Department of Mathematics

MATH 222 - Discrete and Combinatorial Mathematics

Full Solutions to Practice Questions

Contributors:

Elise Marchessault

Ashna Wright

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- Kieka Mynhardt's notes, assignments, and tests for Math 222
- Introduction to Combinatorics and Graph Theory - Custom Edition for the University of Victoria
- Discrete Mathematics: Study Guide for MAT212-S - Dr. Kieka Myndardt
- Discrete Mathematics - Norman L. Biggs
- Applied Combinatorics, fourth edition - Alan Tucker
- Discrete Mathematics, An Introduction to Mathematical Reasoning - Susanna S. Epp
- Discrete Mathematics with Combinatorics - James A. Anderson

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1 Preliminaries

1.1 Sets

Solutions:

1. Yes, $A = B$. We will prove this by showing that $A \subseteq B$ and $B \subseteq A$.

   We begin by showing that $A \subseteq B$. Let $a \in A$, then we know that $a = 2k$ for some integer k. Letting $k = j - 1$, where j is an integer, we see that $a = 2(j-1) = 2j - 2$, so $a \in B$, hence $A \subseteq B$.

   Now we show $B \subseteq A$. Let $b \in B$. This means that $b = 2j - 2$ for some integer j. Picking $j = k + 1$, for some integer k, we can see that $b = 2(k+1) - 2 = 2k + 2 - 2 = 2k$, therefore $b \in A$. Thus $B \subseteq A$ and we can conclude that $A = B$.

2. The powerset of any set is the collection of all possible subsets of that set. With $\emptyset$ representing the *empty set*.

   (a) $A \cup B = \{1, 2, 3, x, y\}$.
   $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{x\}, \{y\}, \{1,2\}, \{1,3\}, \{1,x\}, \{1,y\}, \{2,3\}, \{2,x\}, \{2,y\}, \{3,x\}, \{3,y\}, \{x,y\}, \{1,2,3\}, \{1,2,x\}, \{1,2,y\}, \{1,3,x\}, \{1,3, y\}, \{1,x,y\}, \{2,3,x\}, \{2,3,y\}, \{2,x,y\}, \{3,x,y\}, \{1,2,3,x\}, \{1,2,3,y\}, \{2,3, x,y\}, \{1,3,x,y\}, \{1,2,x,y\}, A \cup B\}$.

   (b) $B \times C = \{(x,u), (x,v), (y,u), (y,v)\}$.
   $\mathcal{P}(B \times C) = \{\emptyset, \{(x,u)\}, \{(x,v)\}, \{(y,u)\}, \{(y,v)\}, \{(x,u),(x,v)\}, \{(x,u), (y,u)\}, \{(x,u),(y,v)\}, \{(x,v),(y,u)\}, \{(x,v),(y,v)\}, \{(y,u),(y,v)\}, \{(x,u), (x,v),(y,v)\}, \{(x,u),(x,v),(y,v)\}, \{(x,u),(y,u),(y,v)\}, \{(x,v),(y,u),(y,v)\}, B \times C\}$.

   (c) $\mathcal{P}(C) = \{\emptyset, \{u\}, \{v\}, C\}$
   $\mathcal{P}(\mathcal{P}(C)) = \{\emptyset, \{\emptyset\}, \{\{u\}\}, \{\{v\}\}, \{C\}, \{\emptyset, \{u\}\}, \{\emptyset, \{v\}\}, \{\emptyset, C\}, \{\{u\}, \{v\}\},$

$\{\{u\}, C\}, \{\{v\}, C\}, \{\emptyset, \{u\}, \{v\}\}, \{\emptyset, \{u\}, C\}, \{\emptyset, \{v\}, C\}, \{\{u\}, \{v\}, C\}, \{\emptyset,$
$\{u\}, \{v\}, C\}\}$.

(d) $B \cap C = \emptyset$.
$A \times (B \cap C) = A \times \emptyset = \emptyset$.
Note: The Cartesian product of any set with the empty set is always the empty set.

(e) $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$.
$(A \times B) \times C = \{((1, x), u), ((1, y), u), ((2, x), u), ((2, y), u), ((3, x), u), ((3, y), u), ((1, x), v), ((1, y), v), ((2, x), v), ((2, y), v), ((3, x), v), ((3, y), v)\}$.

3. Suppose for contradiction that the empty set, $\emptyset$, is not a subset of some arbitrary set, S. Then there exists some element in $\emptyset$ that is not in S, but by definition there are no elements in $\emptyset$, a contradiction. Therefore $\emptyset$ is a subset of every set.

4. The statement is true, so we must prove it.

   Let $x \in A$, then $x \in B$ (since $A \subseteq B$) and $x \in C$ (since $A \subseteq B$). x is in both B and C, thus $x \in B \cap C$. We picked x to be an arbitrary element of A therefore $A \subseteq B \cap C$.

5. To prove this we will show $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ and then $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

   We begin with $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$. Suppose $(x, y) \in A \times (B \cup C)$, certainly $x \in A$, with $y \in B$ or, inclusively, $y \in C$.
   Case 1: Suppose $y \in B$. Then $(x, y) \in (A \times B)$, and certainly $(x, y) \in (A \times B) \cup (A \times C)$.

Case 2: Suppose $y \in C$. Then $(x, y) \in (A \times C)$, and certainly $(x, y) \in (A \times B) \cup (A \times C)$.

Now we will show that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$. Suppose that $(x, y) \in (A \times B) \cup (A \times C)$. This means that either $(x, y) \in A \times B$, or (inclusive) $(x, y) \in A \times C$.
Case 1: Suppose $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. Certainly $y \in B \cup C$, thus $(x, y) \in A \times (B \cup C)$.
Case 2: Suppose $(x, y) \in A \times C$. Then $x \in A$ and $y \in C$. Certainly $y \in C \cup B$, thus $(x, y) \in A \times (C \cup B)$.

Since we have proved that both sets are subsets of each other, we may conclude equality.

6. Suppose that $C \subseteq B - A$. This means for $x \in C$, that $x \in B$ but $x \notin A$. Then no element in C is also in A which means that $A \cap C = \emptyset$.

7. (a) False. Consider this counterexample: Let $A = \{1, 2\}$ and $B = \{a, b\}$. $A \cup B = \{1, 2, a, b\}$. Certainly $\{1, 2, a, b\} \in \mathcal{P}(A \cup B)$, but $\{1, 2, a, b\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

   (b) True. We will show that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$ and then $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

   Suppose that $X \in \mathcal{P}(A \cap B)$. This means that $X \subseteq A \cap B$, hence $X \subseteq A$ and $X \subseteq B$. Then of course $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, so $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

   Suppose $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. This implies that $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, and by definition, $X \subseteq A$ and $X \subseteq B$. Then $X \subseteq A \cap B$, so $X \in \mathcal{P}(A \cap B)$. Now we may conclude equality.

(c) True. Suppose that $A \subseteq B$. Let $X \subseteq A$, then $X \in \mathcal{P}(A)$. But since $A \subseteq B$ we also know that $X \subseteq B$ hence $X \in \mathcal{P}(B)$. Therefore $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

8. We will prove this statement by showing $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and then $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

   Suppose that $x \in \overline{A \cap B}$. This means $x \notin A \cap B$, so either $x \in A$ and $x \notin B$, $x \notin A$ and $x \in B$, or $x \notin A$ and $x \notin B$. In symbols this precisely means that $x \in \overline{A} \cup \overline{B}$ (draw a Venn-Diagram for yourself to clearly see this).

   Now let $x \in \overline{A} \cup \overline{B}$. So either $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then $x \notin A$ and certainly $x \notin A \cap B$, so $x \in \overline{A \cap B}$. If $x \in \overline{B}$ we know that $x \notin B$, so of course $x \notin B \cap A$. This means that $x \in \overline{A \cap B}$.

   We may now conclude equality of the sets and the identity has been proved.

1.2 Relations and Graphs

Solutions:

1. We will verify each property individually.

   (a) **Reflexive**: No.
   $A \cap A = A \neq \emptyset$.

   Symmetric: Yes.
   If $A \cap B = \emptyset$, then $B \cap A = \emptyset$ since $A \cap B = B \cap A$.

   Transitive: No.
   $\{a\} \cap \{b, c\} = \emptyset$ and $\{b, c\} \cap \{a, d\} = \emptyset$, but $\{a\} \cap \{a, d\} = \{a\} \neq \emptyset$.

   Antisymmetric: No.
   $\{a, b\} \cap \{c, d\} = \emptyset$, but $\{c, d\} \cap \{a, b\} = \emptyset$.

   (b) **Reflexive**: Yes.
   Technically everyone is a friend to themselves.
   Note: Be a best friend to yourself, self love is essential!

   Symmetric: Yes.
   If x is a friend of y, then y is a friend of x.

   Transitive: No.
   Suppose that x is a friend of y and y is a friend of z, this does not guarantee that x and z are friends.

   Antisymmetric: No.

   (c) **Reflexive**: Yes.
   $(x_1, x_2) \quad (x_1, x_2)$ since $x_1 = x_1$ and $x_2 \leq x_2$.

   Symmetric: No.
   $(1, 2)\ R\ (1, 3)$ since $2 \leq 3$, but $(1, 3) \not R\ (1, 2)$ as $3 \nleq 2$.

Transitive: Yes.
Suppose $(x_1, x_2)\ R\ (y_1, y_2)$ and $(y_1, y_2)\ R\ (z_1, z_2)$. This means that $x_1 = y_1$ and $y_1 = z_1$, so $x_1 = z_1$. Also, $x_2 \leq y_2$ and $y_2 \leq z_2$, so $x_2 \leq z_2$ hence $(x_1, x_2)\ R\ (z_1, z_2)$.

Antisymmetric: Yes.
If $(x_1, x_2)\ R\ (y_1, y_2)$ and $(y_1, y_2)\ R\ (x_1, x_2)$, $x_2 \leq y_2$ and $y_2 \leq x_2$, thus $x_2 = y_2$, hence $(x_1, x_2) = (y_1, y_2)$.

(d) **Reflexive**: Yes.
$X \subseteq X$.

Symmetric: No.
If $X \subseteq Y$, then $Y \nsubseteq X$ unless $X = Y$ (not necessarily the case).

Transitive: Yes.
Suppose that $X \subseteq Y$ and $Y \subseteq Z$, then certainly $X \subseteq Z$.

Antisymmetric: Yes.
Suppose that $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.

2. To prove that R is an equivalence relation we must show that R is reflexive, symmetric and transitive.

(a) **Reflexive**: Yes.
A positive integer certainly has the same largest prime divisor as itself.

Symmetric: Yes.
If x has the same largest prime divisor as y, then y has the same largest prime divisor as x.

Transitive: Yes.
Suppose that x and y share the same largest prime divisor, as well as y and z. Then x and z share that largest prime divisor.

Therefore R is an equivalence relation.

The equivalence class of $z = 11$ is [11] since 11 is prime. The other positive integers in [11] are the positive integers whose largest prime divisor is 11.

The number of equivalence classes is equal to the cardinality of all the prime numbers, which is infinite.

(b) **Reflexive**: Yes.
$x_1^2 + x_2^2 = x_1^2 + x_2^2$.

