A Fast and Accurate Scheme for Sea Ice Dynamics with Subgrid Resolution

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Introduction and Motivations

- Arctic Sea Ice has been thinning and retreating for decades and the trend is projected to continue [1].
  - Since the advent of satellite observations in 1979, the annual mean Arctic Sea Ice Extent has been steadily decreasing.
  - A number of studies suggest that we may observe the loss of multi-year ice in the Arctic by mid-century.
- Unfortunately, there is significant inter-model scatter in simulations and it is expected that most model projections are *conservative* [1].
- Due to several climatic feedback effects, accurately simulating the evolution of sea ice is synonymous with improving confidence in Global Climate Models (GCMs). The above mentioned feedback effects include, but aren't limited to [2]:
  - affecting the heat exchange between the ocean and the atmosphere by acting as an insulating layer and reflecting sunlight away from the ocean through the albedo affect.
  - affecting ocean circulation through modification of the freshwater content of the upper ocean.
- Addressing the uncertainties and improving our understanding of sea ice physics is one of the "grand challenges of climate science" [1].

Proposed Research

- To simulate sea ice evolution, it is critical that we accurately represent sea ice drift, which hinges on the rheology formulation [2].
  - the most common formulation is the Viscous-Plastic (VP) formulation, introduced by Hibler [3].
- The VP formulation leads to a very non-linear problem with strict numerical stability requirements, e.g. on the order of 0.01 seconds for a grid size of 10 km [4]!
  - This has motivated the use of implicit methods to solve the necessary equations, but these solvers are known to exhibit slow convergence properties! However, recent studies have shown promising improvements through the use of a Jacobian-Free Newton Krylov solver [4].
  - Others have added an artificial elastic term to the VP equations in order relax the stability requirements [5] creating the explicit, Elastic-Viscous-Plastic (EVP) method.
- Although the EVP method has shown promising results, it has been known to introduce numerical noise due to the added elastic term [4], therefore our research aims to build on the recent advances in implicit solvers. **We propose the use of a fully second order JFNK solver with a more consistent discretization of the VP rheology term!**

Key Governing Equations

- **The 2-D Sea Ice Momentum Equation**

$$\rho h \frac{D\boldsymbol{u}}{Dt} = -\rho h f(\boldsymbol{k} \times \boldsymbol{u}) + \tau_a - \tau_w + \nabla \cdot \boldsymbol{\sigma} - \rho h g \nabla H_d \tag{1}$$

- **The Thickness and Concentration Continuity Equations**

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = S_h \tag{2}$$

$$\frac{\partial A}{\partial t} + \nabla \cdot (A\boldsymbol{u}) = S_A \tag{3}$$

- **The Viscous Plastic (VP) Rheology Formulation:** Following Hibler [3] and Lemieux [4].

$$\sigma_{ij} = 2\eta\dot{\epsilon}_{ij} + [\zeta - \eta]\dot{\epsilon}_{kk}\delta_{ij} - P\delta_{ij}/2, \quad i,j = 1,2 \tag{4}$$

$$\dot{\epsilon}_{11} = \frac{\partial u}{\partial x}, \quad \dot{\epsilon}_{22} = \frac{\partial v}{\partial y}, \quad \dot{\epsilon}_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \text{and} \quad \dot{\epsilon}_{kk} = \dot{\epsilon}_{11} + \dot{\epsilon}_{22}$$

$$P = P^* h \, \mathsf{exp}[-C(1-A)]$$

$$\zeta = \zeta_{max}\mathsf{tanh}\left(\frac{P}{2\Delta\zeta_{max}}\right), \quad \zeta_{max} = kP, \quad \eta = \zeta e^{-2}$$

$$\Delta = \left[(\dot{\epsilon}_{11}^2 + \dot{\epsilon}_{22}^2)(1+e^{-2}) + 4e^{-2}\dot{\epsilon}_{12}^2 + 2\dot{\epsilon}_{11}\dot{\epsilon}_{22}(1-e^{-2})\right]^{1/2}$$

