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Econometric Modelling Based on Pattern Recognition *via* the Fuzzy c-Means Clustering Algorithm

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Abstract

In this paper we consider the use of fuzzy modelling in the context of econometric analysis of both time-series and cross-section data. We discuss and demonstrate a semi-parametric methodology for model identification and estimation that is based on the Fuzzy c-Means algorithm that is widely used in the context of pattern recognition, and the Takagi-Sugeno approach to modelling fuzzy systems. This methodology is exceptionally flexible and provides a computationally tractable method of dealing with non-linear models in high dimensions. In this respect it has distinct theoretical advantages over non-parametric kernel regression, and we find that these advantages also hold empirically in terms of goodness-of-fit in a selection of economic applications.

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1. Introduction

The specification and estimation of any econometric relationship poses significant challenges. This is especially true when it comes to the choice of functional form, as the latter is not always suggested or prescribed by the underlying economic theory. Any mis-specification of the functional form of an econometric model can have serious consequences for statistical inference - for example, the parameter estimates may be inconsistent. In response to the rigidities associated with explicit and rigid parametric functional relationships, and to avoid the consequences of their mis-specification, a more flexible non-parametric approach to their formulation and estimation is often very attractive (*e.g.*, Pagan and Ullah, 1999).

Such non-parametric estimation does, however, have its drawbacks. Most notably, in order to perform well, multivariate kernel regression requires a reasonably large sample of data, and it founders when the number of explanatory variables grows. With regard to the latter point there are really two problems. First, there is evidence that the small-sample properties of the multivariate kernel estimator can be quite unsatisfactory (*e.g.*, Silverman, 1986). Second, the rate of convergence of the kernel estimator decreases as the dimension of the model grows - this is the so-called "curse of dimensionality". Recently, Coppejans (2000) has dealt with this problem by using cubic B-spline estimation in conjunction with earlier results from Kolmogorov (1957) and Lorentz (1966) that enable a function of several variables to be represented as superpositions of functions of a single variable.

There is considerable appeal in the prospect of finding an approach to the formulation and estimation of econometric relationships that are highly flexible in terms of their functional form; make minimal parametric assumptions; perform well with either small or large data-sets; and are computationally feasible even in the face of a large number of explanatory variables. In this paper we use some of the tools of fuzzy set theory and fuzzy logic in the pursuit of just such an objective.

These tools have been applied widely in many disciplines since the seminal contributions of Zadeh (1965, 1987) and his colleagues. These applications are numerous in such areas as computer science, systems analysis, and electrical and electronic engineering. The construction and application of "expert systems" is widespread in domestic appliances, motor vehicles, and commercial machinery. The use of fuzzy sets and fuzzy logic in the social sciences appears to have been limited mainly to psychology, with very little application to the analysis of economic problems and economic data. There are a few such examples in the field of social choice - for example, Dasgupta and Deb (1996), Richardson (1998), and Sengupta (1999). In the area of econometrics, however, there have been surprisingly few applications of fuzzy set/logic techniques. Indeed, we are aware of only four such examples.

Josef *et al*. (1992) used this broad approach in the context of modelling with panel data. Lindström (1998) used fuzzy logic in a rather different way to model fixed investment in Sweden on the basis of the level and variability in the real interest rate. More specifically, he generated an index for aggregate investment using fuzzy sets and operators to combine the imprecise information associated with the interest rate. He found that this approach provided a simple method of modelling inherent non-linearities in a flexible (non-parametric) manner. It should be noted that Lindström's analysis involved no *statistical inference*, as such. This same technique was used by Draeseke and Giles (1999, 2000) to generate an index of the New Zealand underground economy, and it has now been programmed into the SHAZAM (2000) econometrics package. The advantage of using this particular technique in that context was that the variable of interest is intrinsically unobservable.

Shepherd and Shi (1998) used fuzzy set/logic theory in a rather different manner in their analysis of U.S. wages and prices. Their modelling technique involved clustering the data into fuzzy sets, estimating the relationship of interest over each set, and then combining each sub-model into a single overall model by using the membership functions associated with the clustered data together with the Takagi and Sugeno (1985) approach to fuzzy systems. Their analysis also highlighted the ability of such fuzzy analysis to detect and measure non-linear relationships in a very flexible way, even though the underlying sub-models may be linear. Indeed, this is a specific and attractive feature of the Takagi-Sugeno analysis. In this paper we draw on the approach of Shepherd and Shi and adapt it to model a number of interesting relationships. We discuss some of the issues that arise when adopting this semi-parametric approach in the specification and estimation of an econometric model, and by way of specific practical applications we illustrate some of its merits relative to standard parametric regression and conventional (kernel) non-parametric regression.

The use of fuzzy sets and fuzzy logic is just one of several possible approaches within the general area of expert systems and artificial intelligence. In the context that is of interest to us here, an obvious competitive approach is that based on (artificial) neural networks (*e.g.*, Bishop, 1995; White, 1992; White and Gallant, 1992). Neural networks also facilitate the modelling of nonlinear relationships whose underlying form is not parametrically constrained at the outset. However, our adaptation of the Takagi-Sugeno/Shepherd-Shi (TSSS) methodology has an important and appealing advantage over the use of neural networks. Namely, the basic relationships that are estimated have a direct economic interpretation, and they can be related to the underlying economic theory in a useful way. The same is not generally true in the case of neural networks. Moreover, and as we shall see in more detail below, the TSSS methodology is computationally efficient, especially in as much as it requires only one "pass" through the data. In contrast, neural networks usually have to be "trained", and this involves more protracted computations.

In the next section we outline the background concepts in fuzzy set/logic theory that we will be using. A crucial component of our analysis is the implementation of the so-called "fuzzy c-means" algorithm to partition the data-set in a flexible manner, and this algorithm is considered in some detail in section 3. In section 4 we describe our variant of the TSSS modelling procedure, and section 5 presents the results associated with several illustrative empirical applications of this fuzzy technology. Some concluding remarks are given in section 6.

2. Fuzzy Sets and Fuzzy Logic

As was noted above, fuzzy logic relates to the notion of fuzzy sets, the theoretical basis for which is usually attributed to Zadeh (1965). Under regular set theory, elements either belong to some particular set or they do not. Another way of expressing this is to say that the "degree of membership" of a particular element with respect to a particular set is either unity or zero. The boundaries of the sets are hard, or "crisp". In contrast to this, in the case of fuzzy sets, the degree of membership may be any value on the continuum between zero and unity, and a particular element may be associated with more than one set. Generally this association involves different degrees of membership with each of the fuzzy sets. Just as this makes the boundaries of the sets fuzzy, it makes the location of the centroid of the set fuzzy as well.

To consider an illustrative situation relating to a single economic variable, suppose we wish distinguish between situations of excess supply and those of excess demand in relation to the price of some good. In traditional set theory we would have a situation such as:

$$S_s = \{p: p > p^*\}$$
$$S_d = \{p: p < p^*\}$$
$$S_e = \{p^*\}$$

as the crisp sets representing those prices associated with excess supply, excess demand, and the equilibrium price (p^*}, respectively. Any particular price, say \$5, would definitely be in one and only one of these sets. That is, if its degree of membership with S_i is denoted u_i, then $u_j = 1$ implies that $u_k = 0$, for all $k \neq j$; for $k, j = s, d, e$. In contrast to this, in a fuzzy set framework the sets S_s, S_d and S_e would not have sharp boundaries and a particular price (such as \$5) would be associated to some degree or other with each of these sets. For instance, its degrees of membership might be $u_s = 0.2$, $u_d = 0.3$, $u_e = 0.5$. Note that although the u_i's sum to unity in this example, in general they need not, and they should not be equated with the "probabilities" that the price of \$5 lies in each set.

It should also be noted that in the example above, the concepts involved are defined in a crisp manner: "excess supply", "excess demand", and "equilibrium". More generally, fuzzy set theory is just as capable of handling vague linguistic concepts, such as "a rather high price", or "a very low demand". So, we could broaden the above example to allow for fuzzy sets involving prices associated with "a very high excess supply", "a moderate excess supply", "a small excess supply", and so on. Again, the boundaries of these sets would be fuzzy, and degrees of membership would map prices to the sets.

The application of the inductive premise to fuzzy concepts poses some difficulties – not all of the usual laws of set theory are satisfied. In particular, the "law of the excluded middle" is violated, so a different group of set operators must be adopted. So, for example, the "union" operator is replaced by the "max" operator, "intersection" is replaced by "min", and "complement" is replaced by subtraction from unity. Under these "fuzzy" operators the commutative, associative, distributive, idempotency, absorption, excluded middle, involution, and De Morgan's laws are all satisfied in the context of fuzzy sets. For example if the universal set is U={a, b, c, d} let the fuzzy sets S_1 and S_2 be defined as S_1 = {0.4/a, 0.5/b, 1/d} and S_2 = {0.1/a, 0.2/b, 0.5/c, 0.9/d}. Here, the numbers are the "degrees of membership" which map the elements of U to S_1 and S_2. Then, $A \cup B$ = {0.4/a, 0.5/b, 0.5/c, 1/d} and $A \cap B$ = {0.1/a, 0.2/b, 0/c, 0.9/d}, S_1^c = {0.6/a, 0.5/b, 1/c, 0/d}, and S_2^c = {0.9/a, 0.8/b, 0.5/c, 0.1/d}.

Fuzzy sets can be used in conjunction with logical operators in a manner that will enable us to construct models in a flexible way. More specifically, we can use the degrees of membership that associate the values of one or more input variables with different fuzzy sets, together with logical "**IF**", "**THEN**", "**AND**" operations, to derive membership values that associate an output variable with one or more fuzzy sets. For example, in the case of the demand for a particular good, we might have a fuzzy rule of the form:

> "***IF*** *the own-price of the good is quite low,* ***AND*** *the price of the only close substitute good is rather high,* ***THEN*** *the demand for the good in question will be fairly high*."

Or, to anticipate one of our empirical examples later in this paper:

> "***IF*** *the interest rate is fairly high,* ***AND*** *income (output) is relatively low,* ***THEN*** *the demand for money will be quite low*."

Of course, in general, there will be more than one fuzzy rule, and the output variable may have potential membership in more than one fuzzy set:

Rule 1: "***IF*** *the interest rate is fairly high,* ***AND*** *income (output) is relatively low,* ***THEN*** *the demand for money will be quite low*."

Rule 2: "***IF*** *the interest rate is average,* ***AND*** *income (output) is relatively high,* ***THEN*** *the demand for money will be moderately high*."

In such situations, a fuzzy outcome for the output variable could be inferred by taking account of the relevant membership values, and the MAX/MIN operators associated with fuzzy sets. However, a fuzzy outcome is not usually adequate. For instance, in the last example, there is only limited interest in knowing that the fuzzy model predicts that in a certain period the demand for money will be "moderately high", or "quite low". It would be much more helpful to know (for example) that the model predicted a demand of $10million. In other words, we need a way of "de-fuzzifying" the predictions of the model.

The Takagi and Sugeno (1985) and Sugeno and Kang (1985) approach to dealing with this issue involves modfiying the above methodology to one that involves rules of the form:

Rule 1': “***IF*** *the interest rate (r) is fairly high,* ***AND*** *income (Y) is relatively low,* ***THEN*** *the demand for money is* $M = f_1 (r, Y) = M_1$.”

Rule 2': “***IF*** *the interest rate (r) is average,* ***AND*** *income (Y) is relatively high,* ***THEN*** *the demand for money is* $M = f_2 (r, Y) = M_2$.”

Here, f_1 and f_2 are crisp, and generally parametric, functions which yield numerical values for M. In their simplest form these functions would be linear relationships. The various values (M_1, M_2, *etc*.) emerging from these rules can then be combined by taking a weighted average, based on the degrees of membership associated with the input variables and the fuzzy input sets. This is described in more detail in section 4 below. First, however, we need to give further consideration to the definition and construction of the fuzzy input sets, and this involves grouping or “clustering” the input data appropriately.

3. The Fuzzy c-Means Algorithm

3.1 Overview of the Algorithm

In our modelling analysis, which is described in detail in the next section, we need to determine the partitioning of the sample data for each explanatory (input) variable into a number of clusters. These clusters have “fuzzy” boundaries, in the sense that each data value belongs to each cluster to some degree or other. Membership is not certain, or “crisp”. Having decided upon the number of such clusters to be used, some procedure is then needed to locate their mid-points (or more generally, their centroids) and to determine the associated membership functions and degrees of membership for the data-points. To this end, Shepherd and Shi (1998) used a variant of the “fuzzy c-means” (FCM) algorithm. (The latter is sometimes termed the fuzzy k-means algorithm in the literature.) The FCM algorithm is really a generalization of the “hard” c-means algorithm. It appears to date from Ruspini (1970), although some of the underlying concepts were explored by MacQueen (1967). The FCM algorithm is closely associated with such early contributors as Bezdek (1973) and Dunn (1974, 1977), and is widely used in such fields as pattern recognition, for instance.

The algorithm provides a method of dividing up the “*n*” data-points into “*c*” fuzzy clusters (where $c < n$), while simultaneously determining the locations of these clusters in the appropriate space. The data may be multi-dimensional, and the metric that forms the basis for the usual FCM is “squared error distance”. The underlying mathematical basis for this procedure is as follows. Let

$\boldsymbol{x}_k$ be the k'th (possibly vector) vector data-point (k = 1, 2, ..., n). Let $\mathbf{v}_i$ be the center of the i'th. (fuzzy) cluster (i = 1, 2, ..., c). Let $d_{ik} = \| \boldsymbol{x}_k - \boldsymbol{v}_i \|$ be the distance between $\boldsymbol{x}_k$ and $\boldsymbol{v}_i$, and let u_{ik} be the "degree of membership" of data-point "k" in cluster "i", where :

$$\sum_{i=1}^{c} (u_{ik}) = 1.$$

The objective is partition the data-points into the "c" clusters, and simultaneously locate those clusters and determine the associated "degrees of membership", so as to minimize the functional

$$J(U,v) = \sum_{i=1}^{c} \sum_{k=1}^{n} (u_{ik})^m (d_{ik})^2 .$$

There is no prescribed manner for choosing the exponent parameter, "m", which must satisfy $1 < m < \infty$. In practice, $m = 2$ is a common choice. In the case of crisp (hard) memberships, $m = 1$.

In broad terms, the FCM algorithm involves the following steps:

1. Select the initial location for the cluster centres.
2. Generate a (new) partition of the data by assigning each data-point to its closest cluster centre.
3. Calculate new cluster centres as the centroids of the clusters.
4. If the cluster partition is stable then stop. Otherwise go to step 2 above.

In the case of fuzzy memberships, the Lagrange multiplier technique generates the following expression for the membership values to be used at step 2 above:

$$u_{ik} = 1 / \left\{ \sum_{j=1}^{n} [(d_{ik})^2 / (d_{jk})^2]^{1/(m-1)} \right\} .$$

Notice that a singularity will arise if $d_{jk} = 0$ in the above expression. This occurs if, at any point a cluster centre exactly coincides with a data-point. This can be avoided at the start of the algorithm and generally will not arise subsequently in practice due to machine precision. If the memberships of data-points to clusters are "crisp" then

$$u_{ik} = 0 \; ; \; \forall \; i \neq j,$$

$$u_{jk} = 1 \; ; \; \text{j s.t. } d_{jk} = \min.\{d_{ik}, \text{i} = 1, 2, ..., c\}.$$

The updating of the cluster centres at step 3 above is obtained via the expression

$$v_i = [\sum_{k=1}^{n} (u_{ik})^m x_k] / [\sum_{k=1}^{n} (u_{ik})^m] \ ; i = 1, 2, \ldots., c.$$

The fixed-point nature of this problem ensures the existence of a solution. See Bezdek (1981, Chapter 3) for complete and more formal mathematical details. As will be apparent, the FCM algorithm is simple to program. We have chosen to do this using standard commands in the SHAZAM (2000) econometrics package - this makes it convenient to compare our results with those from the other standard regression modelling techniques that are available in that package.

