2-Limited Broadcast Domination in Grid Graphs

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Introduction

Let $G = (V, E)$ be a graph. Suppose there is a transmitter located at each vertex $v \in V$ capable of **broadcasting** at strength $0, 1, \dots, k$, where strength 0 corresponds with not broadcasting. A vertex v broadcasting at strength s is **heard** by all vertices within distance s of v. Our goal is to assign strengths to the transmitters such that every vertex not transmitting hears the broadcast by one that is. The result is a dominating k-limted broadcast on G. The **cost** of such a broadcast is the sum of the strengths of the transmitters. $\gamma_{b,k}(G)$ denotes the least cost of a k-limited broadcast on G. Observe that $\gamma_{b,1} = \gamma$.

Limited broadcast domination is a restriction of broadcast domination (where vertices can broadcast at any strength) introduced in [Erw01]. Broadcast domination is known on grid graphs [BS09], however k-limited broadcast on grid graphs is unknown. We provide tight bounds on the 2-limited broadcast domination number of grid graphs.

2-Limited Broadcasts Domination

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

$$x_{i,k} = \begin{cases} 1 & \text{if vertex } i \text{ is broadcasting at strength } k, \\ 0 & \text{otherwise} \end{cases}$$

Formulation of $\gamma_{b,2}(G)$ as an Integer Linear Program (ILP):

Minimize: $\sum_{k=1}^{2} \sum_{i \in V} k \cdot x_{i,k}$

Subject to: $\sum_{d(i,j) \le k} x_{i,k} \ge 1$, for each vertex $j \in V$

Example 1: Red diamonds represent vertices at their center broadcasting at strength 1 or 2.

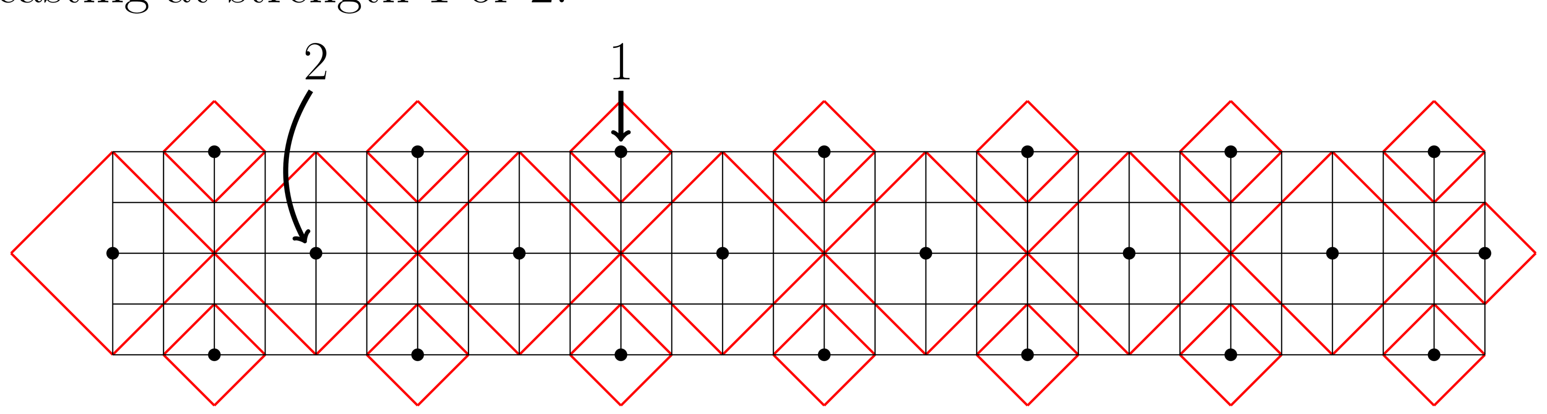


Figure 1: Optimal 2-limited broadcast on $P_5 \square P_{28}$, $\gamma_{b,2}(P_5 \square P_{28}) = 29$.

