate value	0.942	2.610×10^{-5}	0.767	70.387
	(NA)	(NA)	(NA)	(NA)
Log max value	31189.6			
N	502			

Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

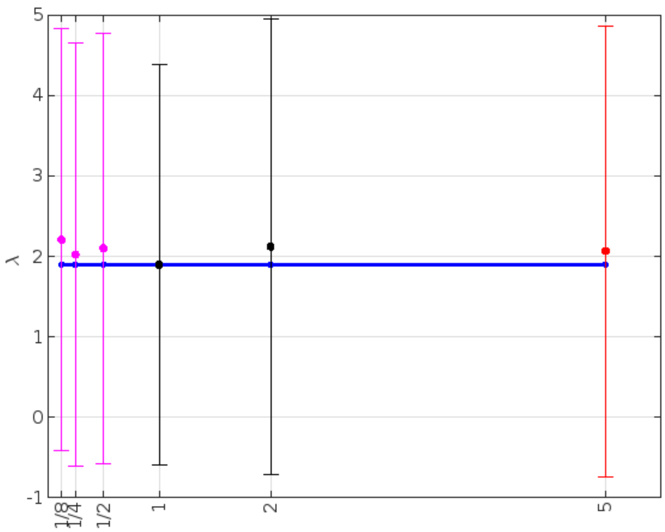
Table 8: 10-year hourly simulation results

True value	λ (1.094)	α (4.205×10^{-7})	β (0.8657)	γ (556.691)
Initial value	1.300	4.000×10^{-7}	0.800	400.000
Estimate value	1.236	4.346×10^{-7}	0.875	525.816
	(0.711)	(2.640×10^{-8})	(0.006)	(24.661)
Confidence Interval	[-0.156,2.692]	[3.829×10^{-7}, 4.864×10^{-7}]	[0.863,0.888]	[477.482,574.151]
Log max value	-65265.845			
N	20080			
True value	0.000	5.000×10^{-7}	0.7000	500.000
Estimate value	1.237	4.356×10^{-7}	0.875	525.830
	(0.711)	(2.651×10^{-8})	(0.006)	(24.661)
Confidence Interval	[-0.115,2.629]	[3.828×10^{-7}, 4.863×10^{-7}]	[0.863,0.887]	[477.495,574.165]
Log max value	-65265.845			
N	20080			
Initial Value	2.000	0.000	0.6000	300.000
Estimate value	NA	NA	NA	NA
Log max value	objective function is undefined			
N	20080			
Initial Value	3.000	6.000×10^{-7}	0.000	700.000
Estimate value	43.925	7.395×10^{-7}	0.014	600.000
	(2.877×10^{-7})	(5.903×10^{-15})	(1.151×10^{-10})	(3.381×10^{-6})
Log max value	535950.404			
N	20080			

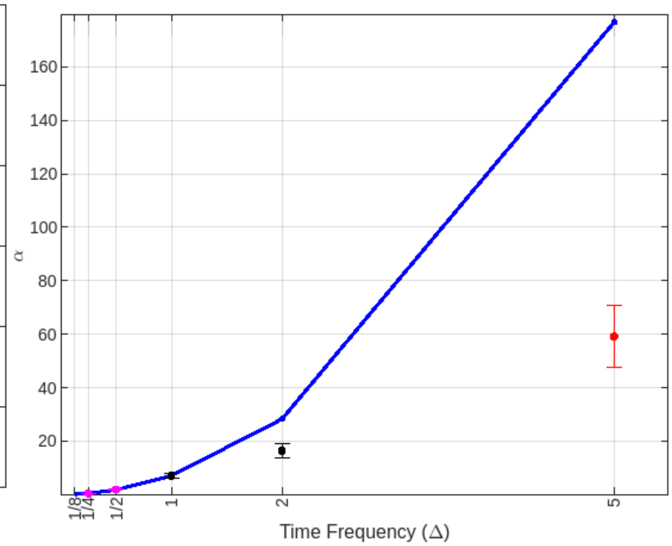
Initial parameter values with the optimal function value (smallest), and the 95% confidence level are reported for all tables. The function value reported is the optimal minimal value of the objective function. Standard error using the gradient of the objective function is reported in parentheses.