3.2 Some Computational Issues

Among the issues that arise in the application of the FCM are the following. First, a value for the number of clusters and the initial values for their centres have to be provided to start the algorithm. It appears that the results produced by the FCM algorithm can be sensitive to the choice of these start-up conditions. Second, being based on a "squared error" measure of distance, the algorithm can be sensitive to noise or outliers in the data.

In relation to the first of these issues, several proposals have been made. Among these are the following. Linde *et al.* (1980) proposed the Binary Splitting (BS) technique to initialize the clusters. The disadvantage of the BS technique is that it requires at least 10 or 20 data points in each cluster, and in practice this may not be attainable. Huang and Harris (1993) extended this approach to the "Direct Search Binary Splitting" technique. Tou and Gonzales (1974) proposed the "Simple Cluster-Seeking" (SCS) method, which deals not only with the initial value issue, but also provides the clusters themselves. However, the SCS technique is *not* invariant to the order in which the data points are considered, and it also depends on certain "threshold" settings.

Yager and Filev (1992) introduced the so-called "Mountain Method" for deciding on the initial values. Their approach is apparently simple, but it becomes computationally burdensome in the context of multi-dimensional data. Consequently, Chiu (1994) proposed a modification of their approach, called "Subtractive Clustering", that does not suffer the same problem. This is the approach used by Shepherd and Shi (1998). Babu and Murty (1993) used genetic algorithms to initialize the clusters, and Katsavounidis *et al*. (1994) proposed a further method (KKZ). More recently, Al-Daoud and Roberts (1996) proposed two initialization methods that appear to be especially useful in the context of very large data sets, and which out-performed both the KKZ and SCS techniques with their test data.

In relation to the second of the above issues (noise and outliers in the data), there have again been a number of contributions. Among these are the following. Following on from earlier (computationally expensive) contributions by Weiss (1988) and Jolion and Rosenfeld (1989), Davé (1991) modified the FCM algorithm (still retaining a "squared error" distance concept) so that noisy data points are effectively allocated to a "noise cluster", rather than being allocated to other clusters, and so "contaminating" the latter. More recently, Keller (2000) suggested another related modification involving the use of a modified objective function with a weighting factor added for each data point - the objective being to assign a kind of influence factor to the single data points. Some authors have taken up the usual techniques used in the statistical literature to deal with the sensitivity of least squares to the problem of outliers. That is to say, methods of "robust estimation" have been incorporated into the FCM algorithm, and these have included "M-estimation", "trimmed least squares", and "least absolute deviations" alternatives to the least squares component of the FCM algorithm. A recent example of this approach is that of Frigui and Krishnapuram (1996), who integrate M-estimation into the FCM algorithm. They provide a computationally attractive generalization of the FCM algorithm that deals with *both* the identification of the number of clusters and the allocation of data points to these clusters simultaneously, and which is robust to noise and outliers.

4. Fuzzy Econometric Modelling

In this section we describe the TSSS methodology that we subsequently apply to a number of econometric model estimation problems. The discussion here is quite expository, and we begin by outlining the analysis in the very simple case where there is a single input variable (other than, perhaps, a constant intercept). So, the fuzzy relationship is of the form:

$$y = f(x) + \varepsilon$$

where the functional relationship will typically involve unknown parameters, and ε is a random disturbance term. While there is no need to make any distributional assumptions about the latter, if this is done then these can be taken into account in the subsequent analysis. If the disturbance has a zero mean, the fuzzy function represents the conditional mean of the output variable, y. To this extent, the framework is the same as that which is adopted in non-parametric kernel regression.

The identification and estimation of the fuzzy model then proceeds according to the following steps:

Step 1: Partition the sample observations for x into c fuzzy clusters, using the FCM algorithm. This generates the membership values for each x-value with respect to each cluster, and implicitly it also defines a corresponding partition of the data for y.

Step 2: Using the data for each fuzzy cluster separately, fit the models:

$$y_{ij} = f_i(x_{ij}) + \varepsilon_{ij} \quad ; \; j = 1, \ldots, n_i \; ; \; i = 1, \ldots, c$$

In particular, if the chosen estimation procedure is least squares, then

$$y_{ij} = \beta_{i0} + \beta_{i1} x_{ij} + \varepsilon_{ij} \quad ; \; j = 1, \ldots, n_i \; ; \; i = 1, \ldots, c$$

Step 3: Model and predict the conditional mean of y using:

$$\hat{y}_k = \left[\sum_{i=1}^{c} (b_{i0} + b_{i1} x_k) u_{ik} \right] / \left[\sum_{i=1}^{c} u_{ik} \right] \quad ; \; k = 1, \ldots, n$$

where u_{ik} is the degree of membership of the k'th. value of x in the i'th. fuzzy cluster, and b_{im} is the least squares estimator of β_{im} (m = 0, 1) obtained using the i'th. fuzzy partition of the sample.

Step 4: Calculate the predicted "input-output relationship" (*i.e.*, the derivative) between x and the conditional mean of y:

$$(\partial \hat{y}_k / \partial x_k) = \left[\sum_{i=1}^{c} (b_{i1} u_{ik}) \right] / \left[\sum_{i=1}^{c} u_{ik} \right] \quad ; \; k = 1, \cdots\cdots, n$$

So, the fuzzy predictor of the conditional mean of *y* is a weighted average of the linear predictors based on the fuzzy partitioning of the explanatory data, with the weights (membership values) varying continuously through the sample. This latter feature enables non-linearities to be modelled effectively. In addition, it can be seen that the separate modelling over each fuzzy cluster involves the use of fuzzy logic of the form "**IF** the input data are likely to lie in this region, **THEN** this is likely to be the predictor of the output variable", *etc*.. The derivative of the conditional mean with respect to the input variable also has this weighted average structure, and the same potential for non-linearity.

Note that this modelling strategy is essentially a semi-parametric one. The parametric assumptions could be relaxed further by using kernel estimation to fit each of the cluster sub-models at Step 2 above, in which case the estimated *derivatives* (rather than coefficients) would be weighted at Steps 3 and 4, instead of the parameter estimates. However, some limited experimentation with this variation of the modelling methodology, in the context of the empirical applications described in the next section, yielded results that were inferior to those based on least squares.

Of course, in general we will be concerned with models that have more than one input (explanatory) variable:

$$y = f(x_1, x_2, \ldots\ldots, x_p) + \varepsilon$$

In such cases the steps in the above fuzzy modelling procedure are extended as follows, assuming a linear least squares basis for the analysis for expository purposes:

Step 1': Separately partition the n sample observations for each x_r into c_r fuzzy clusters (where r = 1, 2, ...,p), using the FCM algorithm. This generates the membership values for each observation on each x variable with respect to each cluster.

Step 2': Consider all c possible combinations of the fuzzy clusters associated with the p input variables, where

$$c = \prod_{r=1}^{p} c_r$$

and discard any for which the intersections involve negative degrees of freedom ($n_r < p$). Let the number of remaining cluster combinations be c'.

Step 3': Using the data for each of these c' fuzzy clusters separately, fit the models:

$$y_{ij} = \beta_{i0} + \beta_{i1} x_{1ij} + \beta_{i2} x_{2ij} + \cdots\cdots + \beta_{ip} x_{pij} + \varepsilon_{ij} \quad ; \; j = 1, \ldots., n_i \; ; \; i = 1, \ldots., c'$$

Step 4': Model and predict the conditional mean of y by using:

$$\hat{y}_k = \left[\sum_{i=1}^{c'} (b_{i0} + b_{i1} x_{1k} + \cdots\cdots + b_{ip} x_{pk}) w_{ik} \right] / \left[\sum_{i=1}^{c} w_{ik} \right] \qquad ; \; k = 1, \ldots., n$$

where

$$w_{ik} = \prod_{r=1}^{p} \delta_{ij} u_{rjk} \quad ; \quad i = 1, \cdots\cdots, c'$$

Here, δ_{ij} is a "selector" that chooses the membership value for the j'th. fuzzy cluster (for the r'th. input variable) if that cluster is associated with the i'th. cluster combination (i = 1, 2, …., c'); and b_{im} is the least squares estimator of β_{im} obtained using the i'th. fuzzy partition of the sample.

Step 5': Calculate the predicted "input-output relationships" (*i.e.*, the derivatives) between the conditional mean of y and each input variable, x_r:

$$(\partial \hat{y}_k / \partial x_{rk}) = \left[\sum_{i=1}^{c'} (b_{ir} w_{rik}) \right] / \left[\sum_{i=1}^{c'} w_{ik} \right] \qquad ; \; r = 1, \cdots\cdots p; \; k = 1, \cdots\cdots, n$$

Comparing these steps with those in the case of a single input variable, it is clear that the computational burden associated with the fuzzy modelling increases at the same rate as in the case of multiple linear regression as additional explanatory variables are added to the model. Under very mild conditions on the input data and the random error term, fitting the sub-models over each fuzzy cluster yields a weakly consistent predictor of the conditional mean of the output variable. The partitioning of the sample into fuzzy clusters, and the determination of the associated membership functions, involves using only the explanatory variable data in a non-stochastic manner. If the explanatory variables are exogenous then so will be the membership values that are used to construct the weighted averages of the least squares predictors in Step 4 above. Then, the fuzzy predictor of the conditional mean of *y* at Step 4 will be weakly consistent.

5. Some Applications

We have applied the fuzzy econometric modelling described above to a number of illustrative estimation and smoothing problems, and the associated results are described in this section. As was noted in the context of the FCM algorithm above, the computation for these applications was undertaken by writing command code for the SHAZAM (2000) econometrics package. We have chosen a number of simple examples that involve both cross-section and time-series data, and which involve various degrees of model complexity in terms of the number of explanatory variables involved.

5.1 Modelling the Earnings-Age Profile

In our first application the relationship of interest is one that explains the logarithm of earnings as a function of age, the latter being a proxy for years of work experience. Typically, in the labour economics literature, this relationship has been modelled by using standard parametric regression, typically with both age and its square being included as regressors. This quadratic relationship allows for the fact that earnings would be expected to increase with age through much of the working life, but that the rate of increase declines with age (*e.g.*, Heckman and Polachek, 1974; Mincer, 1974). Mincer relates the concave quadratic function to the behaviour that is implied by the optimal distribution of human capital investment over an agent's life-cycle, but more recently Murphy and Welsch (1990) have provided evidence that a quartic relationship performs more satisfactorily than a quadratic one.

Our own application involves a cross-section data-set relating to the earnings and ages of a sample 205 Canadian individuals, all of whom had the same number of years of schooling. These data come from the 1971 Canadian Census Public Use Tapes, and have been used in various parametric and non-parametric studies by Ullah (1985), Singh *et al*. (1987), Pagan and Ullah (1999, 152-157) and van Akkeren (2001). As Pagan and Ullah discuss, simple polynomial models result in a smooth concave relationship, but when a non-parametric kernel estimator is used the fitted relationship exhibits a “dip” around the age of forty to forty five. Interestingly, the same effect is obtained by van Akkeren (2001, Table 3.5) via their data based information theoretic

(DBIT) estimation procedure. A possible explanation for this dip is offered by Pagan and Ullah (1999, 154):

> ....(it may be due to) "the generation effect, because the cross section data represent the earnings of people at a point in time who essentially belong to different generations. Thus the plot of earnings represents the overlap of the earnings trajectories of different generations. Only if the sociopolitical environment of the economy has remained stable intergenerationally can we assume these trajectories to be the same. But this is not the case; one obvious counterexample being the Second World War. Therefore, the dip in the nonparametric regression might be attributed to the generation between 1935 and 1945."

Figure 1 illustrates this effect with the Canadian earnings data. There, we show the results of fitting a quadratic relationship by the method of least squares, as well as a non-parametric kernel estimate. The latter uses a Normal kernel with the approximately optimal bandwidth described by Silverman (1986, 45). Also shown in that figure is the result of applying our fuzzy regression analysis with three fuzzy clusters and $m = 2$. (The results were not sensitive to the latter choice of value for the exponent parameter in the application of the FCM algorithm.) Some of the details associated with the clustering of the data and the estimation of sub-models appear in Table 1, together with a comparison of the quality of the overall "fit" of the fuzzy model as compared with the non-parametric kernel and quadratic least squares models. In part (a) of that table we see that the sub-model estimates (based on each of the three fuzzy clusters) are fundamentally different from each other. When these are combined on the basis of the associated membership functions (which are depicted in Figure 2) we are able to fit a flexible non-linear relationship. In part (b) of Table 1 we see that the fuzzy model fits the data better than the non-parametric model and virtually as well as the quadratic least squares model, on the basis of percent root mean squared error. This quadratic penalty function actually favours least squares, and the fuzzy model emerges as the clear winner when the comparison is based on percent mean *absolute* error.

The sensitivity of the fuzzy econometric modelling results to the chosen number of fuzzy clusters is illustrated in Figure 3, where the results for $c = 3$ and $c = 4$ are compared (with $m = 2$). The latter results appear to "over-fit" the data to some degree, and our preference is for the results shown in Figure 1. The general shape of the fit of the fuzzy regressions reinforces the principal result of the non-parametric estimation (and the DBIT results of van Akkernen (2001)). We see an even more pronounced "dip" than in the case of the non-parametric results, with the minimum occurring at an age of forty one (rather than forty four), and the adjacent peaks occurring at ages of thirty three and forty eight (rather than thirty eight and fifty one).