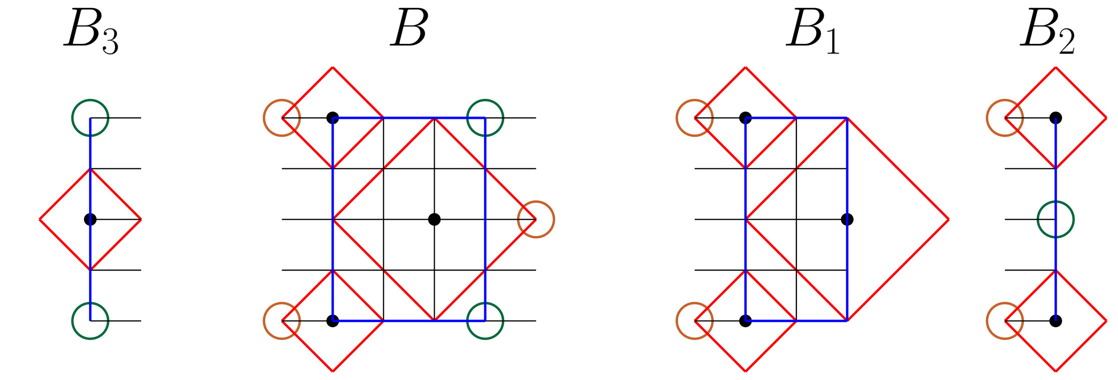
Upper Bounds

For $P_m \square P_n$, where $2 \le m \le 12$, we create upper bounds using the following methodology.

Methodology:

1. Fix m and use an ILP solver [CO17] to determine $\gamma_{b,2}(P_m \square P_n)$ for small values of n (≤ 50),
2. Manually inspect for **patterns** in the broadcast structure,
3. Create general constructions based on these patterns.

Example 2: Referring to Example 1, we observe a pattern in the optimal 2-limited broadcast on $P_5 \square P_{28}$. Using this pattern, we repeatedly **tile** $P_5 \square P_n$ with a **main** tile B and complete the ends of the broadcast with B_1, B_2, or B_3 based on $n \pmod 4$.



$n \equiv 0 \pmod 4$: $B_3 + \underbrace{B + \cdots + B}_{\frac{n-4}{4}} + B_1$,

$n \equiv 1 \pmod 4$: $\underbrace{B + \cdots + B}_{\frac{n-1}{4}} + B_2$,

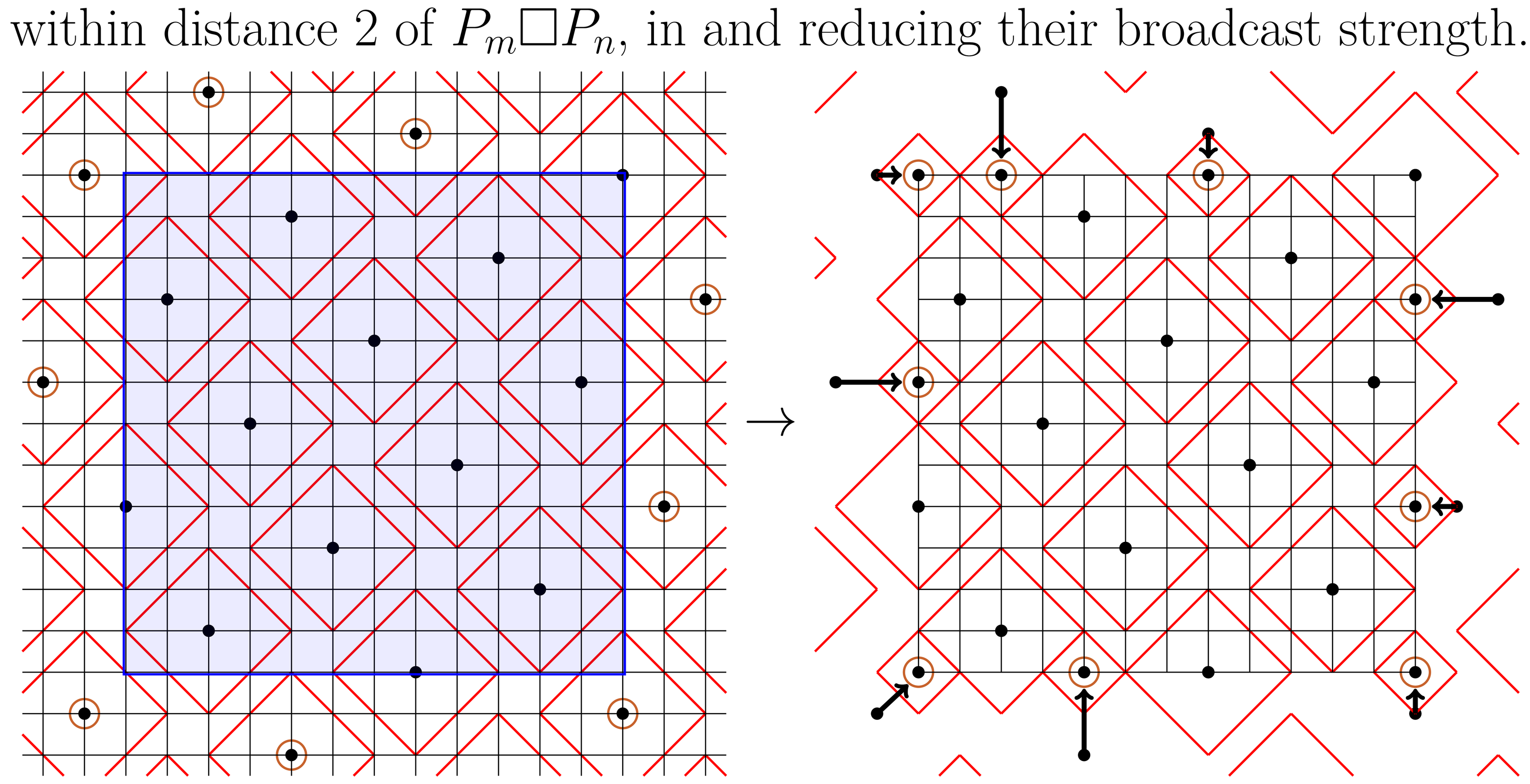
$n \equiv 2 \pmod 4$: $B_3 + \underbrace{B + \cdots + B}_{\frac{n-2}{4}} + B_2$,