Symmetric: Yes.
Suppose that $x_1^2 + x_2^2 = y_1^2 + y_2^2$, then of course $y_1^2 + y_2^2 = x_1^2 + x_2^2$.

Transitive: Yes.
Suppose $x_1^2 + x_2^2 = y_1^2 + y_2^2$ and $y_1^2 + y_2^2 = z_1^2 + z_2^2$, then certainly $x_1^2 + x_2^2 = z_1^2 + z_2^2$.

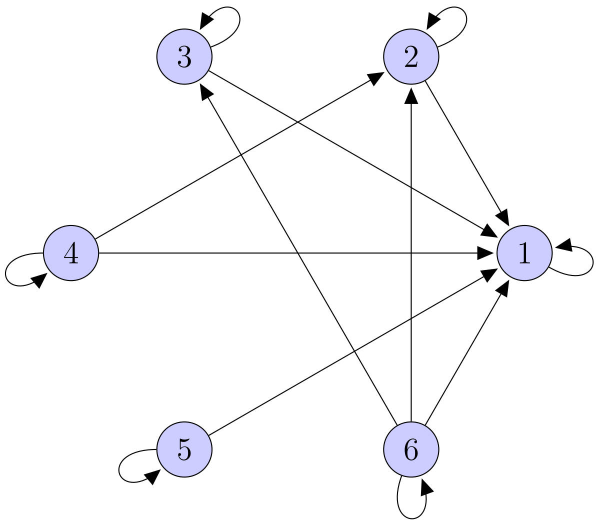
Therefore R is an equivalence relation.

The equivalence class of $z = (2, 5)$ is [29]. This equivalence class includes all ordered pairs of real numbers such that $x_1^2 + x_2^2 = 29$.

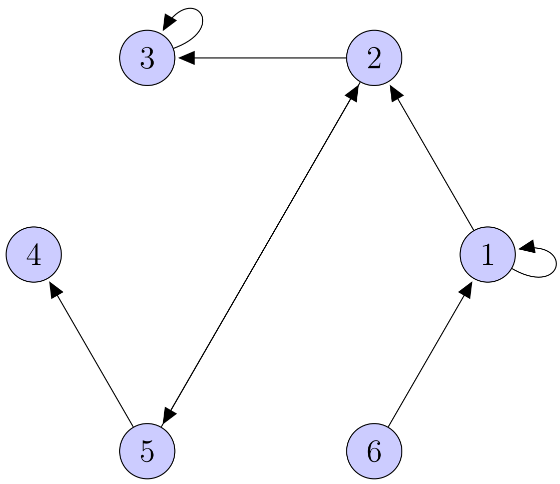
There are infinitely many equivalence classes of R, one for each positive real number that can be written as the sum of the squares of two real numbers.

3. *Hint*: We put an arrow from x to y if $x\ R\ y$.

(a)

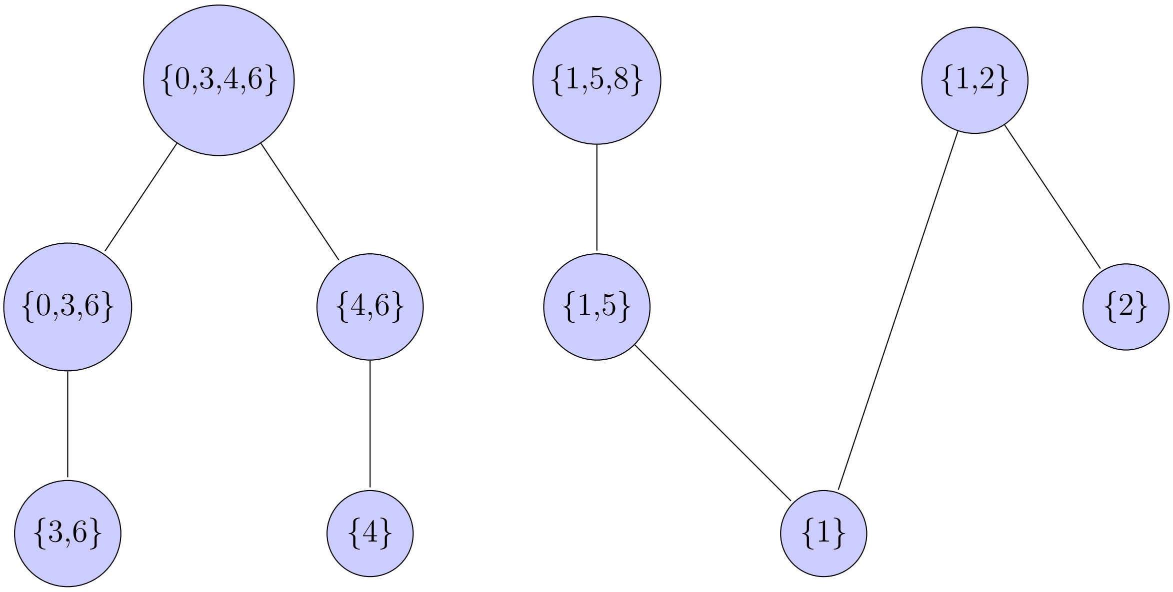


(b)



4. When determining this relation we must ensure that it is reflexive, symmetric and transitive. Specifically every element in the same equivalence class must have these properties with every other element in its equivalence class and with no other elements.
$R = \{(1,1),(3,3),(6,6),(1,3),(3,1),(1,6),(6,1),(3,6),(6,3),(2,2),(5,5),(2,5),(5,2),(4,4)\}$.

5.



6. A relation is a partial order if it is reflexive, transitive and antisymmetric, therefore we will attempt verify these properties.

Reflexive: Yes.
$x \equiv x \pmod 5$.

Transitive: Yes.
Suppose $x \equiv y \pmod 5$ and $y \equiv z \pmod 5$, certainly $x \equiv z \pmod 5$.

Antisymmetric: No.
$0 \equiv 5 \pmod 5$ and $5 \equiv 0 \pmod 5$, but $0 \neq 5$.

Therefore R is not a partial order.

7. (a) $2^k \cdot 2^{\frac{k^2-k}{2}} = 2^{\frac{k(k+1)}{2}}$.

   $|M| = k^2$. There are k entries along the diagonal, which can have entry 0 or 1, 2^k choices. For any entry in the lower triangle of M, it must match in the corresponding upper triangle, therefore there are two choices, 1 or 0, for these $\frac{k^2-k}{2}$ entries.

   (b) $2^k \cdot 3^{\frac{k(k-1)}{2}}$.

   There are 2^k choices along the main diagonal. There are exactly three disjoint possibilities for any entry, (m_i, m_j), for $i \neq j$. Either (m_i, m_j) is in R, (m_j, m_i) is in R, or neither of (m_j, m_i), (m_i, m_j) is in R. Once the lower half triangle, $\frac{k^2-k}{2}$ entires, has been assigned values the upper half will be designated accordingly. Thus there are three options for these entries which gives us $3^{\frac{k^2-k}{2}}$ choices.

8. Suppose $\frac{a}{b} \in \mathbb{Z}$ and $\frac{b}{a} \in \mathbb{Z}$. Since $a, b \in \mathbb{N}$, the only way this is possible is if $a = b$, hence R is antisymmetric.

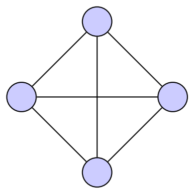
2 Graph Theory

2.1 Graphing Preliminaries

2.2 Definitions and Basic Properties

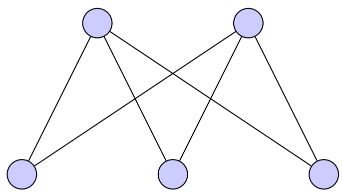
Solutions:

1. (a)



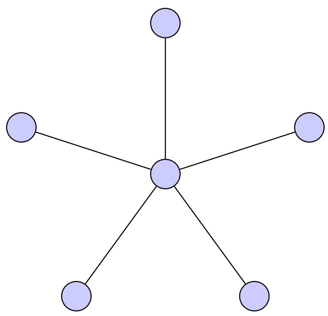
K_4 has 6 edges.

(b)



$K_{2,3} = K_{3,2}$ has 6 edges.

(c)



$K_{1,5} = K_{5,1}$ has 5 edges.

2. (a) $\frac{(n-1)(n)}{2}$ edges.

   This can be seen using the Euler's Theorem. Each vertex in a K_n graph has degree $(n-1)$. And so we have that:

$$\sum_{i=1}^{n}(deg(v_i)) = n \cdot (n-1) = 2 \cdot |E|$$

   Therefore, $|E| = \frac{n \cdot (n-1)}{2}$

   (b) $m \cdot n$ edges.

   In a $K_{n,m}$ graph, there are n vertices of degree m and m vertices of degree n. Using the Euler's Theorem we see:

$$2n \cdot m = 2|E|,$$

   hence

$$n \cdot m = |E|$$

3. Using Euler's Theorem:

$$5 \cdot 4 + 4 \cdot 3 = 2|E|$$

$$|E| = 16.$$

4. (a) Impossible.

   It is not possible to have a graph with a universal vertex and an isolated vertex, as a universal vertex must be adjacent to all other vertices while an isolated vertex cannot be adjacent to any vertices. The existence of one contradicts the definition of the other.

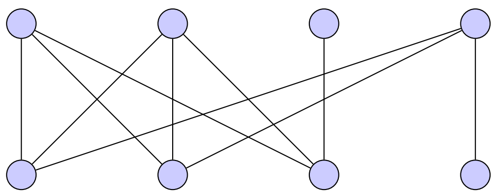
(b) Impossible.

By a corollary to Euler's Theorem we know there must be an even number of odd degree vertices.

(c) Impossible.

The only way to partition 5 vertices would either be with partite sets of size 1 and 4, or partite sets of size 2 and 3. $K_{1,4}$ has $1 \cdot 4 = 4$ edges, while $K_{2,3}$ has $2 \cdot 3 = 6$ edges. Neither complete bipartite graphs on 5 vertices can have more than 6 edges, so 7 edges is not possible.

(d) There are several graphs with these properties, here is one:

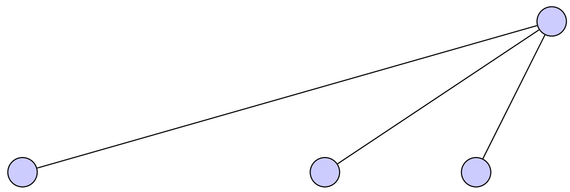


5. A bipartite graph cannot have a K_3 subgraph, as there would be three mutually adjacent vertices in the graph, requiring three different partite sets, contradicting that the graph is **bi**partite.

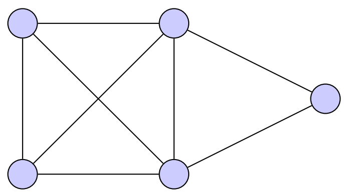
6. (a) No.

   This graph is clearly bipartite so by question 5 we know K_3 cannot be a subgraph.

   (b) Many such subgraphs of size 3 exist, such as:



7. Many such graphs exist, such as K_4 itself. One possible graph is:



8. There are $26 - 5 - 6 - 7 = 8$ vertices of degree x. Applying Euler's Theorem:

$$5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 + 8 \cdot x = 2 \cdot 58,$$

   Rearranging we obtain $x = 3$. Thus the degree of the remaining eight vertices is 3.