Figure 1: Alternative Estimates of the Earnings-Age Profile

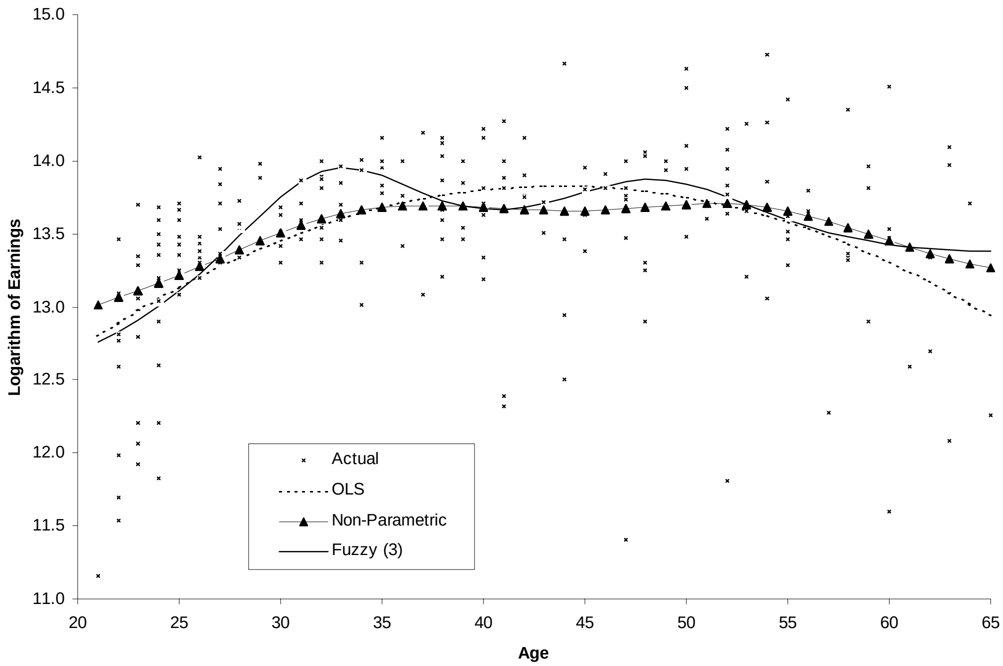


Figure 2: Membership Functions for Fuzzy Regression (c=3; m=2)

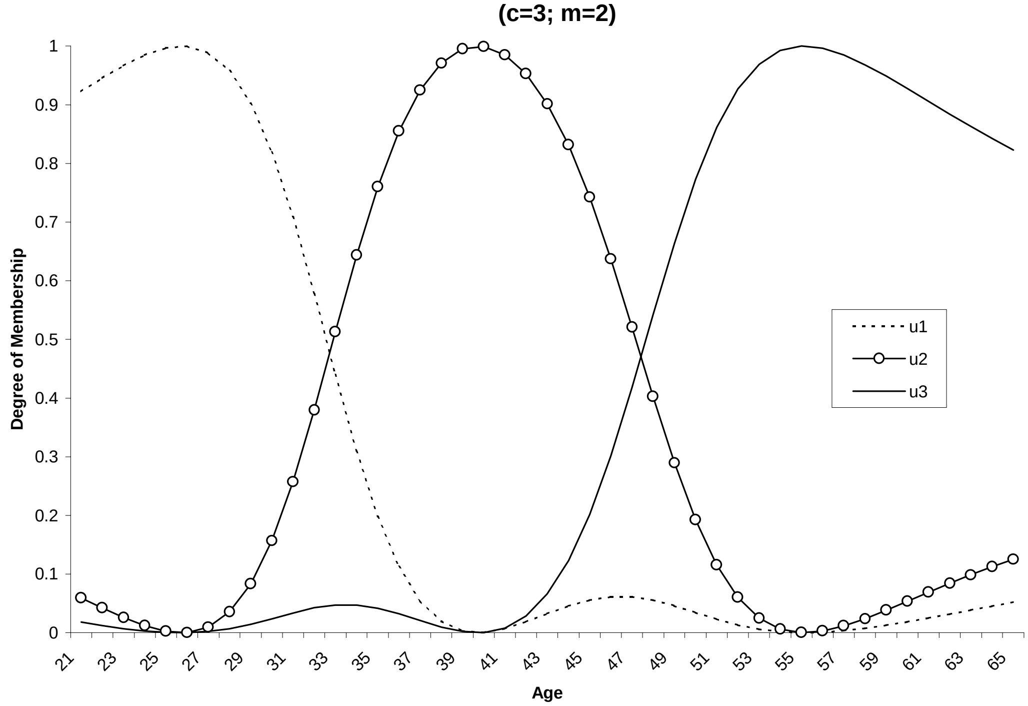


Table 1: Fuzzy Regression Results for Earnings-Age Data

(c = 3; m = 2)

(a) Sub-Model Results

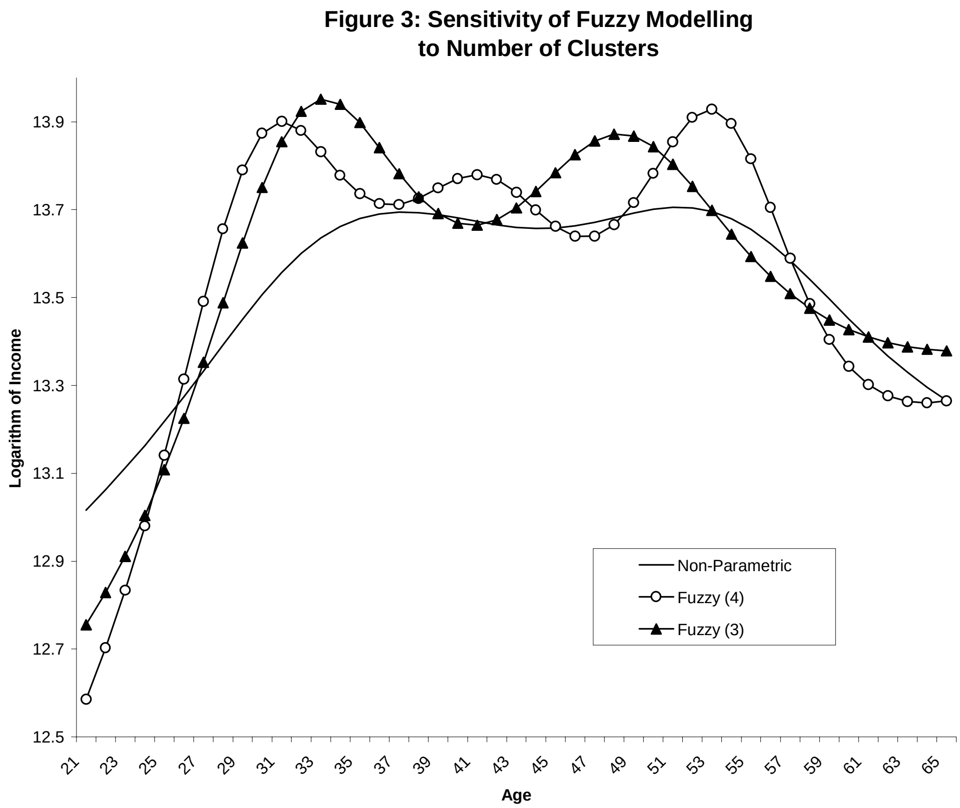
Fuzzy Cluster	**Ranked Observations**	**Cluster Centre**	**Slope (t-value)**	**Intercept (t-value)**
1	1 to 77	25.7421	0.1194	10.1199
	(*21 to 32 years*)		(6.74)	(21.68)
2	78 to 148	39.7241	-0.0180	14.3870
	(*33 to 47 years*)		(-1.31)	(26.22)
3	149 to 205	55.1273	-0.0475	16.2073
	(*48 to 65 years*)		(-2.53)	(15.69)

(b) Comparative Model Performances*

	Least Squares	**Non-Parametric**	**Fuzzy**
%RMSE	4.403	4.459	4.406
%MAE	3.017	3.014	2.992

* %RMSE = Percent root mean squared error of fit.

%MAE = Percent mean absolute error of fit.



5.2 An Aggregate Consumption Function

Our second example involves the estimation of a naive aggregate consumption function for the U.S.A., using monthly seasonally adjusted time-series data for personal consumption expenditures and real disposable personal income. The data are published by the Bureau of Economic Analysis at the U.S. Department of Commerce (2000), and are in real 1996 billions of dollars. The sample period is January 1967 to June 2000 inclusive. The model explains consumption expenditure simply as a function of disposable income. No dynamic effects or other explanatory variables are taken into account.

In this case we found that basing the analysis on four fuzzy clusters produced marginally better results than those based on three clusters. Figure 4 compares the "fit" of the fuzzy model with obtained by simple least squares, and the corresponding comparison is made with a non-parametric kernel fit in Figure 5. The latter was obtained with the same kernel and window choices as in the previous example, and $m = 2$ was used again as the exponent parameter in the application of the FCM algorithm. As can be seen in these two figures, the fuzzy model "tracks" the data much more satisfactorily than do either of its competitors, and this is borne out by the percent root mean square errors and percent mean absolute errors that are reported in Table 2. The close similarity between the results based on three and four fuzzy clusters is clear in Figure 6, and the %RMSE and %MAE values when c = 3 are 1.327 and 1.033 respectively. The membership functions for the Fuzzy (4) model appear in Figure 7.

The economic interpretation of the various fitted models is also interesting. In the linear least squares model the slope parameter is the (constant) marginal propensity to consume (m.p.c.), and is estimated to be 0.9598 (t = 270.39) from our sample. In the case of the non-parametric model the estimated derivatives (which represent the changing m.p.c.) range in value from 0.3134 to 1.2427 as the level of disposable income varies. This is shown in Figure 8, where the income data have been ranked into ascending order. Values of the m.p.c. in excess of unity make no sense in economic terms, and the extreme and unusual pattern of the plot of the non-parametrically estimated m.p.c.'s in Figure 8 suggests that these results should be treated with extreme skepticism. Also given in that figure are the corresponding results for the Fuzzy (4) and Fuzzy (3) models. In both of these cases the plots are much more reasonable than that for the non-parametric model, with the estimated m.p.c.'s for the Fuzzy (3) model lying in a somewhat narrower band than those for the preferred Fuzzy (4) model. The latter m.p.c.'s range from 0.8061 to 1.1705 in value. The few estimates in excess of unity are still troublesome, of course, but apart from these, the derived m.p.c. values are more plausible economically than are those resulting from the non-parametric kernel estimation, or from the (highly restrictive) least squares model.

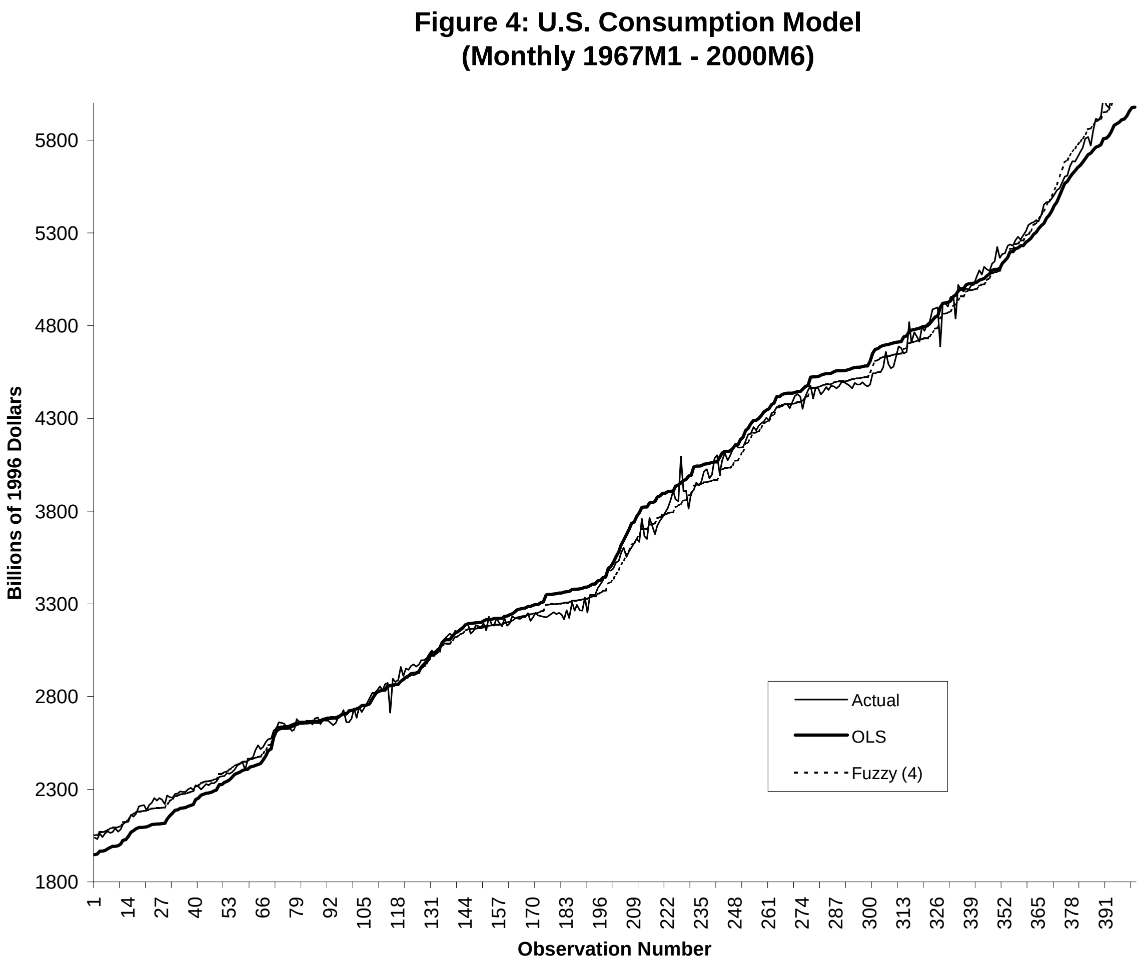


Figure 4: U.S. Consumption Model
(Monthly 1967M1 - 2000M6)
Billions of 1996 Dollars
Observation Number
Actual
OLS
Fuzzy (4)

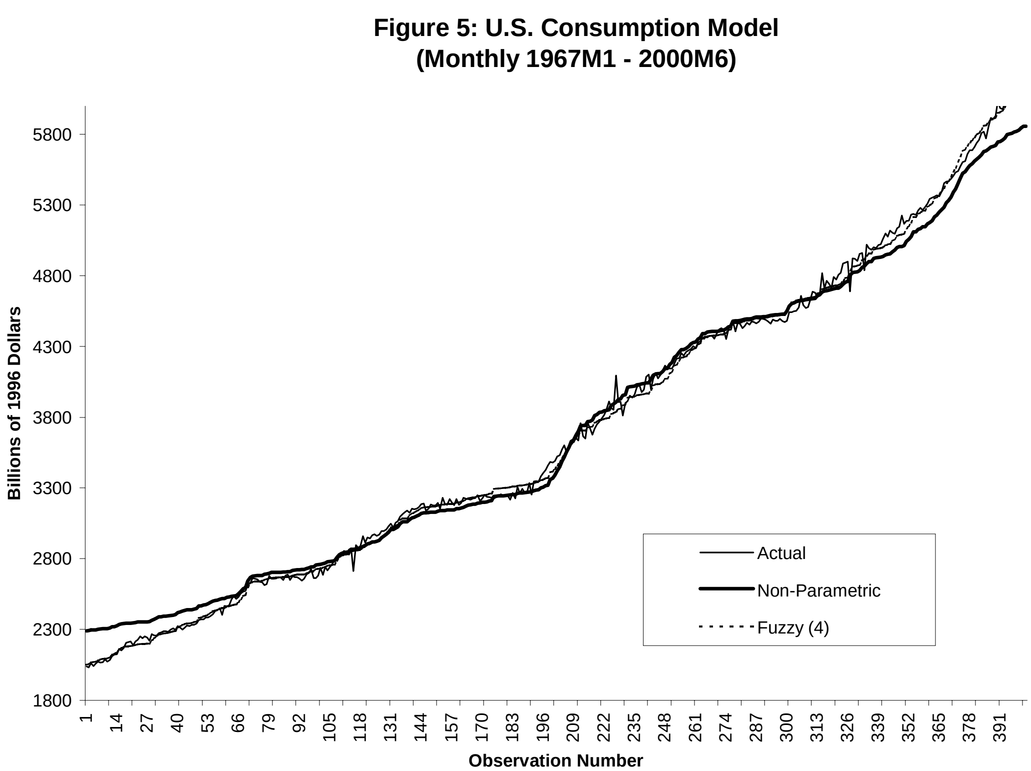


Figure 5: U.S. Consumption Model
(Monthly 1967M1 - 2000M6)
Billions of 1996 Dollars
Observation Number
Actual
Non-Parametric
Fuzzy (4)