$n \equiv 3 \pmod 4$: $\underbrace{B + \cdots + B}_{\frac{n-3}{4}} + B_1$.

Resulting upper bound:

$\gamma_{b,2}(P_5 \square P_n) \le n + 1$.

For $P_{m \ge 13} \square P_{n \ge 13}$, we obtain upper bounds by modifying the 2-limited broadcasts on the plane. Placing $P_m \square P_n$ in the plane (blue rectangle below), we create a valid broadcast by moving broadcasting vertices, within distance 2 of $P_m \square P_n$, in and reducing their broadcast strength.



Through a counting argument on the number of broadcasting vertices in the plane, we establish our generalized upper bound.

Fractional 2-limited Multipacking

The **dual** of the linear programming relaxation of 2-limited broadcast domination is 2-limited multipacking $mp_2(G)$, stated as follows.

Let $G = (V, E)$ be a graph. For each vertex $i \in V$, let

weight packed at vertex $i = y_i \in [0, 1]$.

Formulation of $mp_2(G)$ as a Linear Program (LP):

Maximize: $\sum_{i \in V} y_i$

Subject to (1): $\sum_{d(i,j) \le 1} y_i \le 1$, for each vertex $j \in V$,

(2): $\sum_{d(i,j) \le 2} y_i \le 2$, for each vertex $j \in V$.

Example 3: Red circles depict vertices at their center packed with weight 0.5.

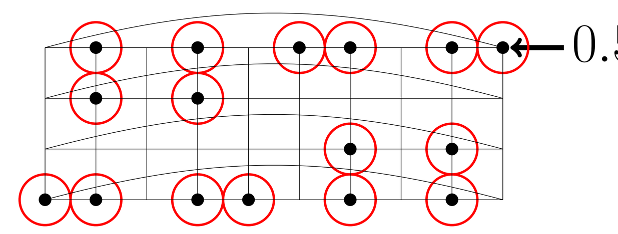


Figure 2: Optimal Fractional 2-Limited Multipacking on $P_4 \square C_{10}$, $mp_2(P_4 \square C_{10}) = 8$.

Lower Bounds

For $P_{2 \le m \le 12} \square P_n$, we create lower bounds by the following methodology.