9. Let x be the number of vertices of degree 3, and y the number of vertices of degree 4. The order of the graph is 24 therefore:

$$5 + 7 + 7 + x + y = 24$$

   Applying Euler's Theorem:

$$5 \cdot 4 + 7 \cdot 1 + 7 \cdot 2 + 3 \cdot x + 4 \cdot y = 3 \cdot 30$$

We now have two equations with two unknowns and can solve this system of equations. Isolating for x in the first equation, $x = 5 - y$, and substituting it into the second equation, with some arithmetic we obtain $y = 4$. Therefore there are exactly four vertices of degree 4.

10. This follows from Euler's Theorem. Let us create a graph where each person is represented by a vertex. We can represent two people 'speaking' by connecting their respective vertices with an edge. Euler's Theorem shows us that there must be an even number of odd degree vertices in any graph (don't forget that zero is an even number). Thus, it follows that there must be an even number of people at any party who speak to an odd number of people.

11. No.

    By Euler's Theorem, since this graph has 31 edges, the sum of its degrees must be 62. There are $13 - 3 - 7 = 3$ vertices with currently undetermined degree, but we know that the sum of the degrees of these three vertices must equal $62 - 3 \cdot 1 - 7 \cdot 4 = 31$. We are now left to determine three positive integers, x_1, x_2, x_3, who sum to 31, which represent the degrees of the undetermined vertices.

    We know that in any graph there must be an even number of odd vertices (currently there is an odd number of odd degree vertices), thus either one or three of these vertices will have an odd degree.

    Case 1: Suppose that only one of the three remaining vertices has odd degree, say x_1, implying there are two vertices with even degree. We know that the vertex of odd degree cannot be degree one or seven by the set up of the problem.

    Case 1a: Suppose $\deg(x_1) = 3$, then $31 = 3 + x_2 + x_3$ which implies that one of the vertices of even degree will have degree ≥ 14. This is impossible since the graph only has 13 vertices.

Case 1b: Suppose $\deg(x_1) = 5$, then $31 = 5 + x_2 + x_3$ which implies that one of the vertices of even degree will have degree ≥ 13. This is impossible given the order of the graph.

Case 1c: Suppose $\deg(x_1) = 9$ is, then $31 = 9 + x_1 + x_2$ which implies that $\deg(x_2) = \deg(x_3) = 12$. Then x_2 and x_3 are universal vertices, but if there are two universal vertices it is impossible to have any vertices of degree one.

Case 2: Suppose that all three of the remaining vertices have odd degree. One of these vertices will have degree at least $\lfloor \frac{31}{3} \rfloor = 11$. Certainly no vertex of this graph may be degree 13 since $|V(G)| = 13$. Thus one vertex must be degree 11, say x_1. We know that:

$$31 = 11 + x_2 + x_3,$$

$$20 = x_2 + x_3.$$

By this set up, $x_2, x_3 \leq 10$, thus their only possible degrees are three, five and nine. Neither combination of two of these numbers adds up to 20.

Therefore there is no possible way for a graph with these parameters to exist.

12. Yes the subgraph of any bipartite graph is also bipartite, just keep the same partite sets, or some subset of them, in the subgraph. This answer does not change if we require non-empty edges sets since then there is just limited variation on the possible partite sets.

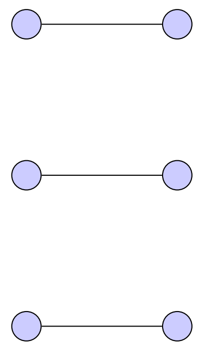
13. This graph G has 15 edges, so the sum of the degrees of the vertices must be 30.

If $d = 1$, the graph has order 30 (disjoint union of 15 K_2's).
If $d = 2$, the graph has order 15 (disjoint cycles C_m, C_n where $n + m = 15$).

If $d = 3$, the graph has order 10.
If $d = 5$, the graph has order 6 (K_6).

These are the only possible cases. Remember that we are looking at simple graphs so multiple edges exist nor can vertices be self-adjacent.

14. (a) Consider three disconnected copies of K_2:



(b) Impossible.

There are an odd number of vertices with even degree (there are other valid arguments).

(c) Impossible.

Notice that this graph has seven vertices, two of which have degree 6. This graph would have two universal vertices, which means that the minimum degree of any vertex must be 2, which is not the case.

15. $2^{\binom{n}{2}} = 2^{\frac{n(n-1)}{2}}$.

We know a complete graph has $\binom{n}{2}$ edges, so we can consider counting the number of spanning subgraphs of K_n. We have the choice of including or excluding each edge of K_n, two options for each edge.

16. $n = 23$.

    The number of vertices will be maximized by minimizing the degree of the vertices.

    First we attempt to create a 3-regular graph. Let n be the number of vertices of this graph, by Euler's Theorem we see:

    $$2 \cdot 35 = 3n.$$

    There is no integral solution for n, so this graph is not possible.

    We next try a graph where every vertex, but one, is degree 3. We see:

    $$70 = 3(n-1) + 4,$$

    Which rearranges to give us $n = 23 \in \mathbb{Z}^+$. Note that this means that there are 22 (an even number of) vertices of (odd) degree 3.

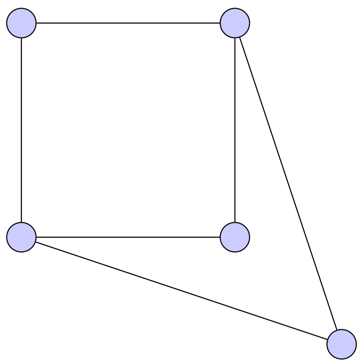
17. Suppose there are n vertices. By Euler's Theorem, $\sum_{i=1}^{n} \deg(v_i) = n \cdot k = 2|E(G)|$. We know $n \in \mathbb{Z}$ and $\frac{2|E(G)|}{k} \in \mathbb{Z}$, therefore k divides $|E(G)|$.

18. $n - 1$.

    All vertices of G, except one, have odd degree. To guarantee there are an even number of odd degree vertices, n must be odd.

If the degree of a vertex in G is k, the degree of that same vertex in $\overline{G}$ is $n-1-k$. $n-1$ is certainly even as n is odd, therefore if k is odd, $n-1-k$ is odd as well. The vertex in G of even degree will still have an even degree in $\overline{G}$ by the same argument. Thus, there are also $n-1$ vertices of odd degree in $\overline{G}$.

19. There are many possible graphs, here is one:



20. Impossible.

    Let each person represent a vertex where an edge between two vertices denotes friendship. Suppose each person has exactly three friends, meaning that the degree of every vertex is 3. Euler's Theorem tells us that there must always be an even number of odd degree vertices, this set up has an odd number of odd vertices which is not possible.

21. Yes this is possible.

    K_n, where n is odd, or any cycle are two such examples. (More exist, try to find some more!)

22. Suppose our graph has n vertices. First we must notice that if a graph has a vertex of degree 0, an isolated vertex, then there is no vertex of degree $n-1$, an universal vertex. Similarly a graph with a vertex of degree $n-1$ cannot have a vertex of degree 0. Therefore the degrees of the vertices of the given graph will be a subset of either $V_1 = \{0, 1, 2, ..., n-2\}$ or $V_2 = \{1, 2, 3, ..., n-1\}$. We can see that $|V_1| = |V_2| = n-1$, but there are n vertices we need to assign degrees to, thus at least two of the vertices must share the same degree meaning it is impossible for every vertex to have a different degree.

    Note: This proof method uses the Pigeonhole Principle.

23. Let G be a graph with n vertices and n edges. By Euler's Theorem we know,

$$\sum_{i=1}^{n} deg(v_i) = 2 \cdot |E(G)| = 2n.$$

By assumption there are no vertices of degree 0 or 1, so $\delta(G) \geq 2$.

Suppose for contradiction that there is at least one vertex with degree more than 2. By Euler's Theorem we see:

$$\sum_{i=1}^{n} deg(v_i) \geq 2(n-1) + 3 > 2n,$$

a contradiction. Therefore we can conclude that the degree of every vertex must be exactly 2.

2.3 Isomorphisms

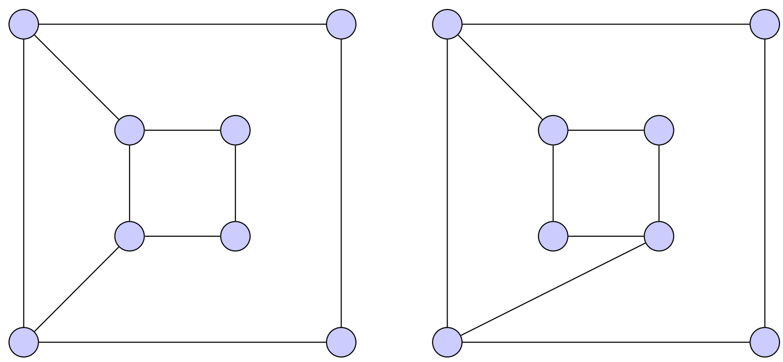
Solutions:

1. Put simply, these two graphs have the same structure.

   More formally, two graphs, G and H, are isomorphic if there exists a bijection between $V(G)$ and $V(H)$, $f : V(G) \longrightarrow V(H)$, where $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Isomorphism is denoted: $G \cong H$.

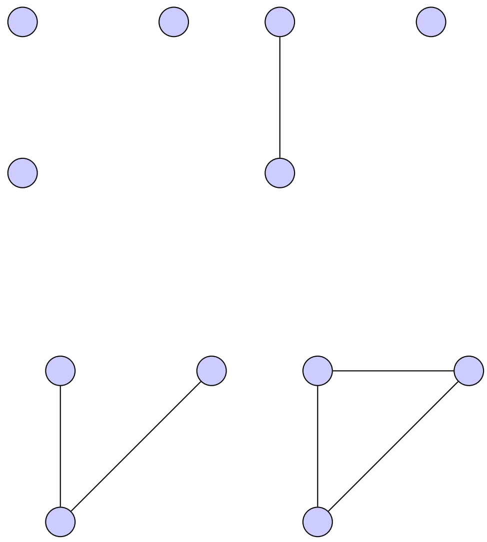
2. The converse of this statement is: "If two graphs G_1 and G_2 have the same number of vertices, same number of edges, and the same degree sequence, then they are isomorphic".