Figure 6: Sensitivity of Fuzzy Consumption Model to Number of Fuzzy Clusters

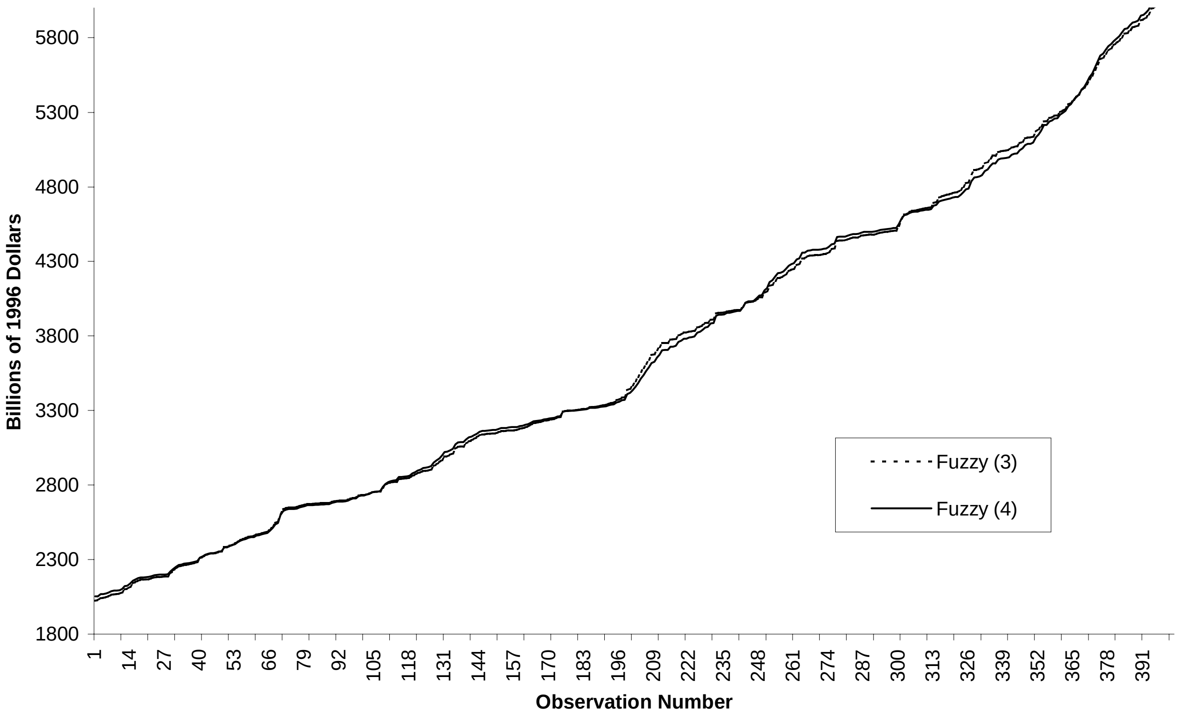


Figure 7: Membership Functions for Fuzzy Regression (c=4; m=2)

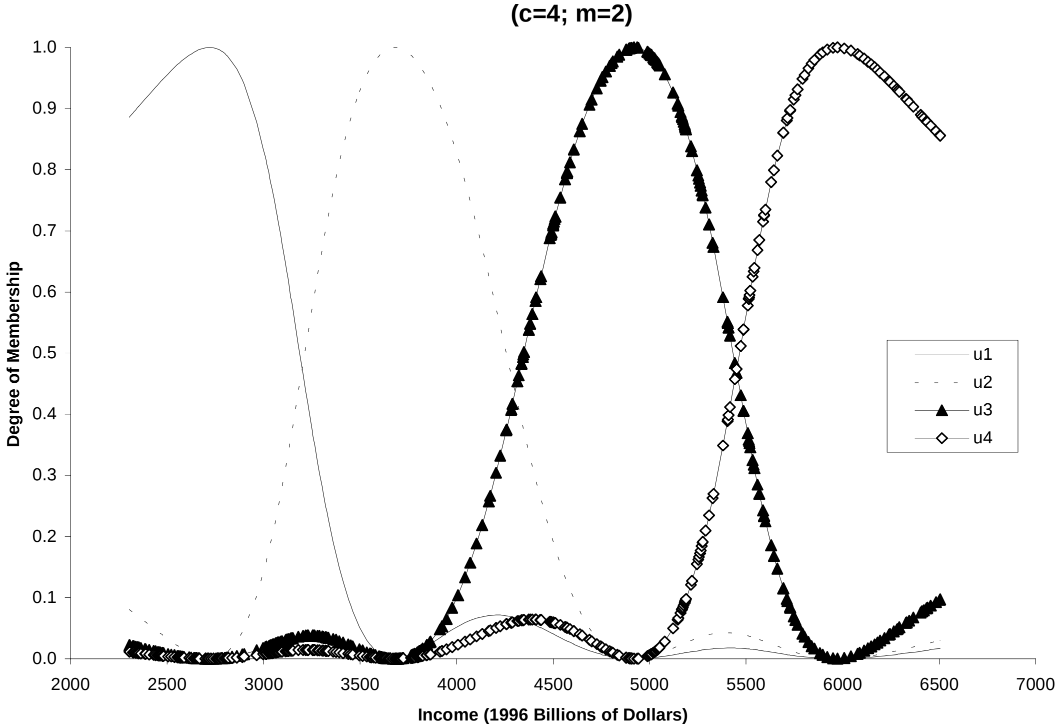


Table 2: Fuzzy Regression Results for Consumption-Income Data

(c = 4; m = 2)

(a) Sub-Model Results

Fuzzy Cluster	**Ranked Observations**	**Cluster Centre**	**Slope (t-value)**	**Intercept (t-value)**
1	1 to 108	2720.247	0.8140	184.45
			(92.18)	(7.55)
2	109 to 217	3681.513	0.8061	258.91
			(59.69)	(5.19)
3	218 to 332	4926.031	0.9740	-393.75
			(63.76)	(-5.26)
4	333 to 402	5973.417	1.1705	-1428.90
			(63.52)	(-13.05)

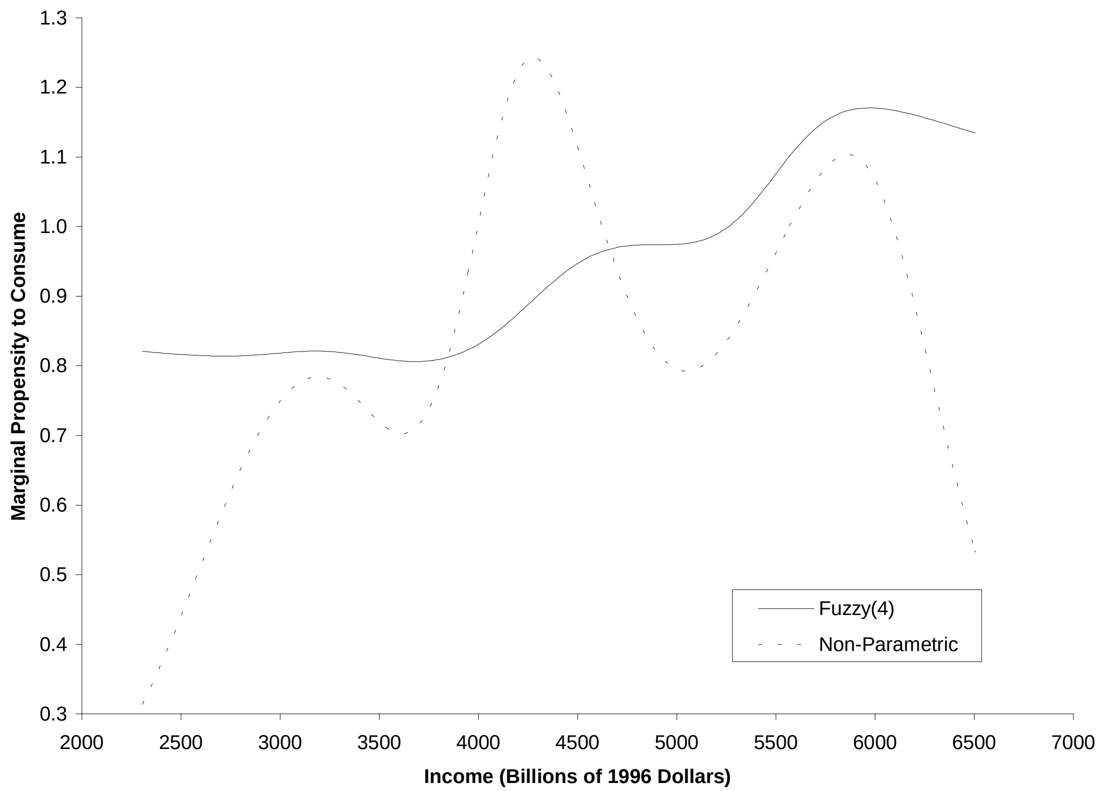
(b) Comparative Model Performances*

	Least Squares	**Non-Parametric**	**Fuzzy**
%RMSE	2.306	3.285	1.201
%MAE	1.828	2.328	0.942

* %RMSE = Percent root mean squared error of fit.

%MAE = Percent mean absolute error of fit.

Figure 8: Predicted Marginal Propensities to Consume



5.3 A Money Demand Model

Our third empirical application involves a simple demand for money model:

$$\log(M_t) = \beta_1 + \beta_2 \log(Y_t) + \beta_3 \log(r_t) + \varepsilon_t$$

where M is the money stock, Y is output, and r is the rate of interest. We have used annual U.S. Department of Commerce (various years) data for the period 1960 to 1983 inclusive, as presented by Griffiths *et al.* (1993, p. 316). We used three fuzzy clusters to analyze each of the two input variables, and then considered all of the resulting nine combinations of the data partitioning. Of these, five resulted in empty or inadequately sized sets - either there were no sample points consistent with the intersection of the fuzzy sets for log(Y) and log(r), or there were insufficient to fit a regression. The sample points associated with the remaining four intersections are shown in Table 3, and the TSSS analysis was applied to this four-way partitioning of the data.

As Figures 9 and 10 indicate, the fit of the fuzzy model clearly dominates that obtained from a non-parametric regression model, and it also dominates an OLS multiple regression analysis – especially at the end-points of the sample. This is confirmed by the %RMSE and %MAE figures that are shown in part (b) of Table 3. Figures 11a and 11b provide details of the membership functions for the income and interest rate input variables respectively, and Figures 12a and 12b show the derivatives of the fuzzy model with respect to each of these input variables at each sample point. As the data are all in (natural) logarithms, these derivatives are elasticities, and in these figures they are compared with their counterparts from the non-parametric model. In each case, not only do the fuzzy elasticities evolve more plausibly with an increase in the input variable, but they also have the anticipated signs. In contrast, the elasticities derived from the non-parametric regression model have signs that conflict with the underlying economic theory for some values of income and the interest rate variable. In the case of the OLS regression model based on the full sample period, the (constant) income and interest rate elasticities are 0.7091 (t = 44.41) and –0.0533 (t = -2.49) respectively. The sample means of their fuzzy counterparts are quite similar in value, being 0.7368 and –0.0606 respectively, and the unsatisfactory nature of the non-parametric results is underscored when we note that the corresponding sample averages of those estimated elasticities are 0.3237 and 0.0666. (The sign of the latter value, of course, conflicts with the prior theory.) In short, the results of the fuzzy modelling dominate both OLS and non-parametric regression analysis in this example.

Figure 9: U.S Demand for Money Model

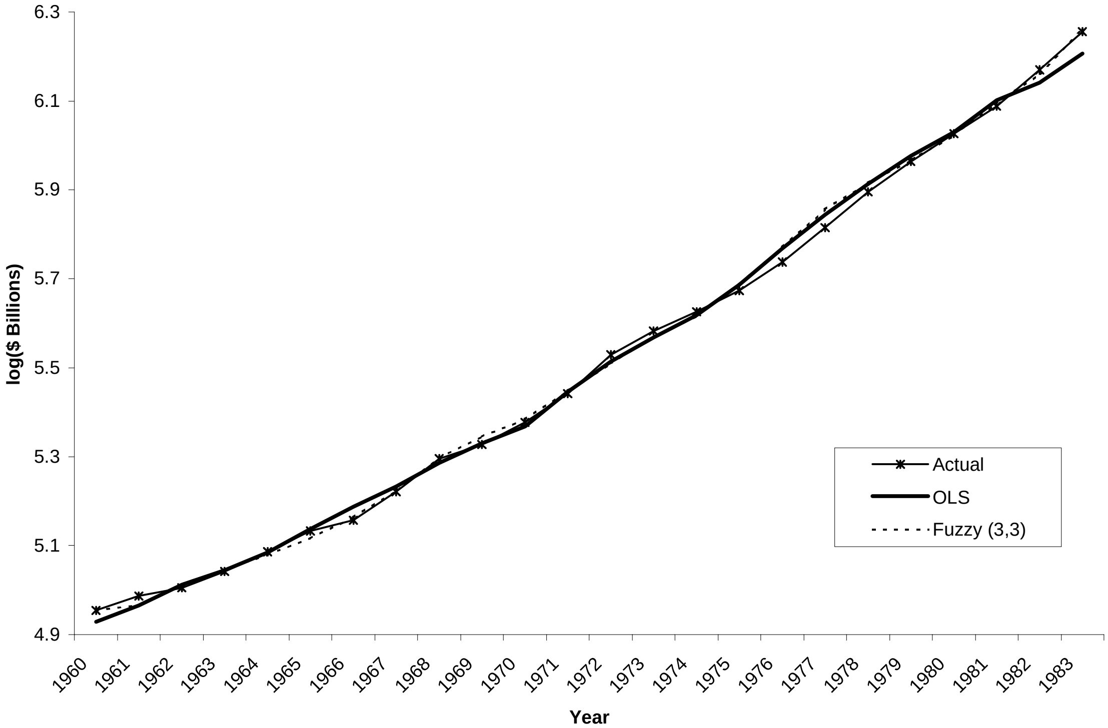


Figure 10: U.S. Demand for Money Model

Table 3: Fuzzy Regression Results for Money Demand Model

($c_1 = c_2 = 3$; $m = 2$)

(a) Sub-Model Results

Fuzzy Cluster	Observations	Cluster Centre	Fuzzy Cluster	Observations	Cluster Centre
	log(Income)			**log(Interest Rate)**	
Y1	1 to 9	6.448	**R1**	1 to 6	1.188
Y2	10 to 18	7.122	**R2**	7 to 19	1.740
Y3	19 to 24	7.882	**R3**	20 to 24	2.348

(b) Cluster Intersection Estimation Results

Fuzzy Intersection	Observations	β_1 (t-value)	β_2 (t-value)	β_3 (t-value)
($Y1 \cap R1$)	1 to 6	1.4501	0.5638	-0.0053
		(3.81)	(8.43)	(-0.11)
($Y1 \cap R2$)	7 to 9	-1.3218	0.9976	-0.0819
		(n.a.)	(n.a.)	(n.a.)
($Y2 \cap R2$)	10 to 18	0.8628	0.6632	-0.0320
		(5.63)	(32.94)	(-1.30)
($Y3 \cap R3$)	20 to 24	-0.4878	0.8718	-0.1465
		(-1.55)	(23.38)	(-5.26)

(c) Comparative Model Performances*

	Least Squares	Non-Parametric	Fuzzy
%RMSE	0.019	0.062	0.015
%MAE	0.015	0.045	0.012

* %RMSE = Percent root mean squared error of fit; %MAE = Percent mean absolute error of fit.
n.a. = not available, as degrees of freedom are exactly zero.