The Numerical Method

The Rheology Term

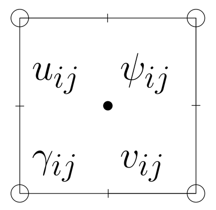
- Traditionally, discretizations of $\nabla \cdot \boldsymbol{\sigma}$ use finite difference approximations for spatial derivatives of viscosities, requiring boundary conditions be imposed on diagnostic variables.
- **To avoid this, we expand the rheology term further and calculate the viscosity derivatives directly** according to

$$\partial_* \zeta = k \, \mathsf{tanh}\left(\frac{1}{2\Delta k}\right)\partial_* P - \frac{P}{4\Delta^3}\left[1 - \mathsf{tanh}^2\left(\frac{1}{2\Delta k}\right)\right]\partial_*(\Delta^2), \tag{5}$$

differentiating P and Δ^2 in terms of the prognostic variables $\boldsymbol{u}$, h, and A.

The Discretized System - A Fully Second Order Approach

- Spatially, we use centered differences and discretize (1) on the Arakawa C-grid, i.e.



where γ and ψ are referred to nodes and tracer points, respectively.
- Previous JFNK solvers use a backward Euler Approach [4], giving first order accuracy in time.
  - **we use the Crank-Nicolson scheme to create a fully second order scheme, in both space and time!**

The Solver

- At time step n, utilizing the above discretization results in the N dimensional, non-linear system:

$$A(\boldsymbol{u}^n)\boldsymbol{u}^n = \boldsymbol{b}(\boldsymbol{u}^n). \tag{6}$$

- To solve (6), the JFNK method adopts iterative approach. Letting $\boldsymbol{u}^k$ be the kth iterate, its associated residual is,

$$\boldsymbol{F}(\boldsymbol{u}^k) = A(\boldsymbol{u}^k)\boldsymbol{u}^k - \boldsymbol{b}(\boldsymbol{u}^k). \tag{7}$$

Taking a first order, multivariate Taylor Expansion about $\boldsymbol{F}(\boldsymbol{u}^k)$ and setting the left hand side to 0 gives us the linear system,

$$J(\boldsymbol{u}^k)\delta\boldsymbol{u}^k = -\boldsymbol{F}(\boldsymbol{u}^k) \quad \text{where} \quad \boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \delta\boldsymbol{u}^k,$$

which is solved via the GMRES method.
- This solver is "Jacobian-Free"; only the Jacobian's ACTION on a vector is needed. **We improve on existing work by partially forming the Jacobian and using a second order approximation for the unformed part**, i.e.

$$J(\boldsymbol{u}^k)\boldsymbol{v} \approx \frac{A(\boldsymbol{u}^k + \epsilon\boldsymbol{v})\boldsymbol{u}^k - A(\boldsymbol{u}^k - \epsilon\boldsymbol{v})\boldsymbol{u}^k}{2\epsilon} + A(\boldsymbol{u}^k)\boldsymbol{v} - D\boldsymbol{b}(\boldsymbol{u}^k)\boldsymbol{v}, \tag{8}$$

Validation

- To validate this solver, we wish to assess its ability to produce a known solution. Unfortunately, this system is too complex to have any known analytical solutions.
- Nevertheless, we can produce an exact solution by adding appropriate forcing to (1) [6] and testing the solver's ability to solve

$$-\rho h\frac{\partial \boldsymbol{u}}{\partial t} - \rho h f(\boldsymbol{k} \times \boldsymbol{u}) + \tau_a - \tau_w + \nabla \cdot \boldsymbol{\sigma} - \rho h g \nabla H_d = \mathcal{L}(\boldsymbol{w}), \tag{9}$$

where $\mathcal{L}(\boldsymbol{w})$ is produced by replacing $\boldsymbol{u}$ in (1) with the 2-D propagating sine wave,

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{10}\mathsf{sin}\left[(\frac{4x}{L_x} - 2)^2 + (\frac{4y}{L_y} - 2)^2 + ct\right] \\ \frac{1}{10}\mathsf{cos}\left[(\frac{4x}{L_x} - 2)^2 + (\frac{4y}{L_y} - 2)^2 + ct\right] \end{bmatrix}. \tag{10}$$