   This statement is false. Here is one of many counter-examples:

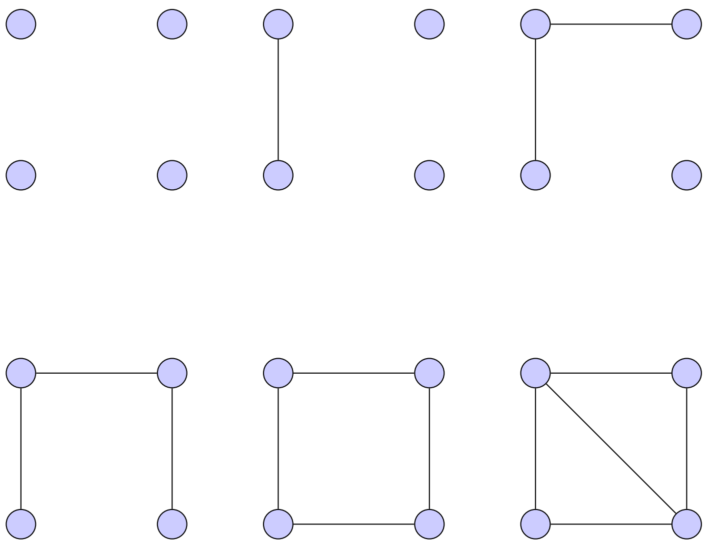


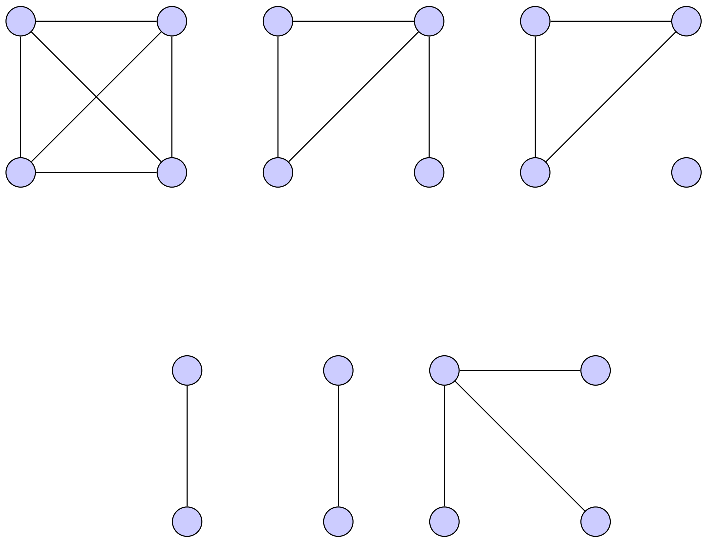
   The above two graphs have the same number of vertices, edges and identical degree sequences, but are not isomorphic since the leftmost graph has no five cycles, while the rightmost graph has two.

3. (a)

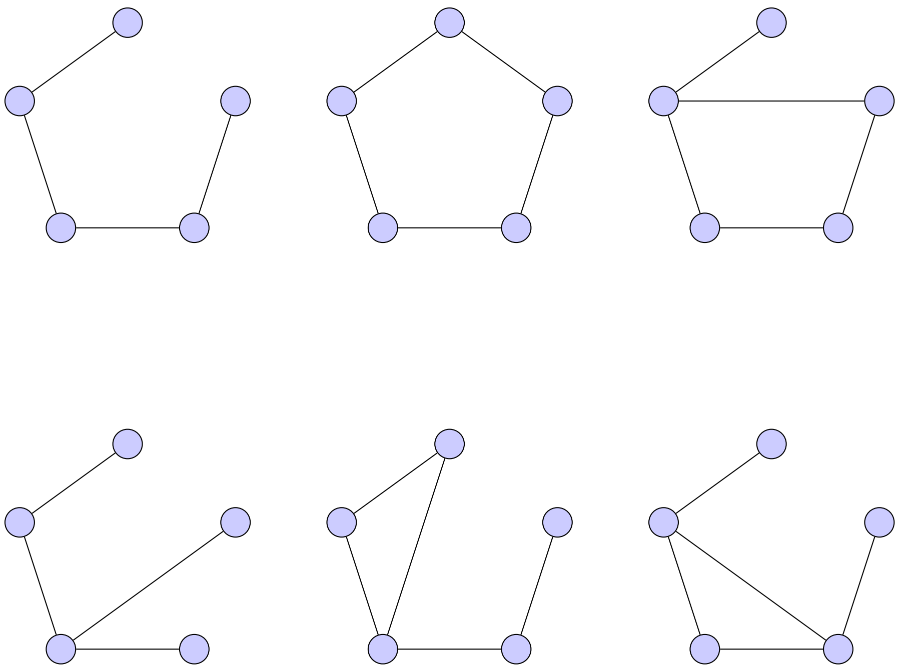


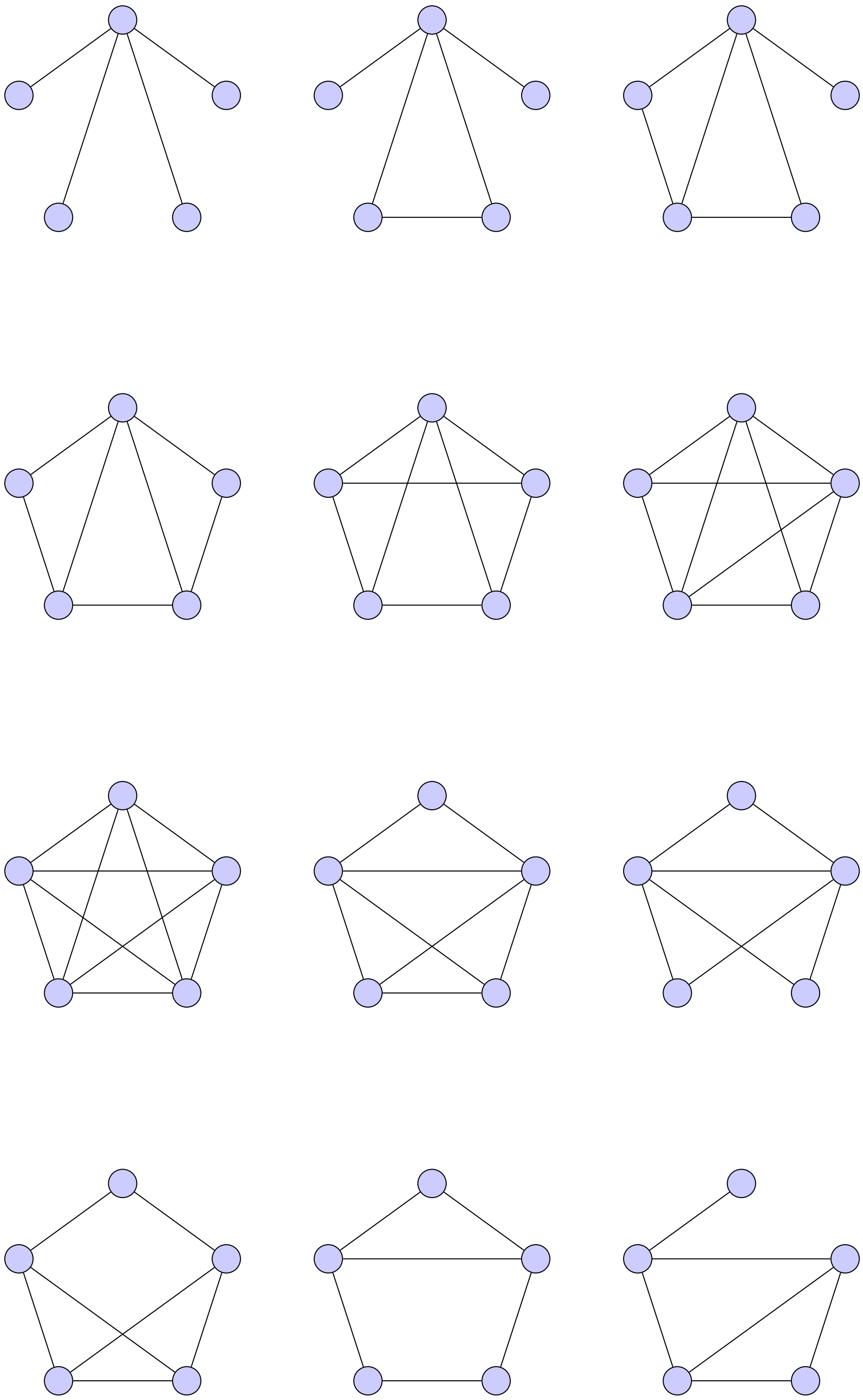
(b)

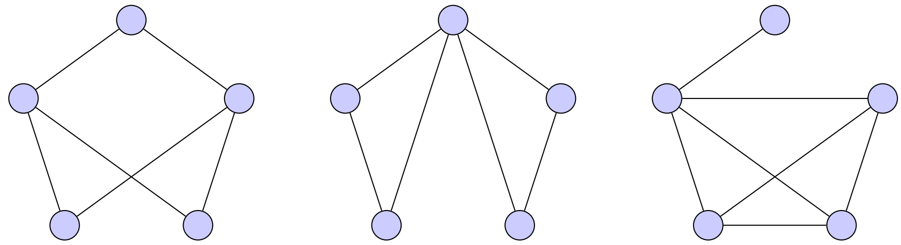




(c)







4. Consider a graph, G, with n vertices. G is certainly isomorphic to some spanning subgraph of K_n since $E(G) \subseteq E(K_n)$, and $V(G) = V(K_n)$. G is also a subgraph, although not spanning, of K_k for $k > n$. Therefore G is isomorphic to some subgraph of every complete graph on least n vertices.

5. We prove this using the definition of isomorphism; if two graphs G and H are isomorphic, there exists a bijection, $f : V(G) \longrightarrow V(H)$, between them that maintains adjacencies.

   Suppose that G has a triangle with vertices a, b, c. The mapping of these vertices to H maintain that they are all pairwise adjacent, that is $f(a), f(b), f(c)$ forms a triangle in H. Thus, each triangle in G corresponds to a triangle in H.

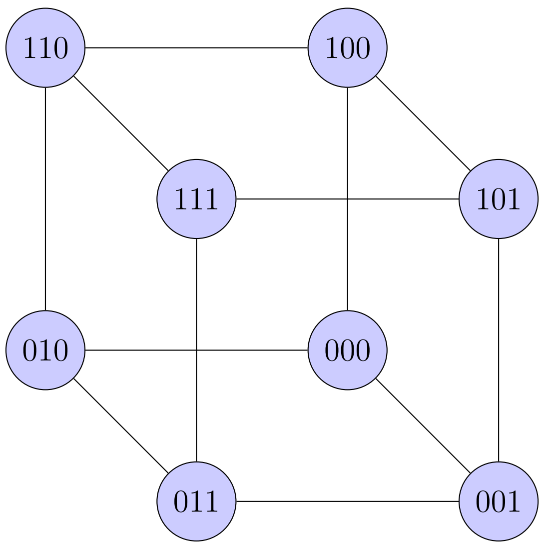
   Since an isomorphism is a bijection, it has an inverse. If a triangle with vertices u, v, w exists in H, then $f^{-1}(u), f^{-1}(v), f^{-1}(w)$ form a triangle in G, as desired.

6. The following is a bijection between the vertices that maintains adjacencies:

$$\begin{aligned} a &\longrightarrow m \\ b &\longrightarrow n \\ c &\longrightarrow p \\ d &\longrightarrow q \\ e &\longrightarrow r \\ f &\longrightarrow s \\ g &\longrightarrow t \\ h &\longrightarrow u \end{aligned}$$

Note: There are multiple correct solutions. Verify yours is correct by following the isomorphism and attempting to draw this graph in the same form as the other.

7. This is best shown with an illustration:



Since we have been able to draw the graph representing the length three binary strings, as outlined by the question, as a cube we may conclude that it is indeed isomorphic to the corners and edges of a cube.

8. (a) False.