8 Appendix B

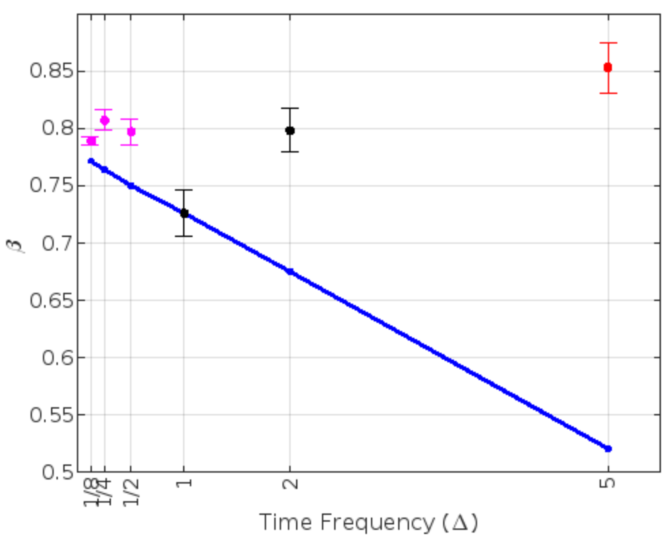
8.1 Plots for Estimated Parameters against Frequency using S&P 500 Data