Methodology:

1. Fix m, given the **main** tile $P_m \square P_x$ used in our 2-limited broadcast construction on $P_m \square P_n$, use an LP solver to determine $mp_2(P_m \square C_x)$,
2. We can repeatedly **tile** $P_m \square P_n$ with this optimal 2-limited multipacking on $P_m \square C_x$ and create a lower bound on $\gamma_{b,2}(P_m \square P_n)$.

For $P_{m \ge 13} \square P_{n \ge 13}$, we use a similar argument as our upper bound and consider the optimal 2-limited multipacking on the plane.

Results

Lower Bounds	$\gamma_{b,2}(P_m \square P_n)$	Upper Bounds
	$\gamma_{b,2}(P_2 \square P_n)$	$= \lceil \frac{n+1}{2} \rceil$
	$\gamma_{b,2}(P_3 \square P_n)$	$= \lceil \frac{2n}{3} \rceil$
$\lceil 8 \lfloor \frac{n}{10} \rfloor \rceil \le$	$\gamma_{b,2}(P_4 \square P_n)$	$\le 8 \lfloor \frac{n}{10} \rfloor + c_4(n)_{\le 8}$
$\lceil 7.703 \lfloor \frac{n}{8} \rfloor \rceil \le$	$\gamma_{b,2}(P_5 \square P_n)$	$\le n + 1$
$\lceil 17.846 \lfloor \frac{n}{16} \rfloor \rceil \le$	$\gamma_{b,2}(P_6 \square P_n)$	$\le 18 \lfloor \frac{n}{16} \rfloor + c_6(n)_{\le 18}$
$\lceil 16.466 \lfloor \frac{n}{14} \rfloor \rceil \le$	$\gamma_{b,2}(P_7 \square P_n)$	$\le 18 \lfloor \frac{n}{14} \rfloor + c_7(n)_{\le 18}$
$\lceil 31.302 \lfloor \frac{n}{22} \rfloor \rceil \le$	$\gamma_{b,2}(P_8 \square P_n)$	$\le 32 \lfloor \frac{n}{22} \rfloor + c_8(n)_{\le 32}$
$\lceil 15.757 \lfloor \frac{n}{10} \rfloor \rceil \le$	$\gamma_{b,2}(P_9 \square P_n)$	$\le 16 \lfloor \frac{n}{10} \rfloor + c_9(n)_{\le 16}$
$\lceil 31.130 \lfloor \frac{n}{18} \rfloor \rceil \le$	$\gamma_{b,2}(P_{10} \square P_n)$	$\le 32 \lfloor \frac{n}{18} \rfloor + c_{10}(n)_{\le 32}$
$\lceil 48.976 \lfloor \frac{n}{26} \rfloor \rceil \le$	$\gamma_{b,2}(P_{11} \square P_n)$	$\le 50 \lfloor \frac{n}{26} \rfloor + c_{11}(n)_{\le 50}$
$\lceil 48.895 \lfloor \frac{n}{24} \rfloor \rceil \le$	$\gamma_{b,2}(P_{12} \square P_n)$	$\le 50 \lfloor \frac{n}{24} \rfloor + c_{12}(n)_{\le 50}$
$\lceil 2 \left(\frac{mn}{13}\right) + 2.48 \left(\frac{m+n}{13}\right) \rceil \le$	$\gamma_{b,2}(P_{m \ge 13} \square P_n)$	$\le 2 \left(\frac{mn}{13}\right) + 4 \left(\frac{m+n}{13}\right) + c_{13}(n)_{\le 2}$

where $c_i(n)_{\le x}$ is a number between 0 and x dependant upon n for all i.

Take Home: We established tight bounds and known optimal values for $\gamma_{b,2}(P_n \square P_m)$. Using similar methods, we have also created tight bounds and known optimal values for $\gamma_{b,2}(P_n \square C_m)$ and $\gamma_{b,2}(C_n \square C_m)$.

References

[BS09] Bostjan Bresar and Simon Spacapan, *Broadcast domination of products of graphs*, Ars Combinatoria **92** (2009).

[CO17] COIN-OR, *CBC: A COIN-OR integer programming solver*, https://projects.coin-or.org/Cbc, 2017.

[Erw01] David John Erwin, *Cost domination in graphs*, Ph.D. Thesis, Western Michigan University (2001).

Acknowledgements: I would like to thank my advisors for their invaluable guidance, this research has been a lot of fun.