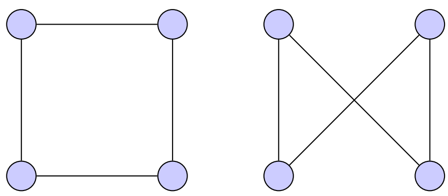
Consider the following counterexample: P_5 and the empty graph, $\overline{K_5}$. Both have five vertices and no cycles but are clearly not isomorphic.

(b) True.

By definition, an isomorphism is a bijection, so certainly the sizes of the vertex sets must be the same. An isomorphism also preserves adjacencies, hence the number of edges of the two graphs are necessarily the same.

(c) False.

It is possible to draw a graph in multiple ways. The key to isomorphism is that the *structures* are identical, not how they are drawn.
For example this is the same graph (hence isomorphic to itself) drawn in two different ways: $K_{2,2}$:



(d) True.

We show that an isomorphism function is an equivalence relation by proving the three properties of an equivalence relation individually.

Reflexive: Yes.

Any graph is certainly isomorphic to itself, just let the isomorphism be the identity function.

Symmetric: Yes.
Suppose $G \cong H$, then there exists a mapping $f : V(G) \longrightarrow V(H)$. By definition, f is a bijection so there exists an inverse mapping, $f^{-1} : V(H) \longrightarrow V(G)$, that maintains adjacencies, telling us that $H \cong G$.

Transitive: Yes.
Let us consider three graphs G, H, I with $G \cong H$ and $H \cong I$. We know there exists a mapping from $V(G) \longrightarrow V(H)$ that maintains adjacencies, and similarly a mapping from $V(H) \longrightarrow V(I)$. Composing these functions we obtain the mapping: $V(G) \longrightarrow V(H) \longrightarrow V(I)$ which maintains adjacencies, hence $G \cong I$.

(e) True.

If this were not true, then it would not be possible to create a bijection that maintains every adjacency.

(f) False.

The degree sequences of these two graphs are different! Two vertices in $K_{3,2}$ have degree 3, while all vertices in C_5 have degree 2.

(g) True.

These graphs are identical in structure.

(h) True.

Let $G \cong H$. If H contains a cycle, in order to maintain all adjacencies in

G, G would contain a cycle too.

9. (a) Non-isomorphic.

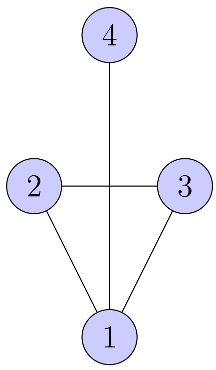
   The degree sequences differ. The graph on the left has two vertices of degree 3, while the graph on the right has four vertices of degree 3.

   (b) Non-isomorphic.

   The graph on the right contains a vertex of degree 4, while the graph on the left does not.

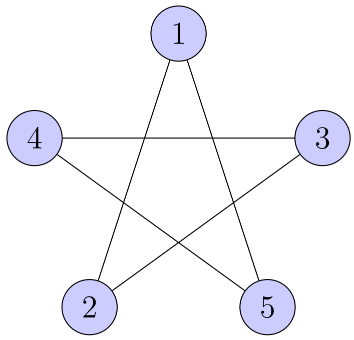
   (c) Isomorphic.

   Here is the rightmost graph drawn in the form of the leftmost graph:



   (d) Isomorphic.

Here is the rightmost graph drawn in the form of the leftmost graph:

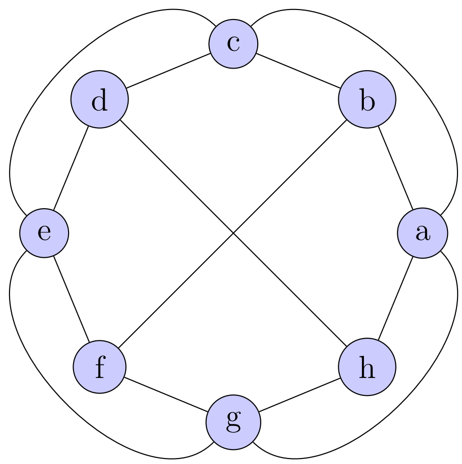


(e) Non-isomorphic.

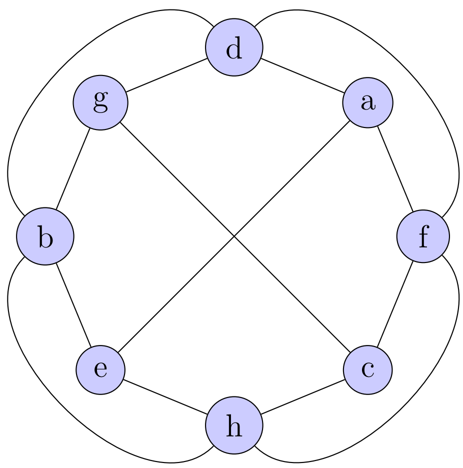
The graph on the left has a triangle formed by vertices c, g, h while the graph on the right is triangle-free.

10. There are exactly ten self-complementary graphs of order 8. Below is one such example (verify for yourself that these two graphs are indeed isomorphic):

G:



$\overline{G}$:



2.4 Eulerian Circuits

Solutions:

1. An Eulerian circuit is a trail that uses every edge exactly once and ends where it began. A Eulerian trail is a trail that goes through every edge, but does not necessarily end where it began.

2. (a) Neither an Eulerian circuit nor an Eulerian trail exist as there are more than two vertices with odd degree.

   (b) No Eulerian circuit exists but there exists an Eulerian trail as there are exactly two odd degree vertices.

   (c) Yes, there exists both a Eulerian circuit and a Eulerian trail as all vertices are of degree 4, meaning all vertices are of even degree.

   (d) Neither an Eulerian circuit nor an Eulerian trail exists as there are more than two vertices with odd degree.

   (e) Neither an Eulerian circuit nor an Eulerian trail exists as there are more than two odd degree vertices.

   (f) Neither an Eulerian circuit nor an Eulerian trail exists as there is only one vertex with odd degree.

3. This new graph will not be Eulerian.

   Both G and H are Eulerian so all of their vertices have even degree. Making vertices from G and H adjacent will result in two odd degree vertices. There will, however, exist a Eulerian trail.

4. Recall the Königsberg Bridge Problem: The city of Königsberg, Prussia, was set on both sides of a river and included two large islands, all connected by seven bridges. Is there a way to walk through the city crossing every bridge exactly once while finishing where you started?

   Let the land masses represent the vertices of a graph and bridges the edges. Solving the problem boils down to finding an Eulerian Circuit in the graph.

5. Assume there exists a path from u to v. By definition, every path is also a walk, hence we have identified a uv walk.

   Conversely, suppose there exists a walk from u to v. We will proceed by induction on the length, k, of the walk.

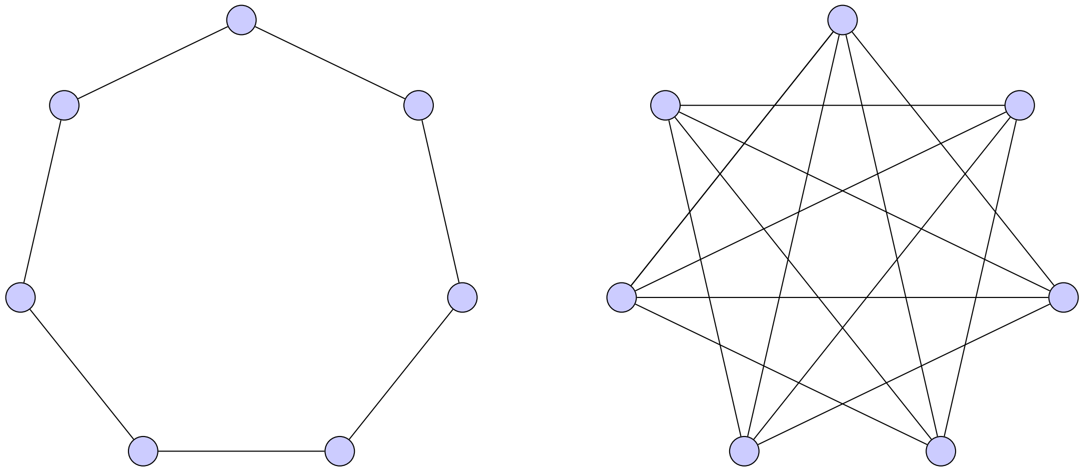
   Base case: Consider a uv walk of length $k = 1$. No vertices are repeated, so this walk is also a path.

   Induction hypothesis: Assume for any uv walk of length $\leq k$, for some positive integer k, that there also exists a uv path.

   Induction step: Consider a uv walk of length $k + 1$. If no vertices are repeated, then this walk is also a path and we are done. Let us assume instead that there is a vertex repeated in this walk, call it x. Then our walk is of the form: $u...x...x...v$. Consider the walk obtained from the first walk by removing all edges travelled between the two x vertices. We now have a walk of length $\leq k$, and so by our induction hypothesis there exists a uv path.

6. We must find a connected graph with all even vertices such that its complement also has even degree and is connected.

   One possible solution:



7. (a) False.

   This graph also must be connected with all vertices a non-zero, even degree.

   Counterexample: Two disjoint K_3's. The degree of every vertex is even, but there is no path that uses every edge exactly once.

   (b) True.

   From question 5, the existence of a uv walk implies the existence of a uv path, so if there exists a closed uu walk, there exists a closed uu path, which is a cycle.

   (c) False.

An Eulerian circuit requires crossing every edge exactly once, but if there are no edges between the components of the graph, there is no way to reach *every* edge.

Counterexample: Two disjiont K_3's.

(d) True.

Consider pairing off the vertices of the graph (which we can do since there are an even number of them), and add an edge between each pair of vertices. Now the degree of every vertex is even and there exists an Eulerian circuit. Removing the edges between the vertex pairs leaves us with k disjoint trails that contain all the edges.

8. $K_{m,n}$ is Eulerian if and only if m, n are both non-zero, even integers.

The m vertices in the first partite set have degree n, while the n vertices in the second partite set all have degree m. Requiring both m, n to be even ensures that all the vertices of the graph have even degree.

9. No, an Eulerian circuit is not necessarily a cycle.

Counterexample: K_5 certainly has an Eulerian circuit however that circuit is certainly not a cycle since vertices have been repeated.

10. (a) Any odd n.

The degree of every vertex will be $n - 1$, an even number.

(b) For all odd values of n there will be a *closed* Eulerian trail (i.e. an Eulerian circuit). The only *open* Eulerian trail occurs when $n = 2$.

11. True.

Let $C = v_0, v_1, ..., v_n, v_0$ be the vertices of the circuit in our graph (remember that vertices may be repeated but not edges). If C has no repeated vertices it is a cycle and we're done.

Suppose that C has a repeated vertex, say v_i, for some $i = 1, ..., n$. We use a similar strategy as the *induction step* from the proof of question 5 and identify the v_0v_0 *path* which is certainly a cycle. Now consider the part of our circuit which starts and ends at the repeated vertex, v_i (the part we ignored to form the v_0v_0 path). If this sub-circuit contains a repeated vertex, repeat the same process as above, if not then it is a cycle and we are done.

12. We show the three properties of an equivalence relation hold.

Reflexive: Yes.
Consider the trivial walk from a vertex to itself.

Symmetric: Yes.
Suppose there exists a uv walk, then follow it in the opposite direction to find a vu walk.