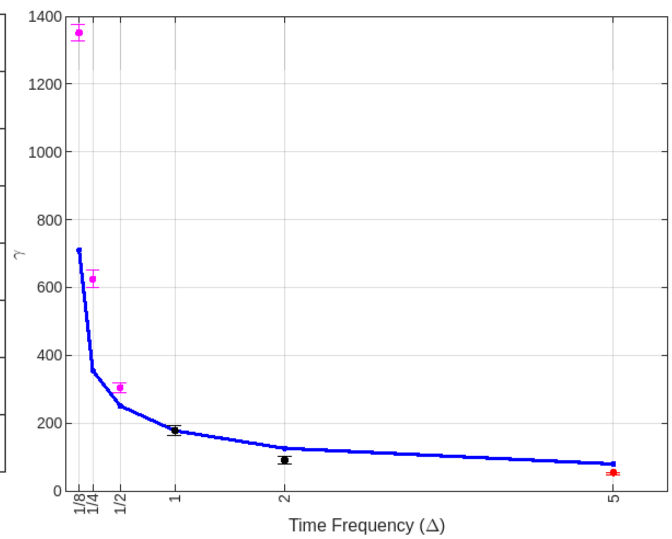
(a) Lambda Plot



(b) Alpha Plot

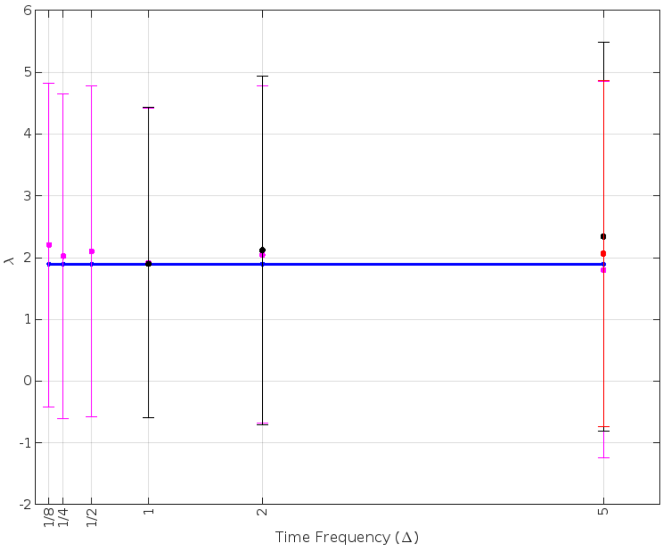


(c) Beta Plot

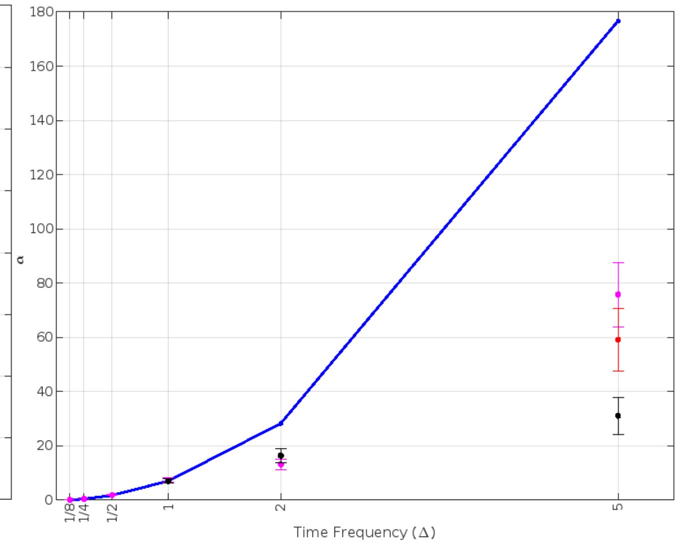


(d) Gamma Plot

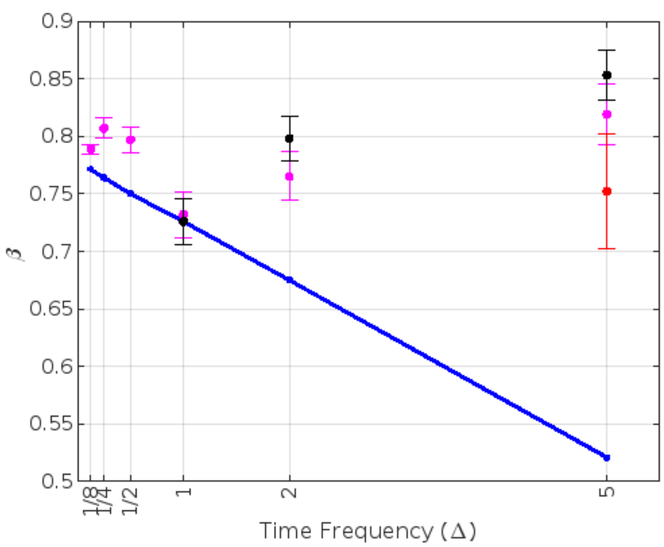
Figure 8: Plots for Proposed Parameters (Blue Line) and Estimated Parameters against Frequency using SPX Data.