Figure 11a: Membership Functions for Income Variable

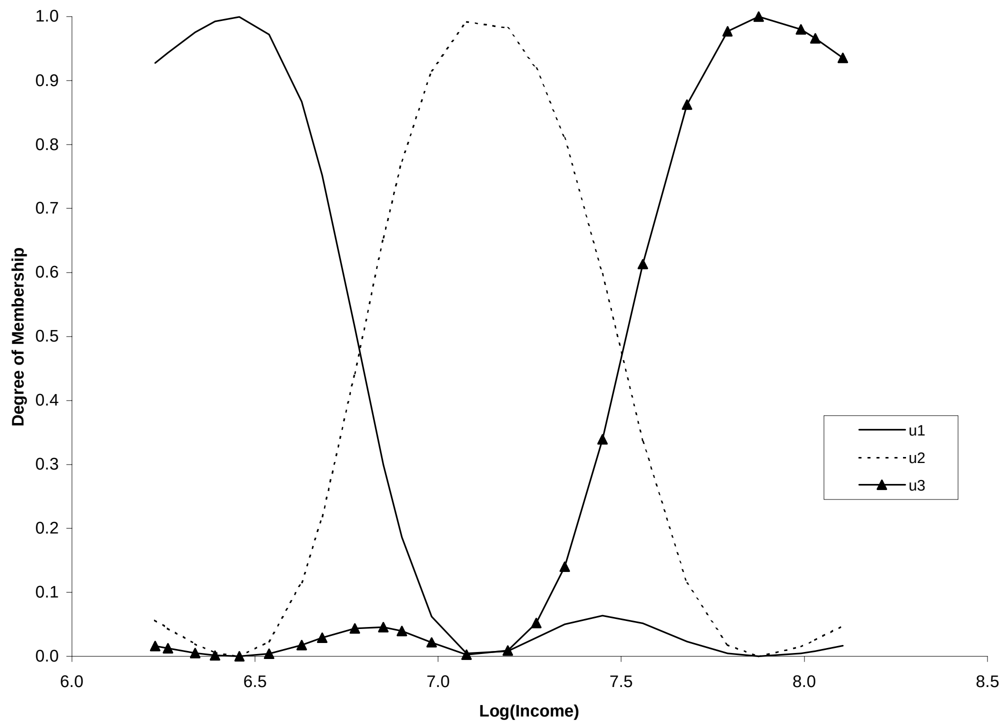


Figure 11b: Membership Functions for Interest Rate Variable

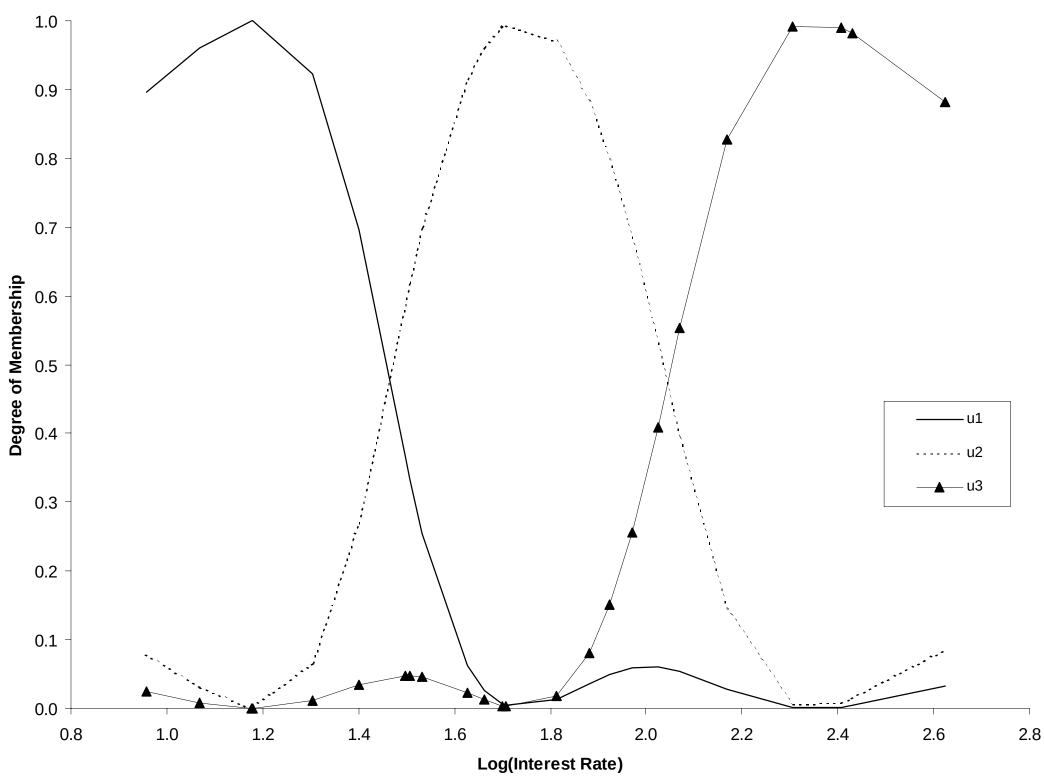


Figure 12a: Predicted Income Elasticities

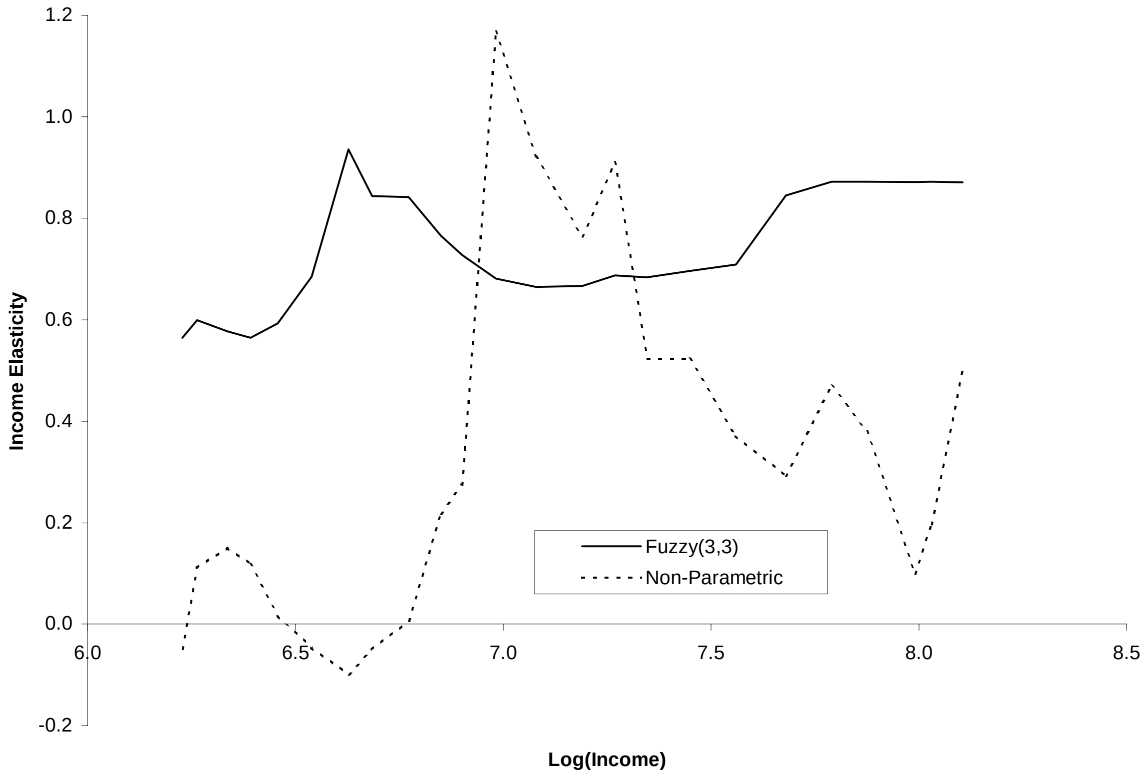
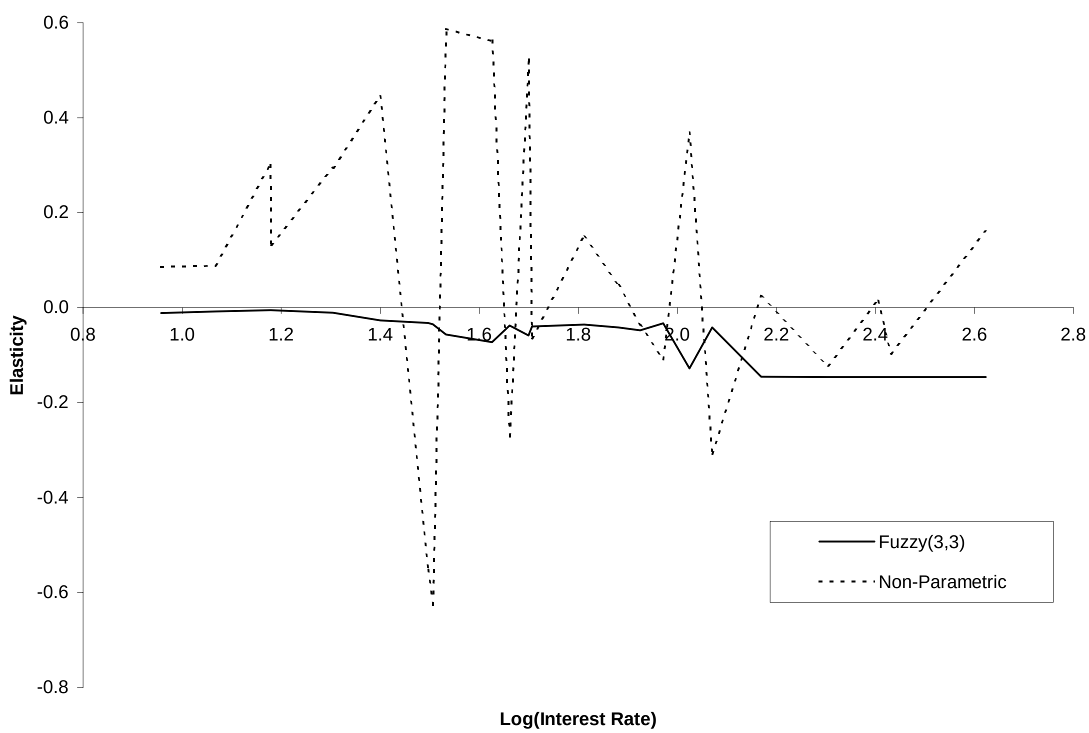


Figure 12b: Predicted Interest Rate Elasticities



5.4 Modelling Kuznets' "U-Curve"

In a seminal contribution, Kuznets (1955) postulated the existence of inverted-U relationship between the degree of income inequality and economic growth. He argued that income inequality (perhaps as measured by the Gini coefficient) increases during the early stages of an economy's growth, reaches a maximum, and then declines as the economy matures. This hypothesis has been subjected to a substantial amount of empirical testing in the development economics literature. The results are somewhat mixed, depending on the type of data used (*e.g.,* time-series, cross-section or panel), the level of development of the country in question, and the method of estimation. In the case of the U.S., the empirical evidence quite clearly rejects Kuznets' hypothesis in favour of U-shaped relationship between income inequality and real *per capita* output. Recently, Hsing and Smyth (1994) found support for this result using aggregate U.S. data, whether an allowance was made for ethnic origin or not. Using the same data, but taking into account the non-stationarity of the data, Jacobsen and Giles (1998) also found support for a U-shaped relationship. They also found differences in the output levels at which the minimum of the U-curve occurred for whites as opposed to blacks and others.

Here, we apply our fuzzy modelling to this same data-set, focussing simply on the income inequality data for "all" ethnic groups, and abstracting from non-stationarity issues. The relationship that we consider for expository purposes is one in which the only explanatory variable is real *per capita* GDP. Both Hsing and Smyth (1994) and Jacobsen and Giles (1998) also considered the percentage of married couple families as an additional explanatory variable, and the latter author also allowed for dynamic effects in the model specification. The benchmark specification that we compare against is a simple but rigid quadratic relationship, estimated by OLS:

$$GA_t = \beta_1 + \beta_2 \, GDPPC_t + \beta_3 \, GDPPC_t^2 + \varepsilon_t$$

where GA is the (%) Gini coefficient for the entire U.S. population, and GDPPC is *per capita* real (1987) GDP. The sample period is 1947 to 1991 inclusive. We also estimate a non-parametric kernel regression that explains GA as a function of GDPPC only, and we apply our fuzzy analysis to the model:

$$GA_t = \beta_1 + \beta_2 \, GDPPC_t + \varepsilon_t$$

with three fuzzy clusters, and $m = 2$. The results were not sensitive to using four fuzzy clusters, as opposed to three.

The estimation results appear in Table 4, and we again see there that the fuzzy model “fits” the data marginally better than its competitors on the basis of either %RMSE or %MAE. In Figure 13 we see that the fuzzy model supports the U-shaped relationship found by either OLS or non-parametric regression, though the minimum of the fitted relationship occurs at slightly different output levels in each case. With quadratic OLS estimation this minimum is at a *per capita* real output level of approximately \$13,300, but at approximately \$12,700 and \$13,500 respectively in the cases of the fuzzy and non-parametric models. The membership functions associated with the fuzzy clustering of the GDP data appear in Figure 14, and the estimated derivatives associated with the fuzzy, OLS and non-parametric models are plotted in Figure 15.