Validation - Cont'd

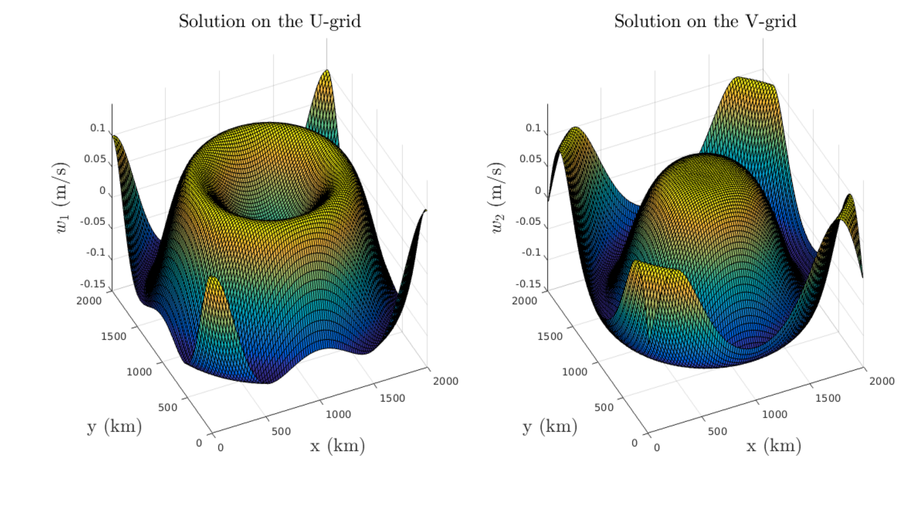


Figure 1: Toy Solution

- We elect to test the solver's ability to produce (10) with:

$$dx = 20 \text{ km}, \; dt = 10 \text{ min}, \; L_x = L_y = 2000 \text{ km}, \; T = 7 \text{ days}$$

Subgrid Resolution With the Distance Function

- **Limiting the Computational Domain:**
To gain efficiency and subgrid resolution, we set our computational domain to regions with ice using

$$\begin{cases} |\nabla\phi(\boldsymbol{x})| = 1 \\ \partial_t\phi + \nabla \cdot (\phi\boldsymbol{u}) = 0 \\ \\ \phi(\boldsymbol{x}) = 0 \quad \text{at the ice-water interface} \\ \phi(\boldsymbol{x}) > 0 \quad \text{in ice} \\ \phi(\boldsymbol{x}) < 0 \quad \text{outside ice} \end{cases} \tag{11}$$

  - For simplicity, advection has been neglected in these tests. When advection is added, (11) will provide subgrid resolution near boundaries by telling us where the ice boundary is with-in a cell; this will be particularly useful near land boundaries (Figure 3).