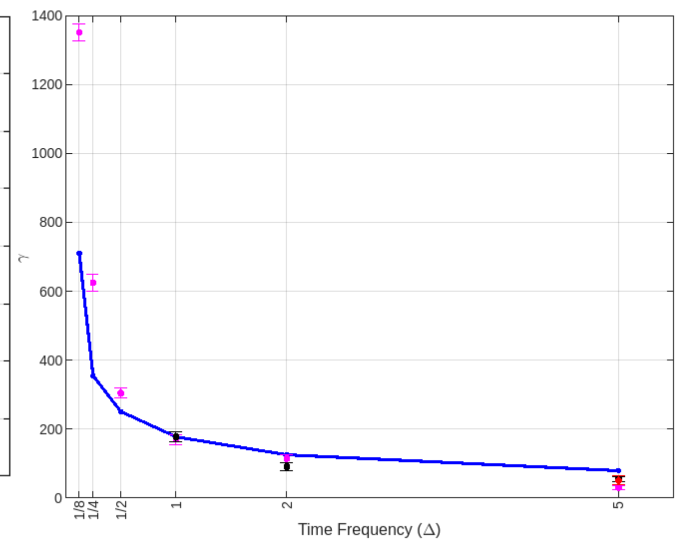
(a) Lambda Plot



(b) Alpha Plot



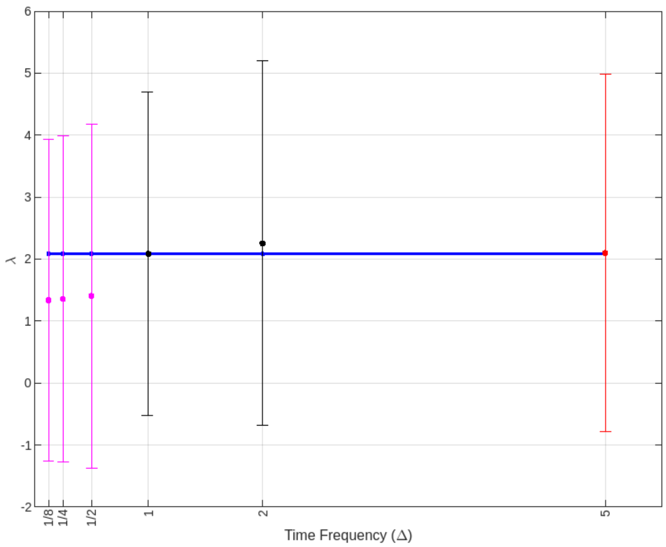
(c) Beta Plot



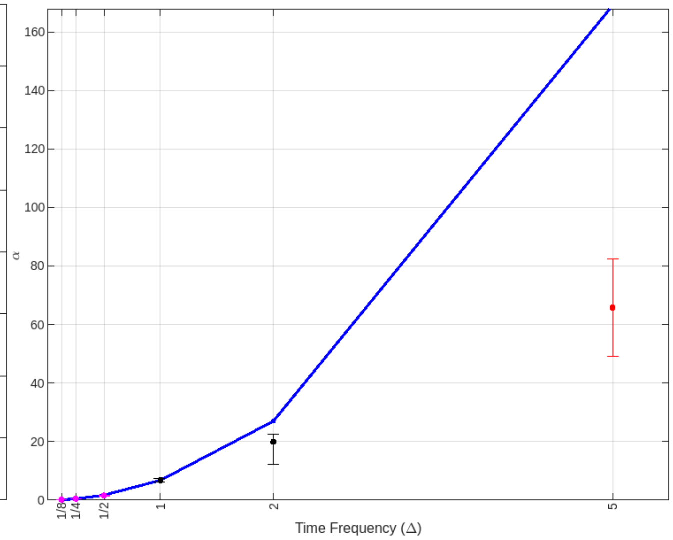
(d) Gamma Plot

Figure 9: Plots for Proposed Parameters (Blue Line) and Estimated Parameters against Frequency using SPX Data (Added Estimation Results using Different Data Sources)

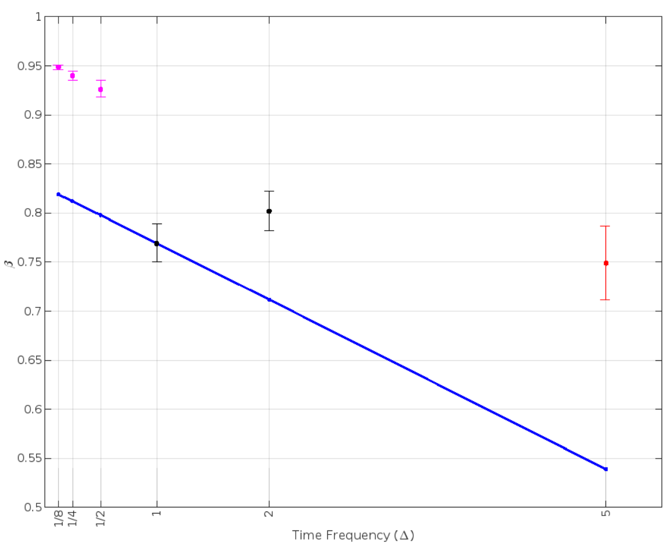
8.2 Plots for Estimated Parameters against Frequency using Dow Jones Data