Transitive: Yes.
Suppose there is a uv walk and a vz walk. Consider first following the uv walk and then continuing to z by the vz walk. This is a uz walk, as desired.

13. If G is connected then we are done.

Alternatively, suppose that G is disconnected (we'd like to show that $\overline{G}$ is connected). Let $u, v \in V(G)$. If $uv \notin V(G)$, then $uv \in V(\overline{G})$ and hence there is a uv path in $\overline{G}$. If $uv \in V(G)$ then certainly u and v are in the same component. We know G is disconnected, so consider some vertex w in a different component than u, v. Certainly $uw, wv \notin V(G)$, so $uw, wv \in V(\overline{G})$, thus there is a uv path in $\overline{G}$.

Since u, v, w were arbitrary vertices of G, and a graph is connected if there exists a uv path between all vertices $u, v \in V(G)$, we may conclude that $\overline{G}$ is connected, as desired.

14. Suppose for a contradiction that G is a disconnected graph of order 9 such that for every pair of distinct vertices, u, v, $deg(u) + deg(v) \geq 8$. Since G is disconnected there must exist vertices, say x, y, such that no xy path exists in G. If no xy path exists then certainly $xy \notin E(G)$ and x, y do *not* share neighbours. Let $deg(x) = k$ for some $k = 0, ..., 8$. Since y is not adjacent to x and they share no neighbours, $deg(y) \leq 8 - k - 1$. Together we have $deg(x) + deg(y) \geq k + 8 - k - 1 = 7$, a contradiction.

Therefore our additional hypothesis that G was disconnected was false, hence G is connected.

15. $|E(G)| \geq n - 1$.

Proof: Consider K_n which has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. We want to remove as many edges as possible without disconnecting the graph. We begin by removing $n-2$ edges from the first vertex, so the first vertex is now a pendant (degree 1) vertex. Every other vertex of the graph is now adjacent to exactly $n-2$ other vertices.

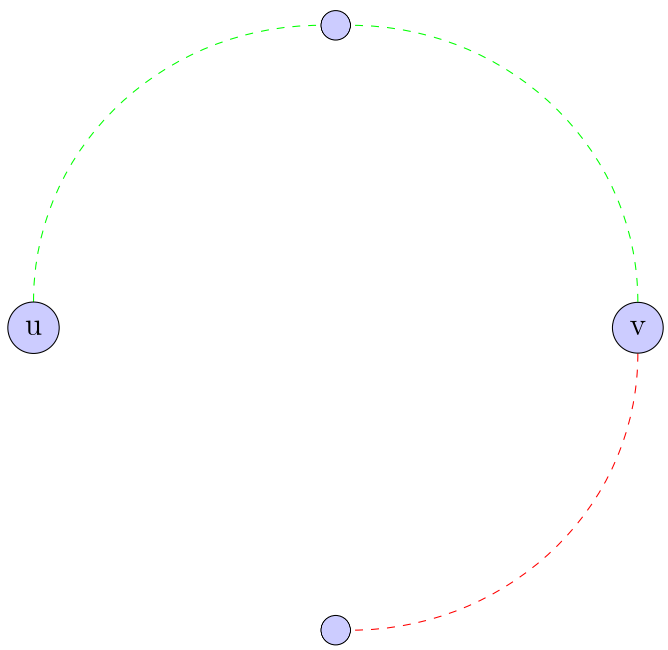
We can now remove $n-3$ edges from the next vertex, $n-4$ from the next, and so on. Hence the maximum number of edges that can be removed without disconnecting the graph is:

$$(n-2)+(n-3)+...+2+1=\frac{(n-2)(n-1)}{2}$$

Therefore the minimum number of edges required for the graph to be connected is exactly:

$$\frac{(n)(n-1)}{2}-\frac{(n-2)(n-1)}{2}=n-1.$$

16. Recall that in any given circuit, there must exist two trails between any given pair of vertices. This is because there are two distinct ways of going around the circuit.



Thus, removing any single edge cannot destroy both trails, as there is no common edge between them, hence there still exists a uv trail.

2.5 Hamiltonian Cycles

Solutions:

1. (a) A Eulerian circuit is a circuit that uses every edge and vertex in the graph. A Hamiltonian cycle is a cycle that uses every vertex. In a Eulerian circuit

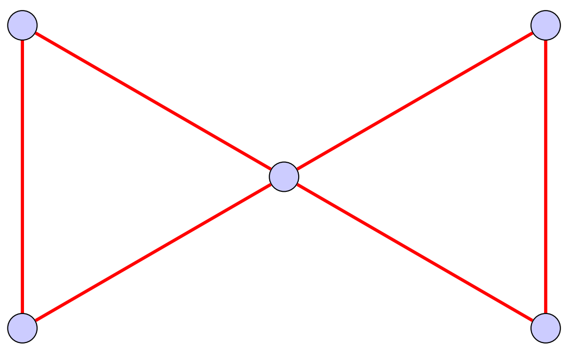
   it is possible to pass through some vertices multiple times while that is not possible in a Hamiltonian cycle. Also, a Hamiltonian cycle may not visit every edge while that is a requirement of a Eulerian circuit.

   (b) A Eulerian trail is a trail that uses every edge and vertex. A Hamiltonian path is a path that uses every vertex.

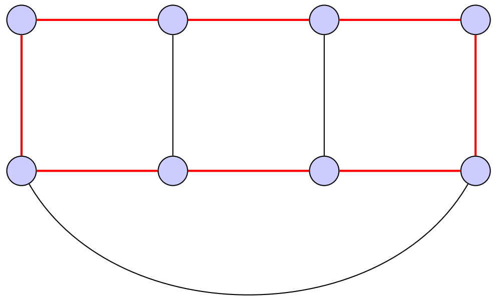
   A Eulerian trail may visit the same vertex multiple times while a Hamiltonian path will not. A Hamiltonian path may not visit every edge in the graph, while that is a requirement of Eulerian trail.

2. (a) This graph is not Hamiltonian. This is because removing either of the vertices of degree 3 in the graph will disconnect it into two components. In a Hamiltonian graph, the graph obtained by removing any non-empty, proper subset of U of the vertices of the graph will have no more than $|U|$ components. This means removing a single vertex cannot disconnect a Hamiltonian graph.

   (b) Yes, the Hamiltonian cycle is highlighted in red.



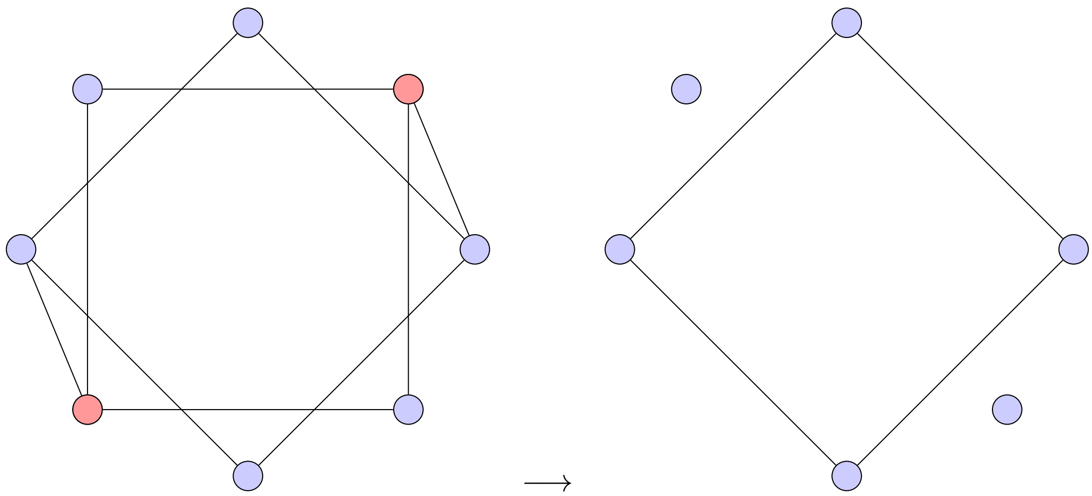
(c) Yes, many Hamiltonian cycles exist. One example is highlighted in red.



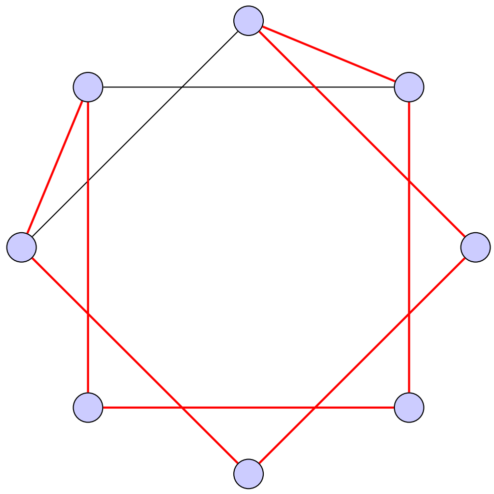
(d) Similar to part (a), removing any vertex of degree 3 will disconnect the graph. This means the graph is not Hamiltonian.

(e) This is the Petersen graph, which is famously non-Hamiltonian.

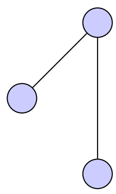
(f) Similar to parts (a) and (d), it is possible to remove two vertices and disconnect the graph into more than two parts, as shown below.



(g) Yes, a Hamiltonian cycle is highlighted in red.

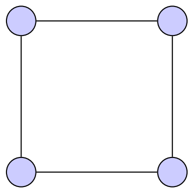


3. The following graph is a sufficient counterexample.



This graph is not Hamiltonian as it is acyclic however the minimum degree is $\frac{n-1}{2} = \frac{3-1}{2} = 1$.

4. Yes, many Hamiltonian and Eulerian graphs exist. The following is a simple example:



5. We can use a graph to represent the relationships of the group, where represent each person is represented with a vertex and each friendship with an edge.

   By Dirac's Theorem, this graph is Hamiltonian as every vertex will have degree at least $\frac{n}{2}$. We can use a Hamiltonian cycle of the graph as the seating plan, which will seat each person next to two friends.

6. If G is complete we are done, as all complete graphs are Hamiltonian. Let us suppose instead that G is not complete.

   Let u and v be two arbitrary non-adjacent vertices in G.

   Let $G' = G - u - v$. The upper limit on the number of edges in G' is the size of K_{n-2}, which is $\binom{n-2}{2}$. This gives the following inequality:

$$\binom{n-2}{2} \geq \binom{n-1}{2} + 2 - (deg(v) + deg(u))$$

This can be rearranged to find a lower limit on $deg(u) + deg(v)$:

$$
\begin{aligned}
deg(v) + deg(u) &\geq \binom{n-1}{2} + 2 - \binom{n-2}{2} \\
deg(v) + deg(u) &\geq \frac{(n-1)(n-2) - (n-2)(n-3)}{2} + 2 \\
deg(v) + deg(u) &\geq n
\end{aligned}
$$

Thus, G satisfies Ore's Theorem which means G is Hamiltonian.

7. (a) False.

   Consider the complete graph K_4. This graph contains many triangles but, as with all complete graphs, is Hamiltonian.

   (b) True.

   This can be seen by simply deleting one edge from a Hamiltonian cycle of the graph. This will leave a path that goes through every vertex, but is not a cycle.

   (c) True.