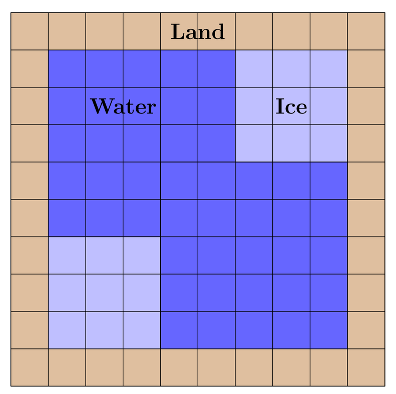


Figure 2: Toy Validation Domain

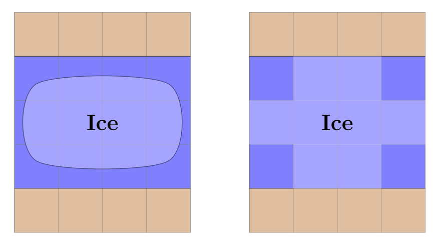
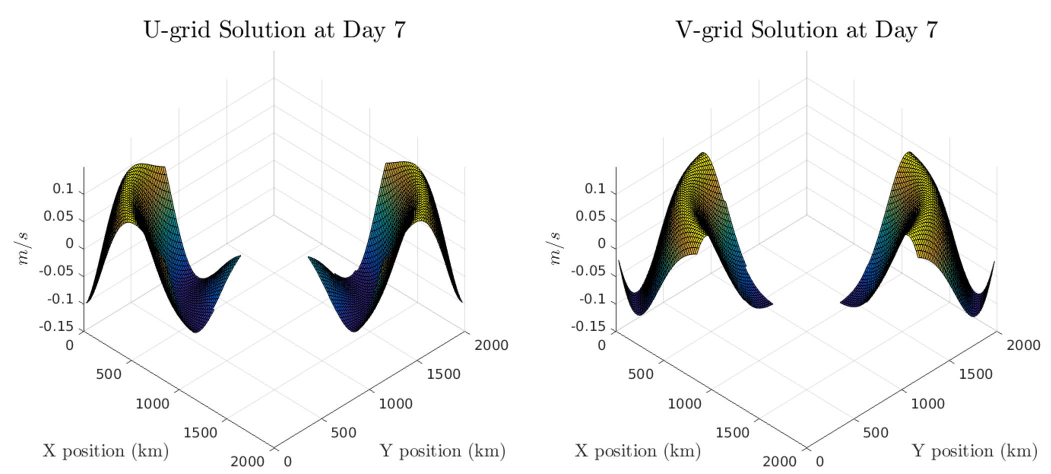
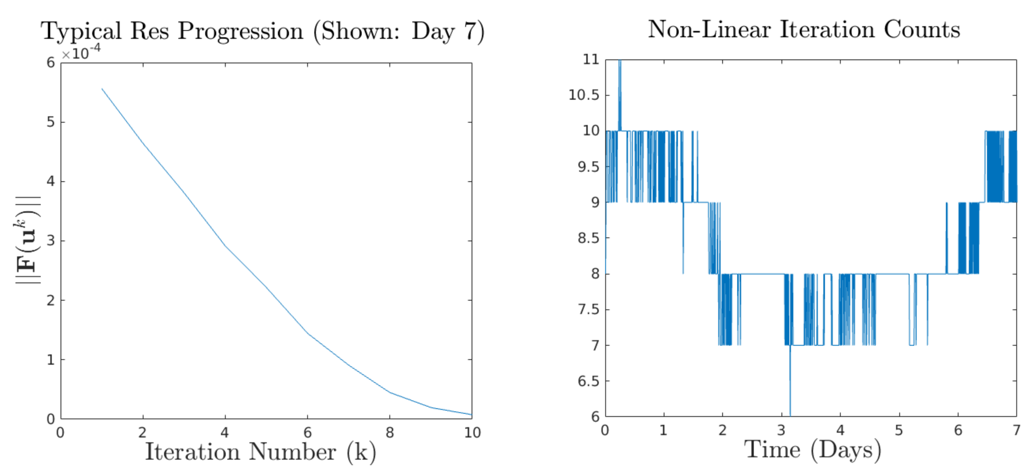


Figure 3: (Right) Config seen with volume cut-off. (Left) Config seen with ϕ

  - For the validation tests, we use (11) to limit our domain to that shown in Figure 2.
- **Results:**