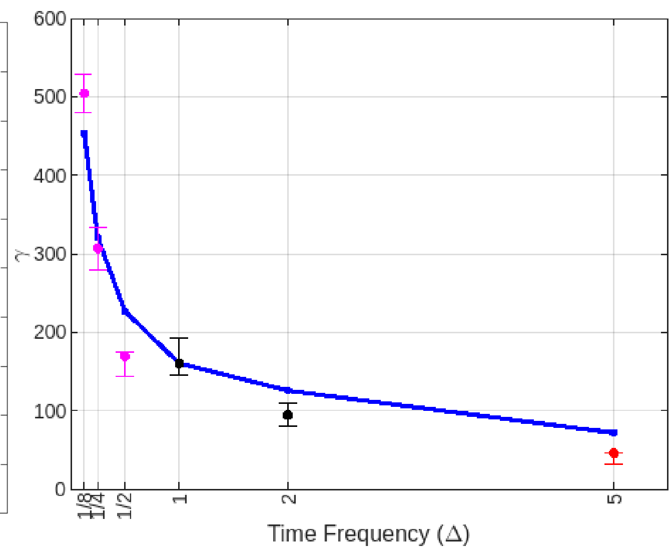
(a) Lambda Plot



(b) Alpha Plot



(c) Beta Plot

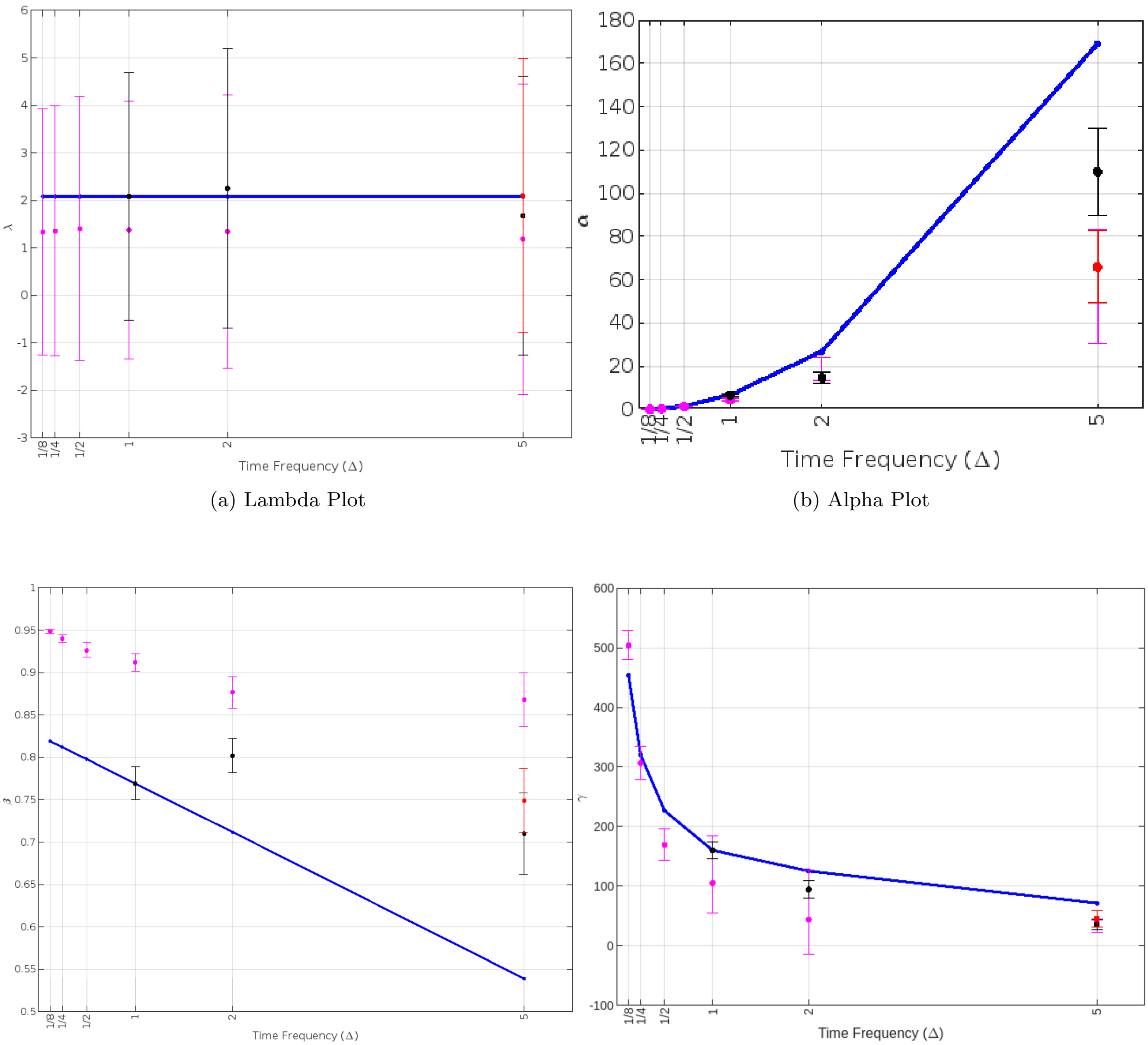


(d) Gamma Plot

Figure 10: Plots for Proposed Parameters (Blue Line) and Estimated Parameters against Frequency using DOW Data.

[3]

[3]We use color coding to indicate the data source: purple for hourly, black for daily, and red for weekly.



(a) Lambda Plot (b) Alpha Plot

(c) Beta Plot (d) Gamma Plot

Figure 11: Plots for Proposed Parameters (Blue Line) and Estimated Parameters against Frequency using DOW Data (Added Estimation Results using Different Data Sources)

8.3 Estimation Sample Code

Listing 1: MATLAB Estimation Code

```
clc
% Import data from Excel file
weekly_data = xlsread('Rus_hr_weekly.xlsx');
% Extract the desired column for returns calculation
column_index = 6;
close_price = weekly_data(:, column_index);
close_price_subset = close_price(1:750);

% Calculate returns for the subset of data
returns = diff(log(close_price_subset));
% Display summary statistics

% Add returns as a new variable in daily_data
daily_data.returns = [NaN; returns];
% Returns simulation, estimation, std error computation
format long

% Settings% Plot the returns
figure;
plot(close_price);
xlabel('Time');
ylabel('Close Price');
title('Russel 2000 weekly Close Price Plot (Using Daily Data) ');

N_ret = 749
% number of return

% optimization options
options = optimoptions('fminunc','MaxFunEvals',15000,'MaxIter',3000,...
    'TolFun',1e-6,'TolX',1e-6);
```

```
%% Simulate return

r=0
h_init = 1e-04

x0 = [0.1,50,8,0.5]
%%
[x_max,f_val4,flag,output] = fminunc(@(x) loss(x,r,h_init,returns),x0,options)
output.message

%std error
gradient = numfirstderiv(@(x)
    hn_lik_vec(inv_rescale(x),r,h_init,returns),x_max,0.00001);
std_error = sqrt(diag(inv(gradient'*gradient)));
estimate_ret = inv_rescale(x_max);
std_error_ret = inv_rescale(std_error);

fprintf('This is returns only estimation result\n')

format short
table(["lambda",
    "alpha","beta","gamma"]',estimate_ret',std_error_ret','VariableNames',["Para",
    "Est_ret", "StdErr_ret"])
format long
% Set the desired confidence level (e.g., 95%)
confidence_level = 0.95;

% Compute the critical value
critical_value = norminv((1 + confidence_level) / 2);

% Compute the margin of error for each parameter estimate
margin_of_error = critical_value * std_error_ret;