   If there is a Hamiltonian path between any two vertices then graph is connected. Take two adjacent vertices u and v. Add the edge connecting them to the Hamiltonian path between them, which will create a Hamiltonian cycle.

8. Notice that G has exactly two fewer edges than K_{13}, as the size of K_{13} is $\binom{13}{2} = 78$. This means the minimum possible degree of a vertex in G is 10, as at most two edges from one vertex in K_{13} could be removed to obtain G.

   As the minimum degree of any vertex in G is 10 then by Dirac's Theorem that

G is Hamiltonian.

The graph is not necessarily Eulerian as G can be obtained from K_{13} by deleting two edges in such a way that G has four vertices of degree 11.

9. A graph $K_{m,n}$ is Hamiltonian if and only if $m = n$, with $m, n \geq 2$.

   Proof:
   $K_{1,1}$ cannot contain a Hamiltonian cycle as it is acyclic. So we consider when $m, n \geq 2$.

   Assume $K_{m,n}$ is Hamiltonian with $m, n \geq 2$. Assume for a contradiction that $m \neq n$. Without loss of generality suppose that $m < n$. A cycle in a bipartite graph is necessarily of even length and will alternate between the two partite sets. Let a cycle begin at some vertex in the partite set of size n. Once the cycle is length $2m$, it will return to the partite set of size n and all vertices in the partite set of size m have been visited. There will however be $n - m > 0$ vertices unreached by this cycle in the partite set of size n. This means $K_{m,n}$ cannot have a Hamiltonian cycle, which is a contradiction.

   Conversely, suppose $K_{m,n}$ has $m = n$. It is easy to see that there exists a Hamiltonian cycle.

10. If a cycle beginning and ending at v contains w then the cycle will have a vertex sequence $\{v, v_1, v_2, ..., v_{k-1}, w, v_k, v_{k+1}, ...v\}$.

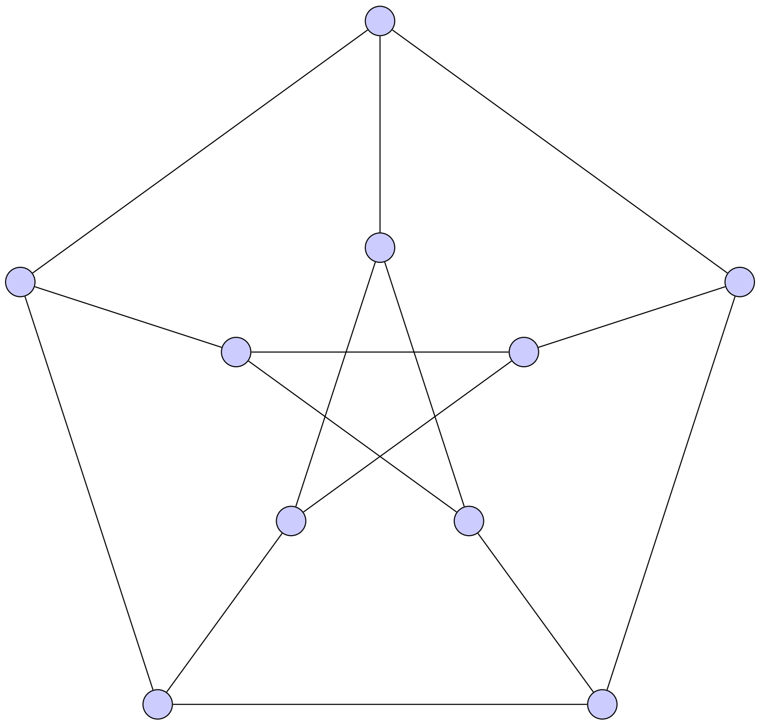
    Thus, there exists edge-disjoint paths from $v - w$ and $w - v$. We can put these path together $w - v + v - w$. This will form a cycle that begins and ends with w with a vertex sequence $\{w, v_k, v_{k+1}, ...v, v_1, v_2, ..., v_{k-1}, w\}$.

11. Let us assume for a contradiction that there exists a Hamiltonian path in a bipartite graph, G, where one partite set contains at least two vertices more

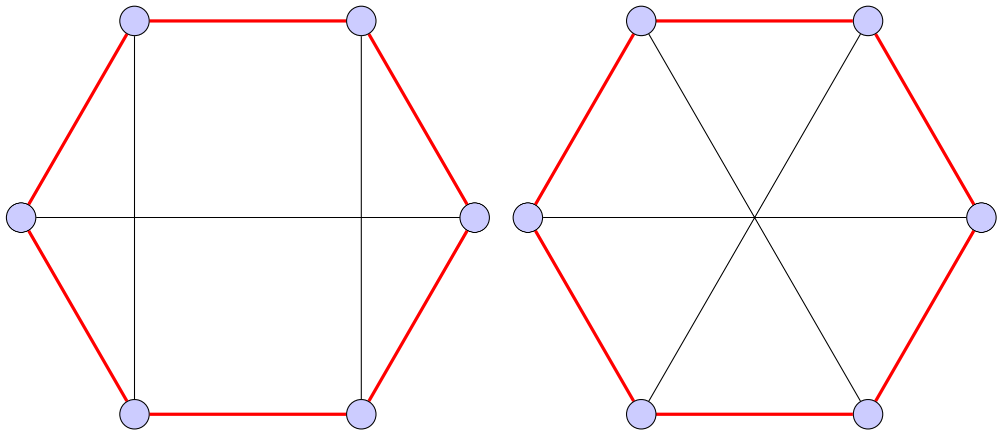
than the other. Let us call the partite sets A and B and assume without loss of generality that $|A| + 2 \leq |B|$.

Certainly if a Hamiltonian path existed it will begin in B and alternate between the two partite sets. Once the path is of length $2|A| + 1$ all vertices in A will be visited, while at least one vertex in B will not be visited. This means it is not possible for a Hamiltonian path to exist.

12. The Petersen graph is such a graph.

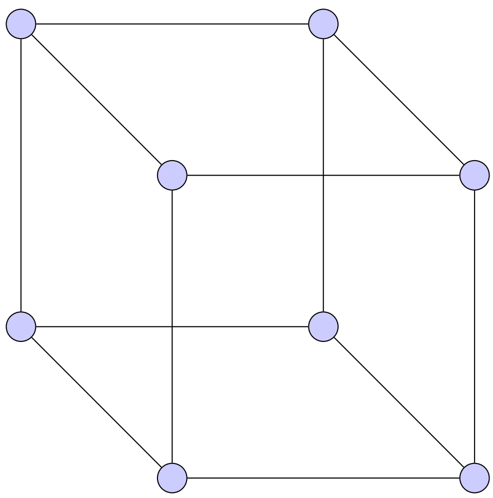


13. Notice that there are only two unique cubic graphs of order 6. A Hamiltonian cycle has been highlighted for each in red.

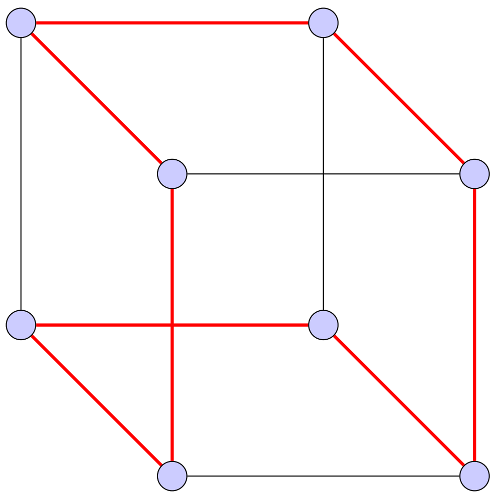


Thus, since both graphs are Hamiltonian it follows that all cubic graphs of order 6 are Hamiltonian.

14. Consider the cube below where the vertices are the corners of the cube and the edges are the edges of the cube.



Highlighted below is one of several Hamiltonian cycles in red:



2.6 Trees and Their Properties

Solutions:

1. Infinitely many possible degree sequences exist. If you can draw a tree with your degree sequence then it is correct.

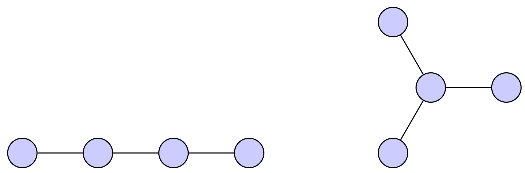
   The sum of your degree sequence must be $2(n-1)$, and there must be at least two vertices of degree 1.

   A possible example is: $3, 1, 1, 1$.

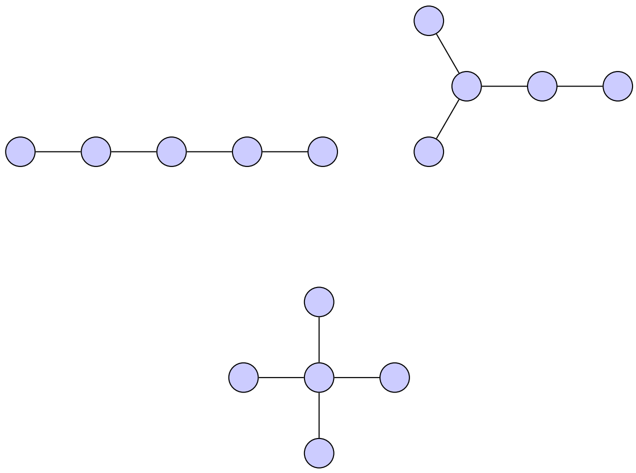
2. The following are equivalent:
   i) G is a tree.
   ii) G is a connected acyclic graph.
   iii) G is a connected graph with $n-1$ edges.
   iv) G is an acyclic graph with $n-1$ edges.

3. Let T be a tree, then T is connected with $n-1$ edges. Consider adding some edge to T, T now has n edges and hence is no longer a tree. Certainly the addition of this edge did not disconnect the graph, hence it must have a cycle.

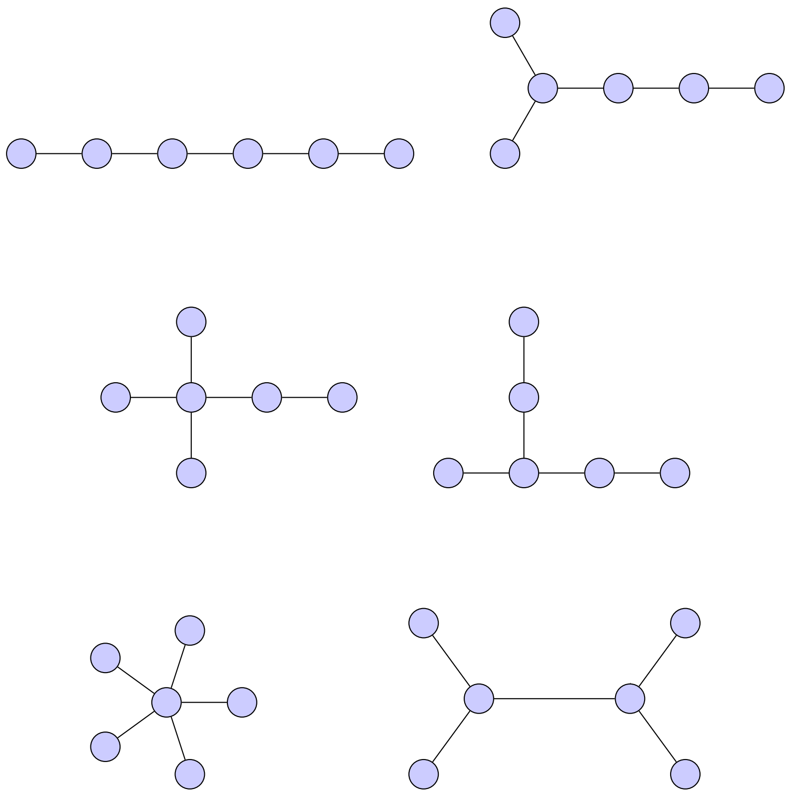
4. (a) $n = 4$:



(b) $n = 5$:



(c) $n = 6$:



5. Yes.

   Every P_n, for $n \in \mathbb{Z}^+$, is a Hamiltonian trail as well as a tree.