Figure 13: Kuznets' U-Curve Model

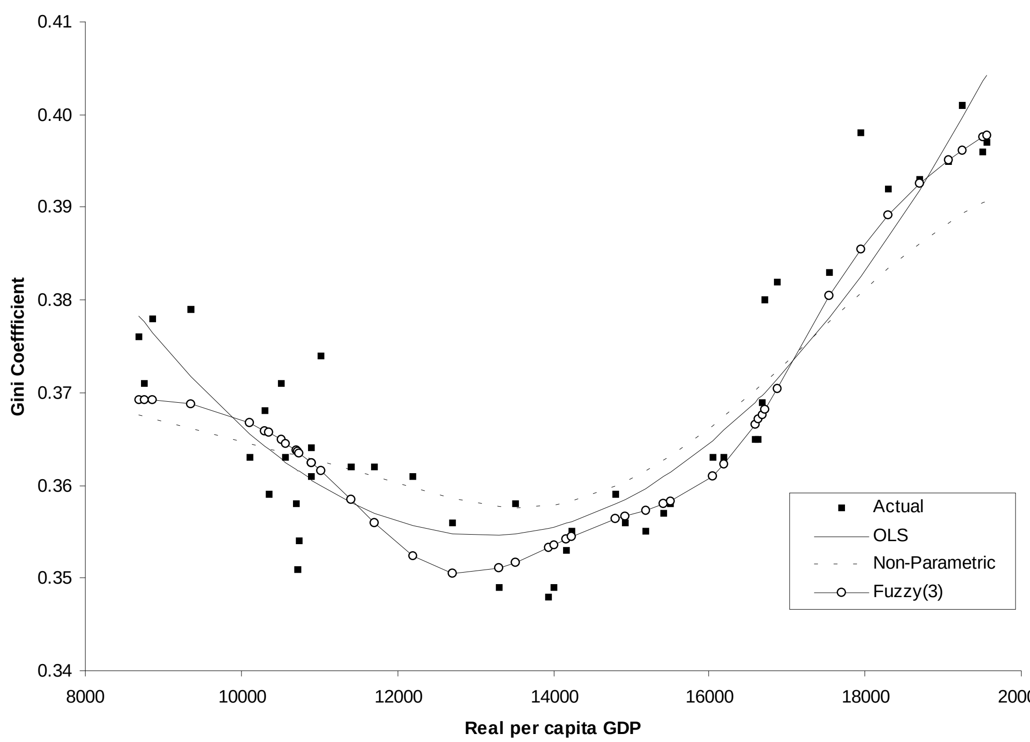


Figure 14: Membership Functions for GDP

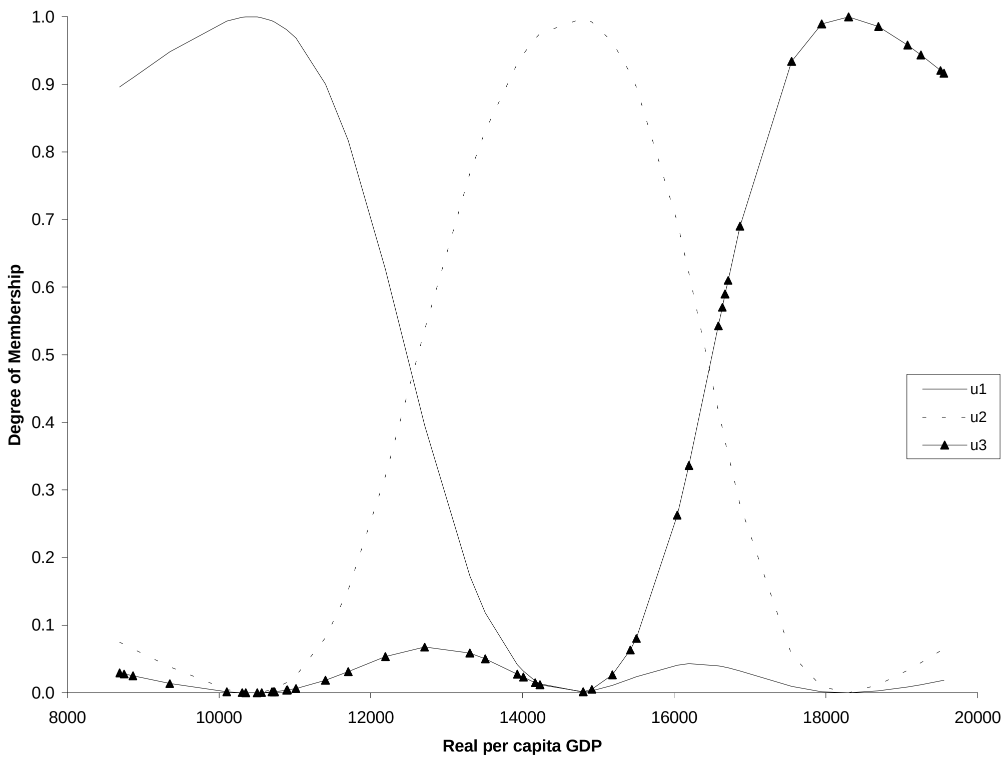


Figure 15: Estimated Gini-GDP Derivatives

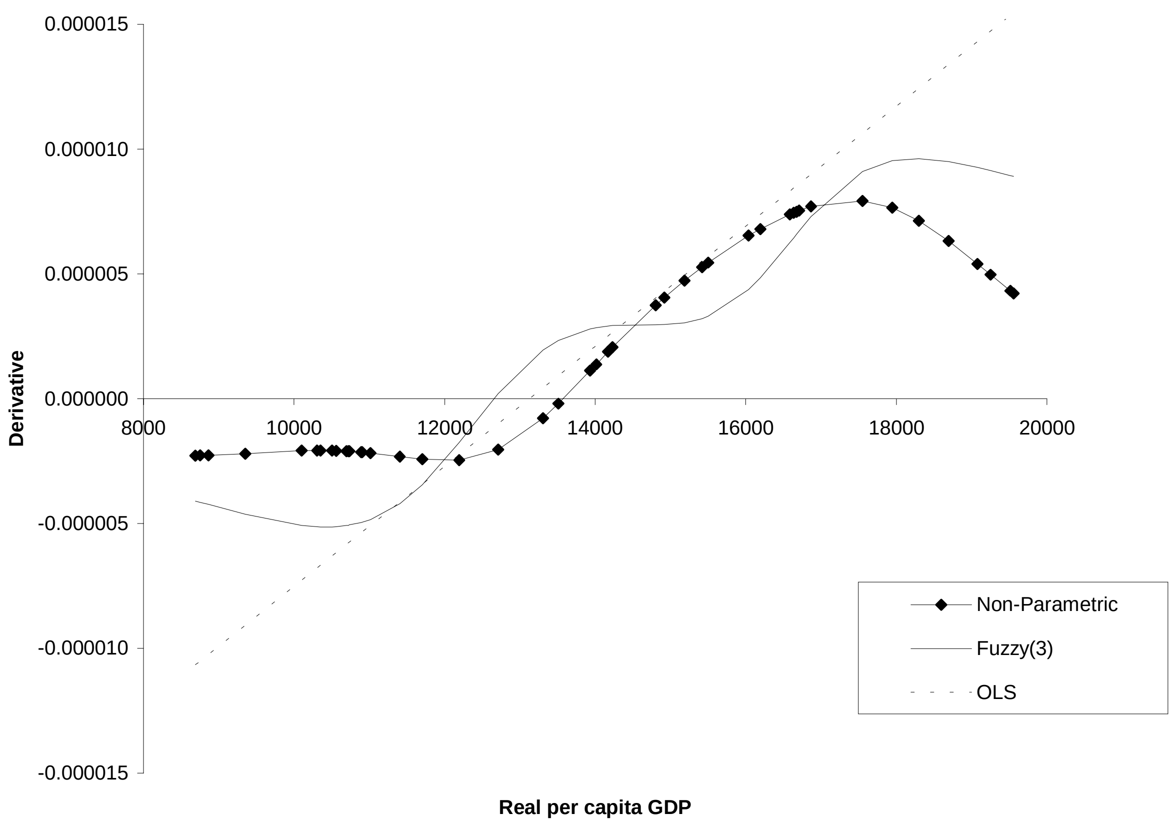


Table 4: Fuzzy Regression Results for Kuznets' U-Curve

(c = 3; m = 2)

(a) Sub-Model Results

Fuzzy Cluster	**Observations**	**Cluster Centre**	**Slope (t-value)**	**Intercept (t-value)**
1	1 to 18	10421.87	$-0.5140*10^{-5}$	0.4190
	(*1947 to 1964*)		(-3.19)	(24.84)
2	19 to 32	14676.64	$0.2956*10^{-5}$	0.3126
	(*1965 to 1978*)		(3.00)	(21.75)
3	33 to 45	18256.73	$0.9617*10^{-5}$	0.2132
	(*1979 to 1991*)		(6.24)	(7.70)

(b) Comparative Model Performances*

	Least Squares	**Non-Parametric**	**Fuzzy**
%RMSE	1.574	1.890	1.570
%MAE	1.288	1.595	1.196

* %RMSE = Percent root mean squared error of fit.
%MAE = Percent mean absolute error of fit.

5.5 Trend Extraction in Time-Series Data

The fuzzy modelling methodology under discussion in this paper can, of course, be applied to a wide range of situations. One that is somewhat different from those considered so far is the detection and extraction of the trend component of an economic time-series. To illustrate this, we have modelled a relationship of the form

$$log(ER_t) = f(t) + \varepsilon_t$$

where ER is the monthly Canada/U.S. exchange rate (noon spot rate, beginning of month). The logarithmic specification recognizes the likelihood of an underlying multiplicative time-series model. In general, such trend extraction would be based on seasonally adjusted data, but the absence of any discernible seasonal pattern in this particular case makes this unnecessary. Table 5 shows the results of our fuzzy modelling with a seven-cluster specification.

As in the previous examples, we compare the fuzzy modelling with non-parametric kernel estimation. Rather than also consider an OLS model where f(t) is some polynomial in time, we have used the Hodrick-Prescott (1980, 1997) filter as a further benchmark for measuring the trend. Our sample covers the period since the (re-)floating of the Canadian dollar, and runs from May 1970 to December 2000. We follow Kydland and Prescott (1990, p.9) and set the smoothing parameter for the Hodrick-Prescott (H-P) filter to $\lambda = 14400$ (*i.e.*, 100 times the square of the data frequency), as we have monthly data. The H-P filter was implemented using with the TSP package (Hall, 1996) using code written by Cummins (1994).

Table 5: Fuzzy Regression Results for the Exchange Rate Trend

(c = 7; m = 2)

Fuzzy Cluster	**Observations**	**Cluster Centre**	**Slope (t-value)**	**Intercept (t-value)**
1	1 to 50	24.07	-0.0010	0.0273
			(-8.38)	(7.70)
2	51 to 104	77.37535	0.0032	-0.2054
			(11.51)	(-9.39)
3	105 to 158	131.5562	0.0014	0.0020
			(10.59)	(0.12)
4	159 to 212	185.7503	0.0014	0.0171
			(5.16)	(0.33)
5	213 to 266	239.6784	-0.0013	0.4860
			(-8.00)	(12.16)
6	267 to 319	293.1247	0.0020	-0.3126
			(8.37)	(-4.35)
7	320 to 368	345.4138	0.0020	-0.3027
			(7.44)	(-3.31)

The membership functions associated with this fuzzy modelling appear in Figure 16, and Figure 17 depicts the associated trend analysis. The corresponding non-parametric and H-P filter trend analyses appear in Figure 18. In this application the H-P filter results are quite robust to the choice of the smoothing parameter. For instance, when λ = 100 (the usual choice for *annual* data) is used, the results are very similar to those shown in Figure 18, but there is undue variation in the extracted trend. On the other hand, in the case of the fuzzy trend analysis a good deal of care has to be taken over the choice of the number of fuzzy clusters if a trend as "flexible" as that produced by the H-P filter is to be obtained. This can be seen in Figure 19, though obviously this sensitivity is at least in part a function of the sample size and the nature of the trend itself.

In many respects, the trend of these exchange rate data that is identified by means of our fuzzy modelling appears to somewhat more reasonable than that resulting from the application of the Hodrick-Prescott filter. In particular, the latter seems to exhibit too much cyclical variation in this case, at least visually. A much more comprehensive comparison of these two filtering procedures would be a fruitful research topic, but it is one that is beyond the scope of the present paper.