% Calculate the lower and upper bounds of the confidence interval
```

```
lower_bound = estimate_ret - margin_of error;
upper_bound = estimate_ret + margin_of error;

% Create a table with the parameter estimates, standard errors, and confidence
    intervals
result_table = table(["lambda"; "alpha"; "beta"; "gamma"], estimate_ret',
    std_error_ret', lower_bound', upper_bound', 'VariableNames', ["Para",
    "Est_ret", "StdErr_ret", "CI_Lower", "CI_Upper"]);

% Display the table
disp(result_table);

% return lik vec only
function [error,z,h] = hn_lik_vec(theta,r,h_init,daily_ret)
    % This function receives parameters theta = (lambda,omega,alpha,beta,
    % gamma), r, h_init and calculates z_1,...,z_N and returns
    % vector of log likelihood, implied residual and h

    N = length(daily_ret);
    lambda = theta(1);

    alpha = theta(2);
    beta = theta(3);
    gamma = theta(4);

    h = zeros(1,N+1);
    h(1) = h_init;
    z = zeros(1,N);

    % calculate h and z implied by stock prices
    for i = 1:N
        z(i) = (daily_ret(i) - r - lambda * h(i)) / sqrt(h(i));
        h(i+1) = beta * h(i) + alpha * (z(i) - gamma * sqrt(h(i)))^2;
    end
```

```
    h = h(1:N);
    % vectorized error
    error = -.5 * (log(2*pi*h(1:N)) + z.^2);
    % error = -.5 * (log(h(1:N)) + z^2);
end

% scaling
function theta = inv_rescale(x)
    % apply the scaling in order to make optimization efficient

    lambda = x(1) * 10^2;

    alpha = x(2) * 10^(-6);
    beta = x(3) * 10^(-1);
    gamma = x(4) * 10^(2);
    theta = [lambda, alpha, beta, gamma];
end

% optimize function
function error = loss(x,r,h_init,ret)
    vec_ret = hn_lik_vec(inv_rescale(x),r,h_init,ret);
    error = -1 * sum(vec_ret(1:end));
end

% first derivative via finite difference
function [gmatrix]=numfirstderiv(func_name, theta, eps)
% This function calculates the numerical first derivative of a function
% The function should return a Nx1 matrix and the parameter vector should
% be kx1. The returned value is a Nxk matrix

for i=1:length(theta)
    theta_forward=theta;
    theta_backward=theta;
```

```
    delta_theta=abs(theta(i)*eps);
    theta_forward(i)=theta_forward(i)+delta_theta;
    theta_backward(i)=theta_backward(i)-delta_theta;
    gmatrix(:,i)=.5*(func_name(theta_forward)-func_name(theta_backward))./delta_theta;
end

if ~isreal(gmatrix)
    disp('WARNING!')
    disp('Complex valued gradient matrix.')
end
```