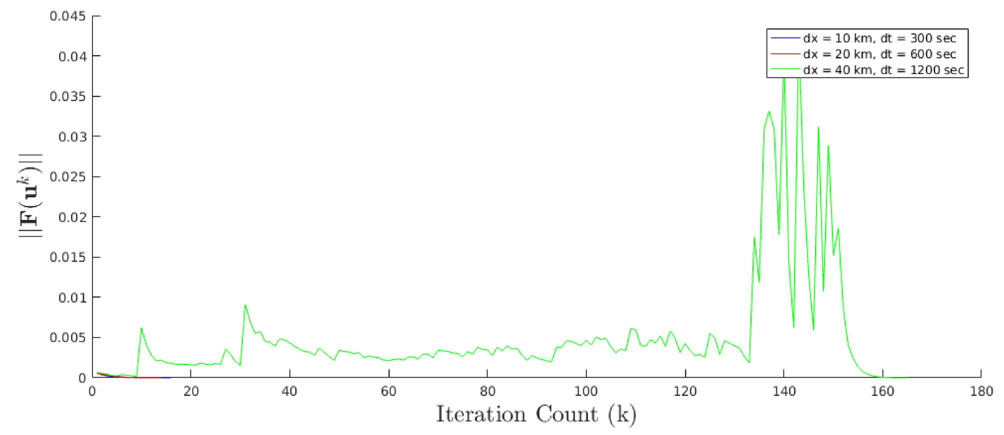
Grid Refinement Tests

Coarse: $dx|dt$ = 40 km|20 min; **Regular:** $dx|dt$ = 20 km|10 min; **Fine:** $dx|dt$ = 10 km|5 min

Day	U-grid Error Norms (m/s)						V-grid Error Norms (m/s)					
	Coarse		Regular		Fine		Coarse		Regular		Fine	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
1	4.0 e-4	3.5 e-3	1.0 e-4	8.5 e-4	7.3 e-5	6.9 e-4	2.9 e-4	4.3 e-3	8.6 e-5	9.4 e-4	1.9 e-4	2.8 e-3
2	1.8 e-3	2.8 e-2	1.0 e-4	6.9 e-4	2.6 e-5	1.7 e-4	1.7 e-3	3.0 e-2	6.4 e-5	3.7 e-4	1.6 e-5	9.4 e-5
3	1.1 e-3	2.0 e-2	1.0 e-4	6.5 e-4	2.4 e-5	1.6 e-4	8.5 e-4	1.7 e-2	6.6 e-5	3.8 e-4	1.6 e-5	1.1 e-4
4	2.5 e-3	4.7 e-2	9.5 e-5	6.7 e-4	2.3 e-5	1.5 e-4	1.5 e-3	2.4 e-2	6.2 e-5	3.6 e-4	1.5 e-5	8.8 e-5
5	1.9 e-3	2.4 e-2	1.1 e-4	-2.1 e-3	9.0 e-5	4.8 e-3	1.8 e-3	2.9 e-2	6.1 e-5	9.6 e-4	5.3 e-5	2.4 e-3
6	1.1 e-3	1.6 e-2	1.0 e-4	7.2 e-4	2.5 e-5	2.0 e-4	1.1 e-3	2.9 e-2	6.0 e-5	3.7 e-4	1.4 e-5	8.0 e-5

Day	U-grid Convergence Rates				V-grid Convergence Rates			
	Coarse → Reg.		Reg. → Fine		Coarse → Reg.		Reg. → Fine	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
1	2.0	2.0	0.5	0.3	1.7	2.2	-1.1	-1.6
2	4.1	5.4	2.0	2.1	4.7	6.3	2.0	2.0
3	3.5	5.0	2.0	2.0	3.7	5.5	2.0	1.8
4	4.7	6.1	2.0	2.2	4.6	6.1	2.0	2.0
5	4.1	3.5	0.3	-1.2	4.8	4.9	0.2	-1.3
6	3.5	4.5	2.0	1.8	4.2	6.3	2.1	2.2

- There is a sensitivity to grid resolution! The fine simulation fails after day 6 and the coarse run seems to be plagued to errors caused by convergence problems (See below).



- For the coarse resolution, some times we observe "spikes" and the solver then "searches around" until it reaches convergence or the max iteration count.

Conclusions and Future Work

- Although there is some sensitivity to resolution, the solver is producing the predicted 2nd order convergence!
- Drastic improvements were noted due to the 2nd order approximation of the Jacobian's action. The solver often converges with-in less than 10 iterations! The change from a 1st to 2nd order approximation is simple to implement.
- Our treatment of the rheology term allows for a more consistent model; boundary conditions only need to be imposed on prognostic variables!
- **Potential Improvements:**
  - *Conditional Termination* - terminate the solver and accept the previous iterate when large increases in residuals are observed. Preliminary results show that this does not introduce added error and it limits wasted computational effort.
  - *Conditional Damping* - activate newton damping instead of terminating solver.
- **Future Work:**
  - Combine our solver with advection solvers and perform physically motivated sea ice simulations.
  - Assess the effects of our sub-grid resolution; particularly in domains with many land boundaries.

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