Figure 16: Membership Functions for Trend

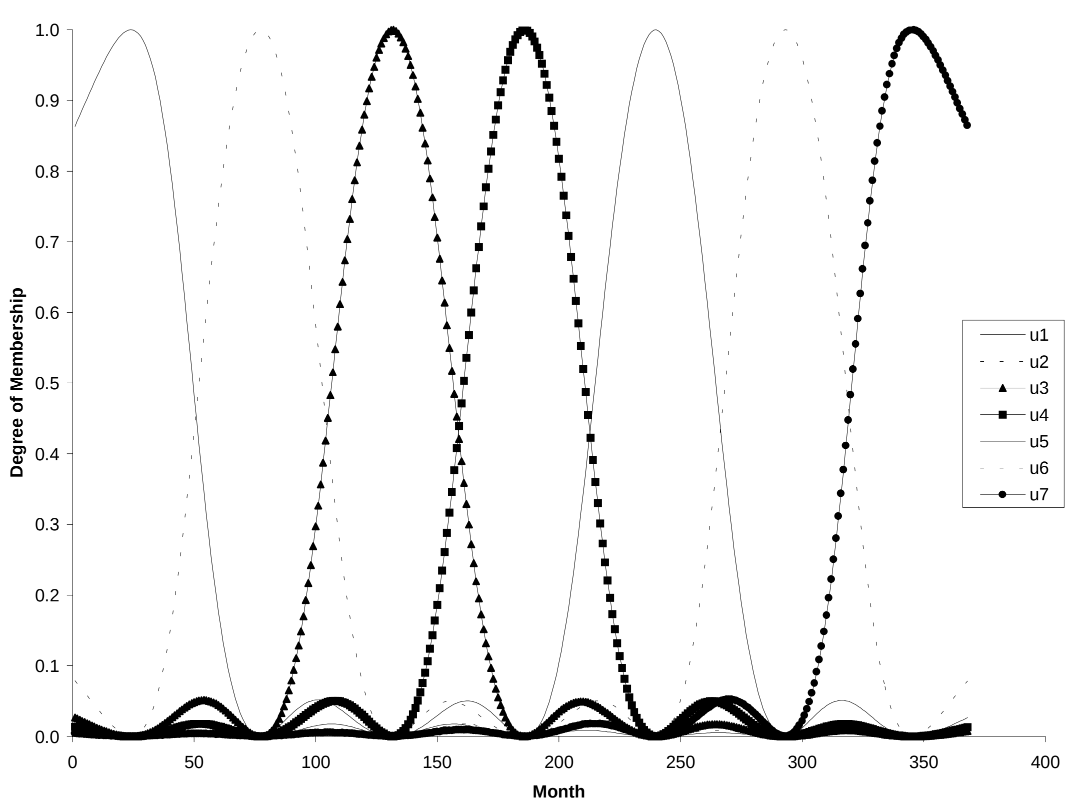


Figure 17: Exchange Rate Fuzzy Trend Analysis

Figure 18: Non-Parametric and H-P Filter Trend Analysis

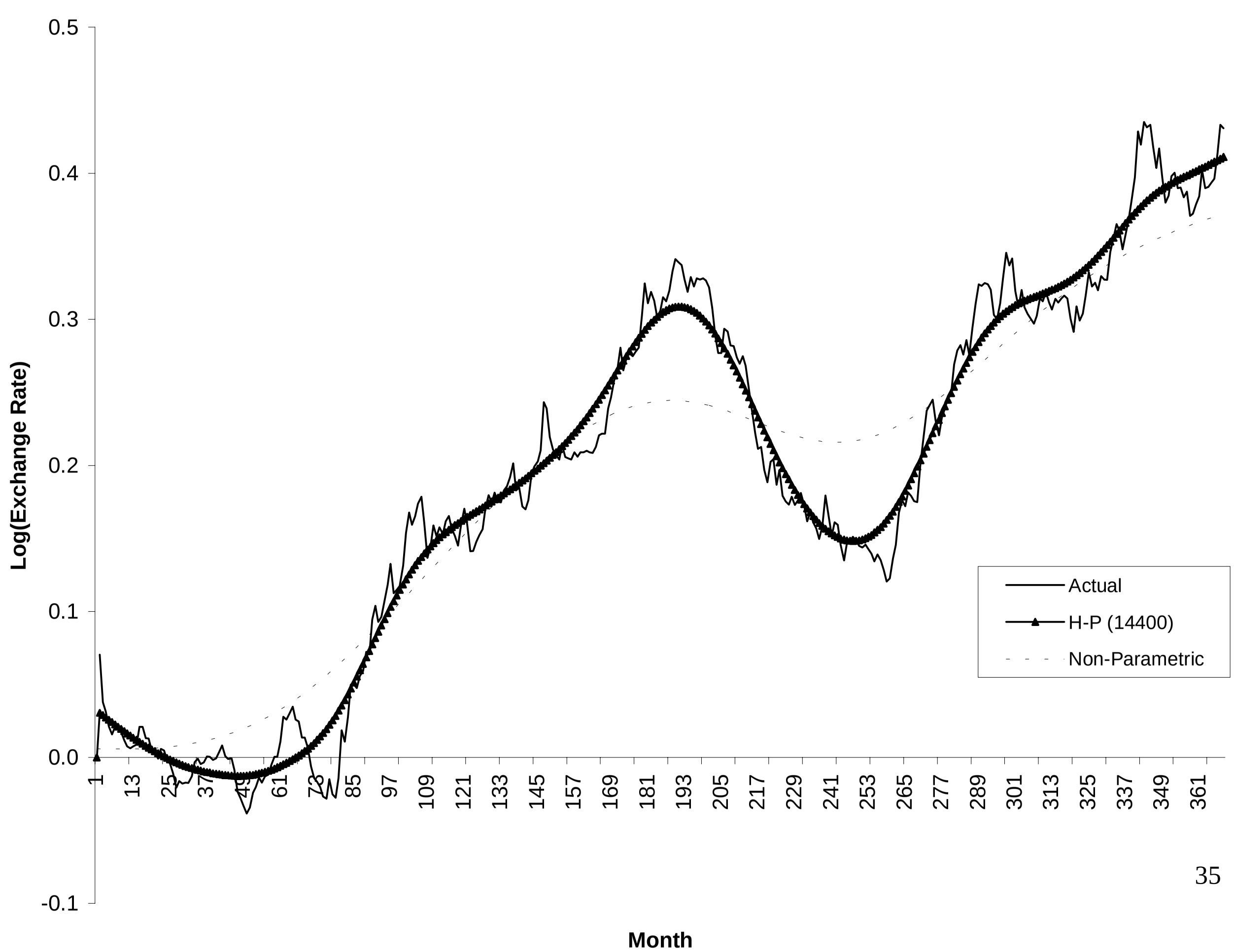


Figure 19: Sensitivity of the Fuzzy Modelling to the Number of Fuzzy Clusters

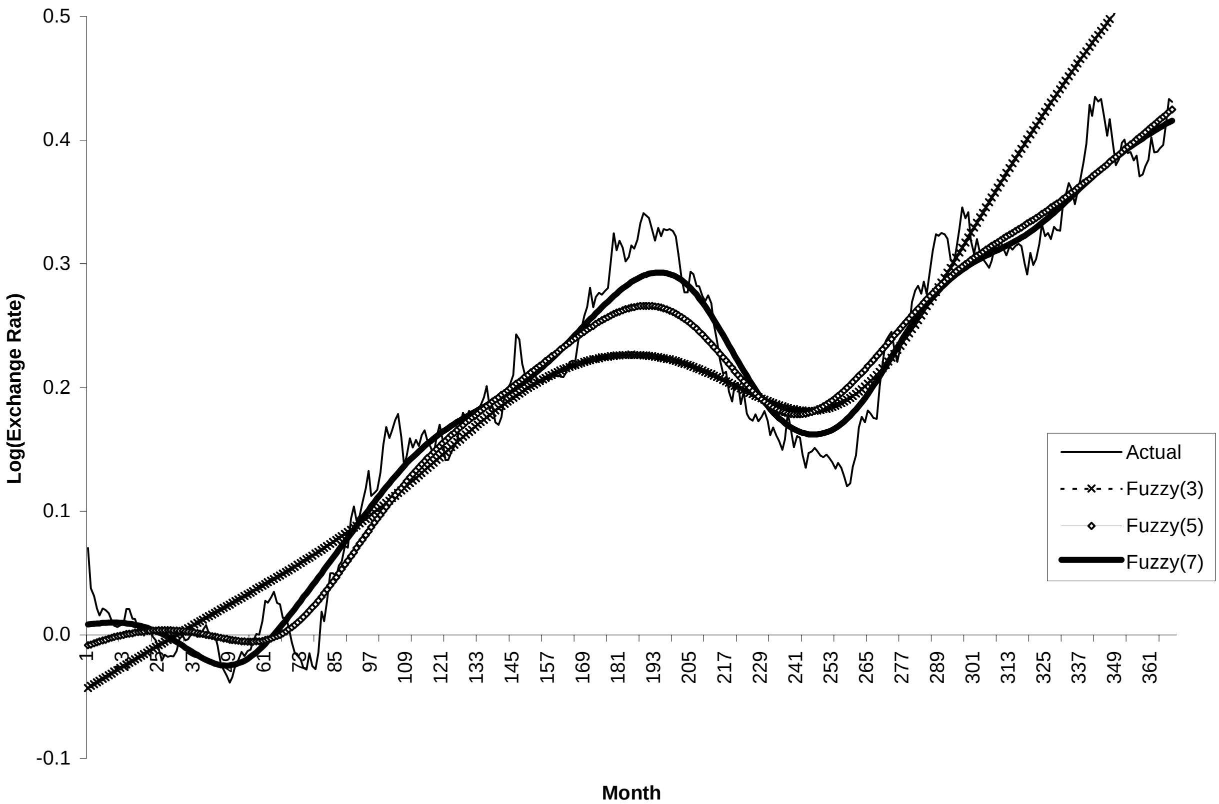
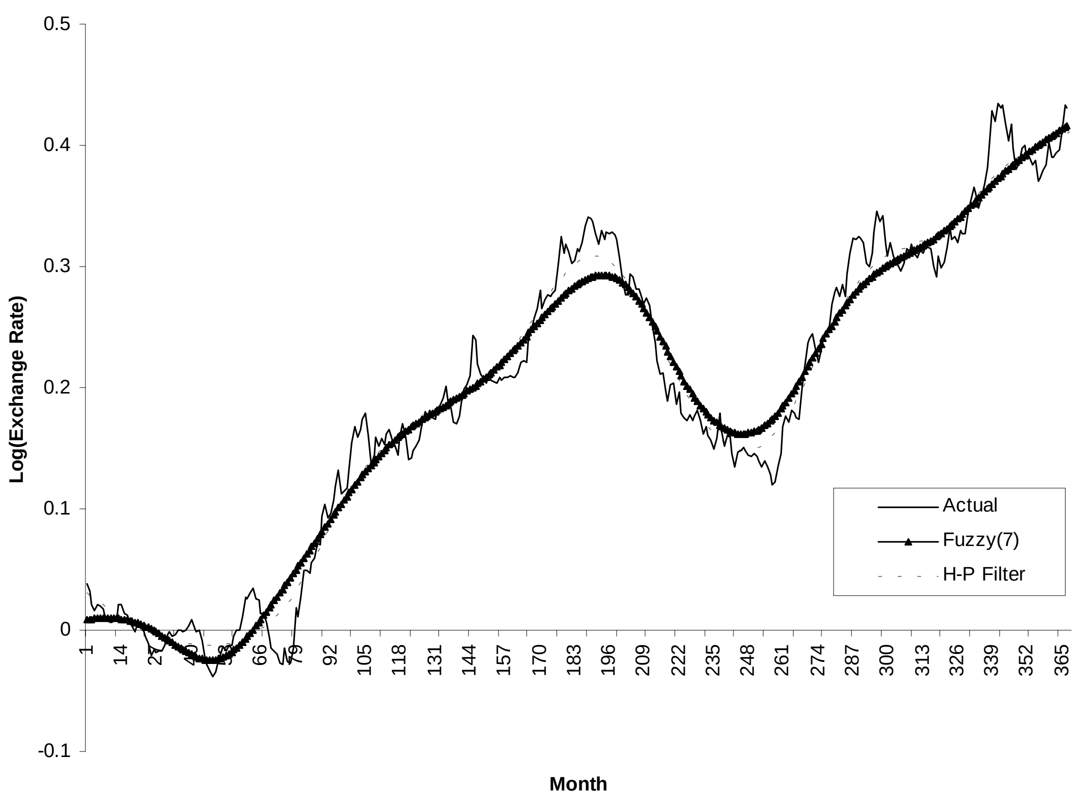


Figure 20: Fuzzy Trend *vs*. H-P Trend



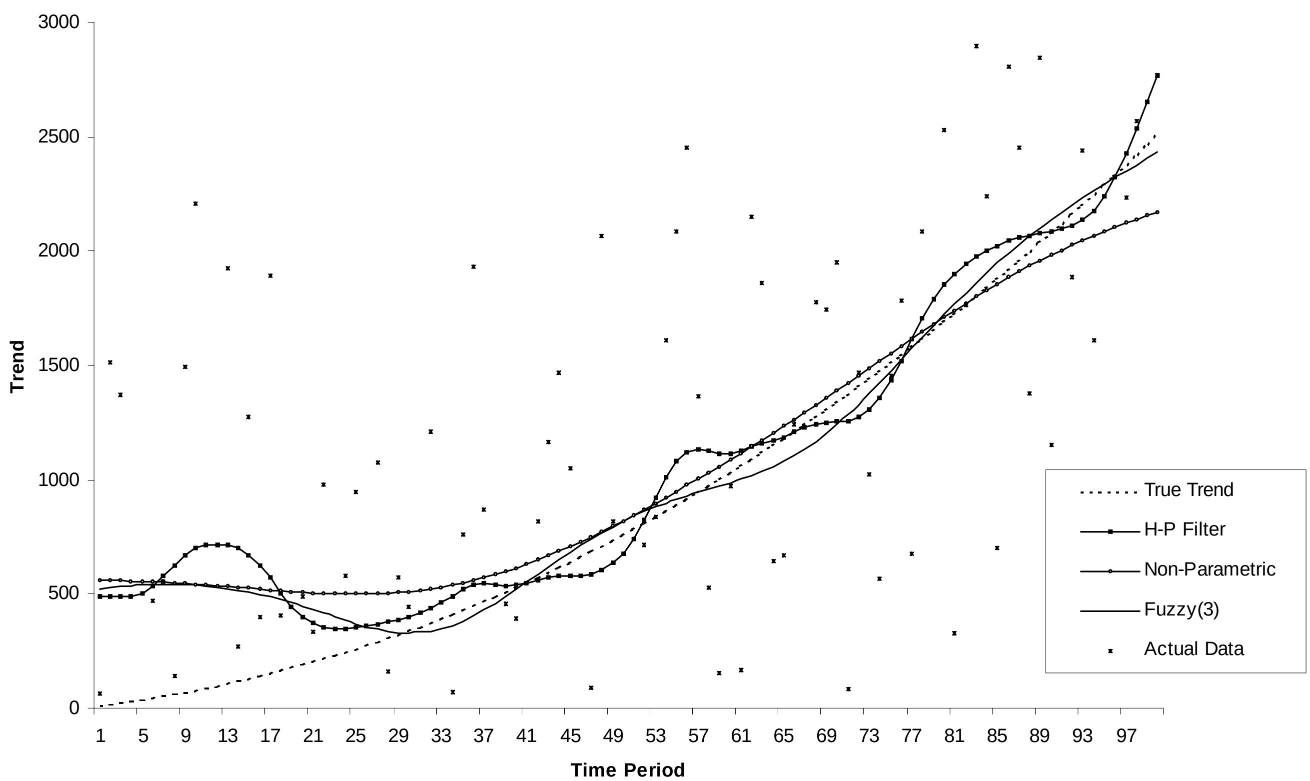
However, by way of a simple experiment the following artificial time-series has been considered:

$$y_t = 10 + 5*t + 0.2*t^2 + 100*\sin(t) + 500*\sin^2(t) + \varepsilon_t$$

where $\varepsilon_t \sim N[0\ ,\ \sigma = 200]$; t = 1, 2, 3, ..., 100.

Figure 21 shows the results of attempting to extract the (known) trend from this series, using a fuzzy model with three membership functions, the H-P filter (with λ = 100), and non-kernel regression, with "time" as the explanatory variable. As can be seen, the data are extremely variable with very little by way of a discernible trend. The fuzzy trend extractor performs well relative to the other methods, and yields a simple correlation of 0.992 with the true trend component. It performs especially well in the latter part of the sample. The corresponding simple correlations for the H-P filter and the non-parametric regression are 0.964 and 0.983 respectively.

Figure 21: Artificial Data - Comparative Trend Analysis



5.6 Modelling the Demand for Chicken

Our final application is based on an example, and data, provided in Studenmund's (1997, pp. 174-175) well known text. This example relates to the demand for chicken, as a function of its own price, disposable income, and the price of a substitute meat. More specifically, the model is of the form:

$$Q_t = f(P^C_t, P^B_t, Y_t) + \varepsilon_t$$

where:

Q = U.S. *per capita* chicken consumption (pounds)

P^C = U.S. price of chicken (cents/pound)

P^B = U.S. price of beef (cents/pound)

Y = natural logarithm of U.S. *per capita* disposable income (dollars)

The sample (Studenmund, 1997, p.199) comprises annual data for 1951 to 1990 inclusive.

As in the earlier regression examples, we consider OLS, non-parametric kernel estimation, and fuzzy modelling. The latter is applied to all three (non-constant) explanatory variables. In each case we use two fuzzy clusters, resulting in 8 potential sub-samples, three of which had positive degrees of freedom. (Three fuzzy clusters implies 27 potential sub-samples, and this resulted in only two with positive degrees of freedom. The resulting fuzzy prediction path exhibited erratic movements at several time-points.)

The basic results are shown in Figure 22 and Table 6. The membership functions were of the same general form as in the previous examples, and are not shown here to reduce space. As can be seen in Figure 22, the fuzzy model performs very creditably, and this is reflected in the %RMSE and %MAE values shown in Table 6. Indeed, if it were not for the relatively poor predictive performance of the fuzzy model during the period 1974 to 1981, its overall performance would be exceptionally good. Our view is that the relatively small sample size disadvantages the fuzzy model in this example, and this may also be an instance where some experimentation is needed with the value for the exponent parameter, "*m*", in the objective functional for the FCM algorithm.