Listing 2: MATLAB Estimation Plots Code

```
clc;

x = [1/8, 1/4, 1/2, 1, 2, 5];
lambda1 = [2.084,2.084 , 2.084,2.084, 2.084, 2.084];
lambda2 = [1.337, 1.357, 1.406, 2.084,2.254, 2.098];




lambda2_lower = [-1.253, -1.274, -1.370, -0.530, -0.685, -0.787];
lambda2_upper = [3.927, 3.988, 4.183, 4.698, 5.194, 4.983];




plot(x, lambda1, 'b.-', 'MarkerSize',6, 'LineWidth', 1.5);
hold on;

errorbar(x(1:3), lambda2(1:3), lambda2(1:3) - lambda2_lower(1:3),
    lambda2_upper(1:3) - lambda2(1:3), 'm.', 'MarkerSize', 8);
errorbar(x(4:5), lambda2(4:5), lambda2(4:5) - lambda2_lower(4:5),
    lambda2_upper(4:5) - lambda2(4:5), 'k.', 'MarkerSize', 8);
errorbar(x(6), lambda2(6), lambda2(6) - lambda2_lower(6), lambda2_upper(6) -
    lambda2(6), 'r.', 'MarkerSize', 8);
```

```
xlabel('Time Frequency (\Delta)');
ylabel('\lambda ');

grid on;


xticks(x);
xticklabels({'1/8', '1/4', '1/2', '1', '2', '5'});
xtickfont = gca;
xtickangle(90);
xtickfont.FontSize = 7;
ytickfont.FontSize=7;

xlim([0, 5.5]);




hold off;


saveas(gcf, 'SPX_alpha!!.png');
```