Figure 22: Chicken Consumption Model

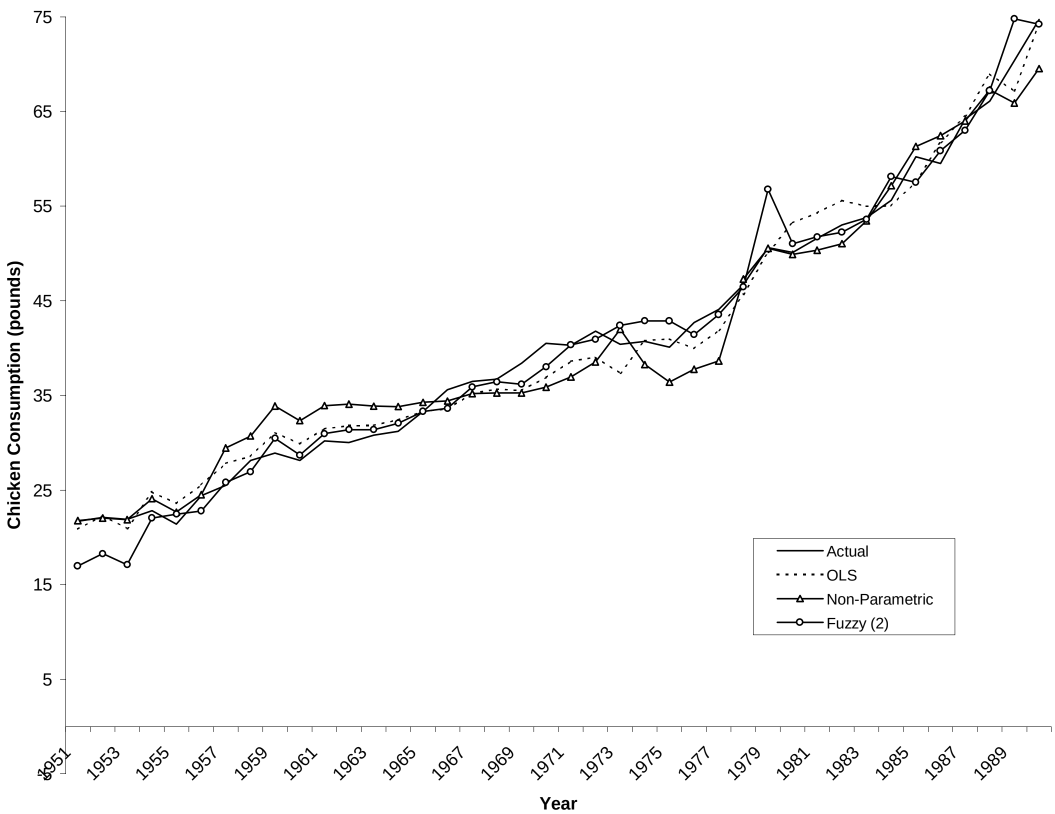


Figure 23 a: Predicted Own-Price Elasticities

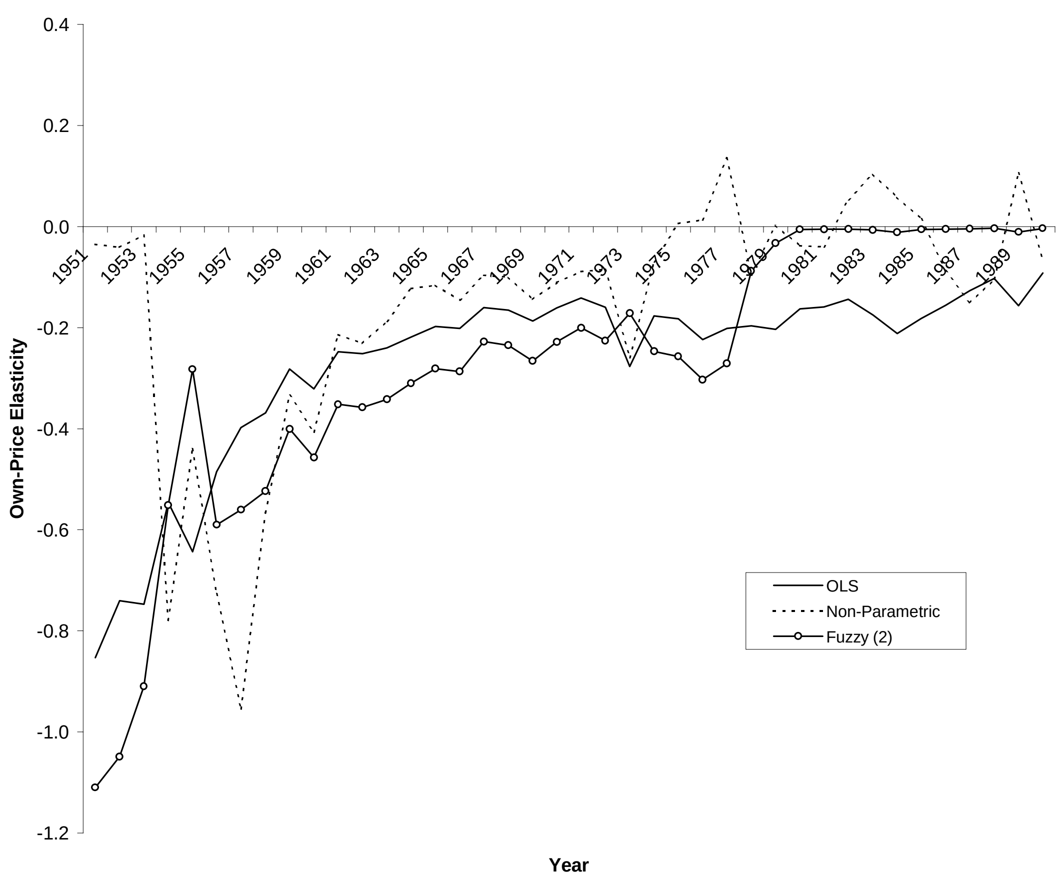


Figure 23 b: Predicted Cross-Price Elasticities

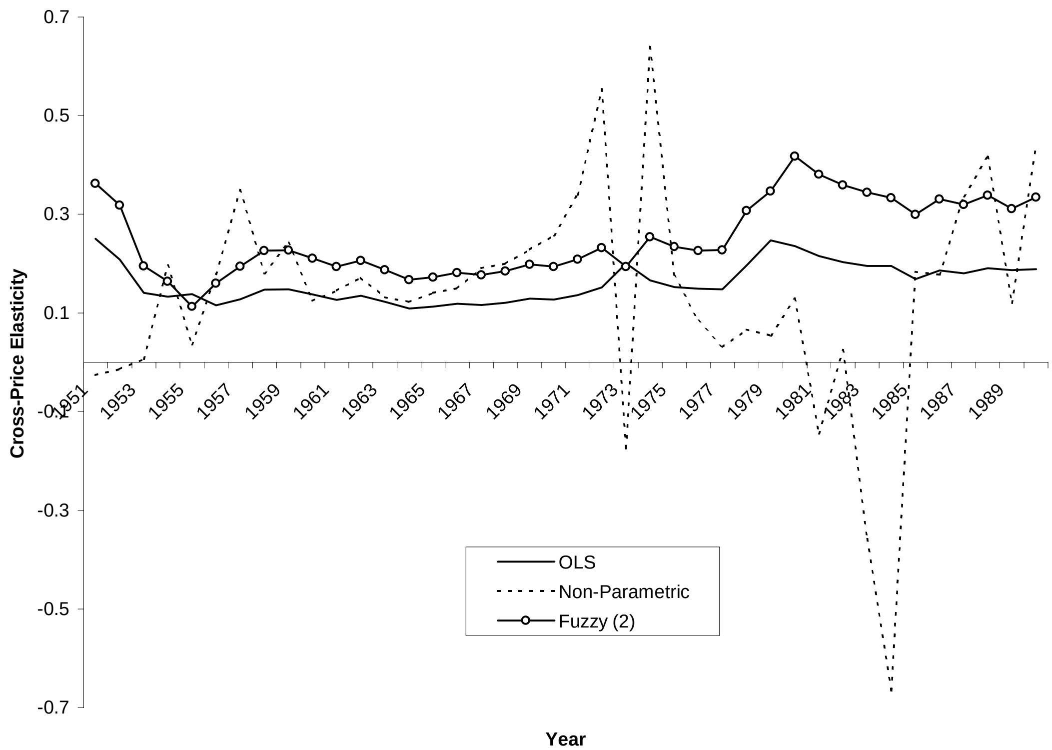


Figure 23 c: Predicted Income Elasticities (Multiplied by 1,0000)

Table 6: Fuzzy Regression Results for Chicken Consumption Model

($c_1 = c_2 = c_3 = 2$; m = 2)

(a) Sub-Model Results*

Fuzzy Cluster	Cluster Centre	Fuzzy Cluster	Cluster Centre	Fuzzy Cluster	Cluster Centre
Price of Chicken		**Price of Beef**		**log(Income)**	
P^C1	10.420	**P^B1**	23.463	**Y1**	2875.560
P^C2	18.166	**P^B2**	60.183	**Y2**	12175.260

(b) Cluster Intersection Results

Fuzzy Intersection	Obs.	β_1 (t-val.)	β_2 (t-val.)	β_3 (t-val.)	β_4 (t-val.)
($P^C1 \cap P^B1 \cap$ Y1)	7 to 22,	30.4720	-1.0523	0.2901	0.0026
	24 to 27	(9.90)	(-5.34)	(1.84)	(3.70)
($P^C1 \cap P^B2 \cap$ Y2)	29 to 33,	9.7358	-0.0243	0.3360	0.0024
	36 to 38, & 40	(1.42)	(-0.07)	(4.29)	(13.36)
($P^C2 \cap P^B2 \cap$ Y1)	1 to 6	15.1140	-0.2595	0.1433	0.0061
		(0.70)	(-0.91)	(0.82)	(0.58)

(c) Comparative Model Performances**

	Least Squares	Non-Parametric	Fuzzy
%RMSE	5.130	7.692	6.879
%MAE	4.407	6.001	4.559

** β_1, β_2, β_3 and β_4 are the coefficients of the intercept, P^C, P^B, and Y respectively.

* %RMSE = Percent root mean squared error of fit; %MAE = Percent mean absolute error of fit.

Figure 23 displays the predicted own-price, cross-price and income elasticities of demand (by year in the sample) implied by the OLS, non-parametric, and fuzzy models. Those relating to the non-parametric estimation are quite unsatisfactory. In particular, in each case there are values that have a sign opposite to that implied by the underlying economic theory. There are no such aberrations in the case of the fuzzy and OLS results. The latter generally appear somewhat more plausible as they exhibit fewer marked fluctuations than do the fuzzy elasticities. This weakness of the fuzzy elasticities is related to a similar feature in the predicted time-paths, noted above in relation to Figure 22. It is also interesting that the fuzzy income elasticities always exceed those from OLS estimation, and this is almost always the case with the cross-price elasticities.

5. Concluding Remarks

In this paper we have discussed the possibility of using various analytical tools from fuzzy set theory and fuzzy logic to model non-linear econometric relationships in a flexible, and essentially semi-parametric way. More specifically, we have explored the use of the fuzzy c-means clustering algorithm, in conjunction with the Takagi and Sugeno (1985) approach to fuzzy systems, in this particular context. The general modelling strategy that we have considered involves identifying interesting and important fuzzy sets or fuzzy clusters of the multidimensional data, in a totally non-parametric way; fitting separate parametric regression models to each fuzzy cluster (or sub-sample) of the data; and then combining the estimates of these separate models in a very flexible way, based on the "degree of membership" of each sample point to each fuzzy cluster. It is this last step in the analysis that facilitates the overall procedure's ability to capture intrinsic non-linearities, because the "weights" that are effectively assigned to each sub-model vary continuously from data-point to data-point.

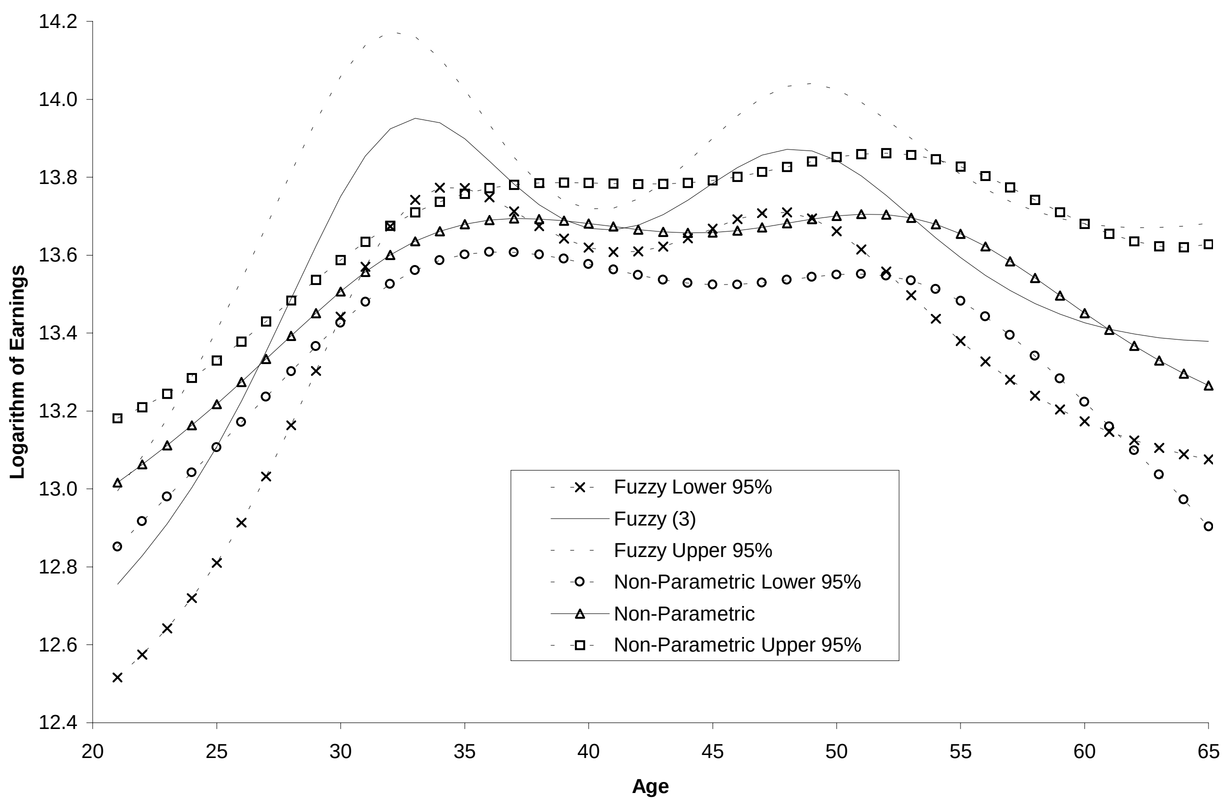
This general fuzzy modelling procedure has a wide range of applications, and we have provided several empirical applications to illustrate this. Not only can it be used to model and smooth data in ways that compete directly with established techniques such as non-parametric kernel regression, or splines, but it can also be used for trend extraction in competition with methods such as the Hodrick and Prescott (1980, 1997) filter. In all of the cases that we have examined, the fuzzy modelling approach performs extremely well. Moreover, it has significant practical advantages over both kernel regression and spline analysis. Spline analysis requires that the location of the "knots" be known in advance, whereas the first stage of our fuzzy modelling procedure determines the relevant data groupings empirically. Non-parametric kernel regression suffers from the so-called "curse of dimensionality" that severely limits its application to relatively simple models. In contrast, the fuzzy modelling structure that we have outlined is readily applicable to quite complex models, without undue computational burden. Indeed, it should also be noted that although our illustrative applications have all used ordinary least squares regression at the second stage of the analysis to fit models to each fuzzy cluster, in fact any relevant estimation technique could be used at that stage. For instance (and depending on the context), logit or probit models could be fitted, or non-linear or instrumental variables regression could be used, without disrupting the general style of the fuzzy modelling strategy that we have described.

We recognize that the bulk of the discussion in this paper addresses issues relating to curve fitting and data smoothing, and relatively little attention has been paid to strictly inferential issues. It was argued heuristically at the end of section 4 that the fuzzy predictor emerging from our modelling approach will be a (weakly) consistent estimator of the conditional mean of the dependent variable. While this is minimally helpful, it will hold only under suitable conditions, and our work in progress explores this issue (and other aspects of the sampling properties of the fuzzy predictor) more thoroughly.

In this context it is also worth noting that the various "fuzzy predictions" that we have presented in our examples are simply "point predictions", and the corresponding confidence intervals are also extremely important. While it is not immediately clear how straightforward it would be to construct exact finite-sample prediction intervals within this framework, they could certainly be approximated by means of bootstrap simulation. Asymptotic prediction intervals can be constructed relatively easily, as follows. It will be recalled that once the sample has been partitioned into fuzzy clusters (typically with a different number of data points in each cluster), our modelling procedure involves estimating a regression over each cluster, and then combining the results using weights based on the membership values. The precise details are given in "Step 3" in section 4 above. These weights vary continuously, but are exogenous if the regressors also satisfy this property. Rather than estimate each cluster regression separately, by least squares, another option is to estimate the group of cluster regressions as a "seemingly unrelated regressions" (SUR) model. The only complication is that the SUR model is "unbalanced", in the sense that the samples associated with each equation are different from one another. However, unbalanced SUR models can be estimated quite readily (*e.g.*, see Srivastava and Giles, 1987, pp. 339-346, for details). This then provides a complete estimated asymptotic covariance matrix for *all* of the estimated parameters in all of the cluster regressions. This information (together with the membership values) can then be used to construct observation-by-observation asymptotic standard errors for the prediction of the conditional mean of the dependent variable.

By way of illustration we have undertaken this analysis in the case of the earnings-age profile data analyzed in section 5.1 above, and the results appear in Figure 24. SHAZAM code was written to produce these results, which are compared with their counterparts from non-parametric kernel regression. As can be seen, the fuzzy modelling procedure produces intervals that are very comparable to those from non-parametric regression, in terms of their shape and width. Given that our illustrative applications are based on relatively small samples, we have not provided corresponding asymptotic confidence intervals for our other "fuzzy predictions".

Figure 24: Confidence Intervals for Earnings-Age Profile Predictions



Clearly, much remains to be done in order to validate the real worth of the type of "fuzzy econometric modelling" has been described and illustrated in this paper. None the less, we view the results presented here as being rather promising. Certainly, further exploration of this approach to modelling and smoothing non-linear economic data seems to be justified.

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