6. False.

   Consider the tree, P_2. Removing the only edge produces a disconnected subgraph which is clearly not a tree.

7. Nine vertices with degree 2.

   Let n denote the number of vertices of this tree and x the number of vertices with degree 2 or 4. Thus, $n = 100 + 20 + x$. Applying Euler's formula knowing that our graph is a tree ($|E| = n - 1$), we obtain the equality:

$$100 + 20(6) + \frac{1}{2} \cdot 4x + \frac{1}{2} \cdot 2x = 2(n-1)$$

   Substituting n we get,

$$100 + 120 + 3x = 200 + 40 + 2x - 2$$

   With basic arithmetic we get $x = 18$. There are $\frac{x}{2} = 9$ vertices of degree 2.

8. $x = 5$.

   T is a tree with 35 vertices, hence it has 34 edges. Euler's Theorem gives:

$$2 \cdot 34 = 25(1) + 2(2) + 3(4) + 2(6) + 3(x)$$

   With some algebra we get $x = 5$.

9. We proceed by induction on the order of the graph.

   Base case: The graph with one vertex and no edges is the trivial tree.

   Induction hypothesis: Suppose that any connected graph of order k with $k-1$ edges is a tree, for some $k \geq 1$.

   Induction step: Consider an arbitrary connected graph, G, of order $k+1$ and size k.
   Suppose every vertex in G has degree at least 2, then by Euler's Formula we have:

   $$2 \cdot k \geq 2(k+1) = 2k+2,$$

   which is a contradiction. Thus there must exist some vertex of degree 1, say u. Now consider $G-u$, which is a connected graph with order k and size $k-1$, so by the induction hypothesis $G-u$ is a tree. Readding the vertex u and its single adjacency cannot form a new cycle, so G is a connected, acyclic graph with $k+1$ vertices and k edges and hence a tree.

   The result follows from the Principle of Mathematical Induction.

10. T has 102 vertices.

    Let $n = |V(G)|$, from Euler's Theorem we get the equation:

    $$2(n-1) = 25(5) + (n-25)(1),$$

    which rearranges to give $n = 102$.

11. There are two vertices of degree 5.

    Let x represent the number of degree 5 vertices. Using Euler's Theorem we see:

    $$2 \cdot 20 = 15(1) + 1(6) + x(5) + (21 - 15 - 1 - x)(3).$$

    With some algebra we get $x = 2$.

12. Suppose T is a tree on n vertices. Then we know that T has $n - 1$ edges and is simultaneously connected and acyclic. Consider removing one edge of T, T is no longer a tree since it has $n - 2$ edges and n vertices. The removal of this edge certainly did not add a cycle, so the graph must be disconnected.

    The components of this new graph are trees.

13. Consider a tree with at least three vertices. We know that a graph is bipartite if and only if it contains no odd cycles, and that every tree is acyclic. Certainly there are no odd cycles in a tree, so every such tree is bipartite.

14. Let T be a tree of order n with two vertices of degree 3. Let x be the number of leaves of T. We know that there are $n - x - 2$ vertices that are not degree 1 or 3, these vertices will have degree at least 2. Applying Euler's Theorem we see:

    $$2(n - 1) \geq 2(3) + x + (n - x - 2)(2) = 2n - x + 2$$

    With some basic arithmetic we obtain the inequality $x \geq 4$, as desired.

15. A tree, T, is a complete bipartite graph if and only if $T = K_{1,n}$ for some positive integer n.

    We know that every tree has at least one leaf. The only way for a complete bipartite graph to have vertices of degree 1 is if one of the partite sets has only one vertex.

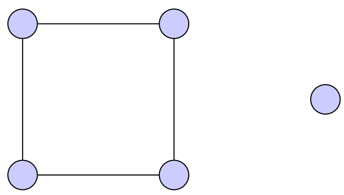
16. $|E(G)| = n - c$.

Let c_i represent the number of vertices in the i^{th} component of the forest, for $i = 1, ..., c$. Certainly $n = c_1 + ... + c_c$, with each component is a tree, thus each component has $c_i - 1$ edges. The total number of edges in the forest is

$$c_1 - 1 + c_2 - 1 + ... + c_c - 1 = n - c$$

17. There are many possible graphs satisfying these properties. If your graph is disconnected or has a cycle then it is not a tree.

Consider the following disconnected cyclic graph on five vertices with four edges that is not a tree:



18. (a) Impossible.

In any tree,

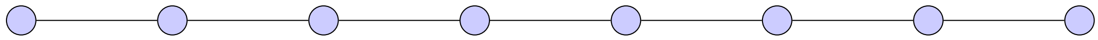
$$\sum_{i=1}^{n}(deg(v_i)) = 2(n-1)$$

(b) Impossible.

In any tree, there are always $n-1$ edges.

(c) Many possible such trees.

Here is one:



(d) Impossible.

See reasoning in (a).

19. (a) There is no such spanning tree.

Vertices f and g are the only vertices of G with degree 6. The appropriate T would require all vertices adjacent to f and g. f and g share the neighbour, m, hence the graph induced by vertices f, g and their neighbours will create a triangle, meaning T was not a tree.

(b) The subgraph induced by vertices b, e, f, g is one such induced 4-cycle.

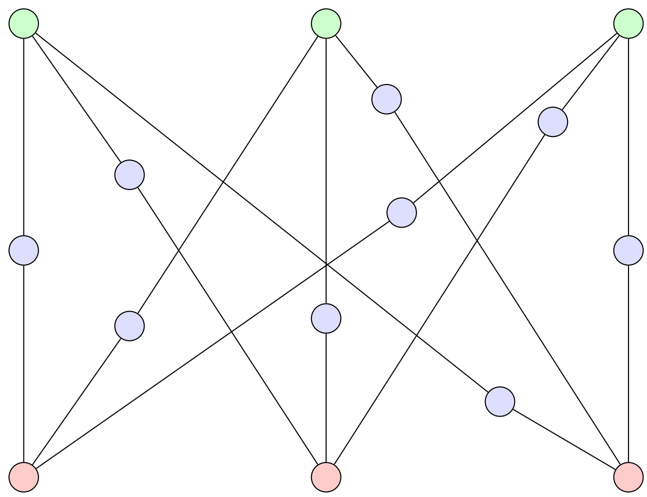
2.7 Planar Graphs

Solutions:

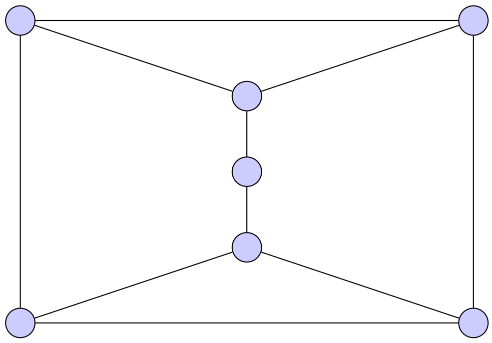
1. A graph is 'planar' if it can be drawn in the plane so that edges only intersect at vertices. Any graph that is not planar is called 'nonplanar'.

2. Two graphs are homeomorphic if one is a 'subdivision' of the other. A 'subdivision' of a graph is a graph that results from a, possibly empty, sequence of subdivisions of edges.

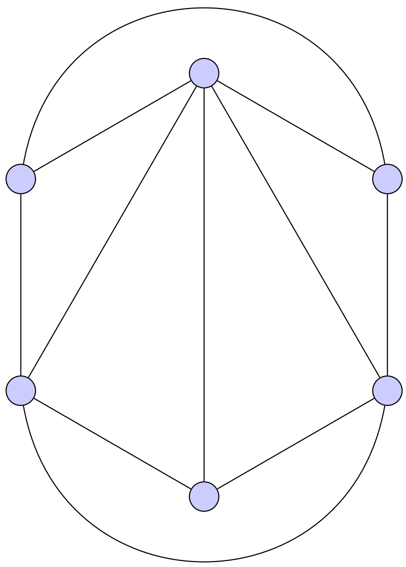
3. (a) This graph is nonplanar as it is homeomorphic to $K_{3,3}$, as shown below.



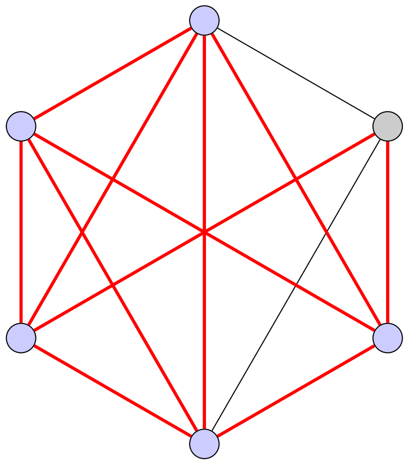
(b) This graph is planar, as shown in the redrawing below.



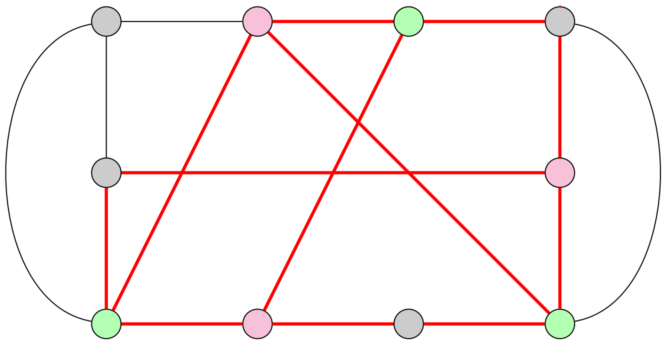
(c) This graph is planar, as shown in the redrawing below.



(d) This graph is nonplanar as it is homeomorphic to K_5, as shown below.



(e) This graph is nonplanar as it is homeomorphic to $K_{3,3}$, as shown below.



4. We proceed by induction on the order of the tree.

   Base case: A tree with one vertex will have no edges and so is trivially planar.

   Induction hypothesis: Suppose that any tree with order k is planar, for some $k \geq 1$.

   Induction step: Let T be a tree with $k+1$ vertices. As T is a tree, it must contain at least two leaves. Let us call one of these leaves a.

   Let $T' = T - a$. Then T' is a connected graph of order k. By the Induction Hypothesis, T' is planar.

Since T' is a acyclic, we can add a to T' without create any edge crossings. Thus T is planar.

The result follows by the Principle of Mathematical Induction.

5. As G is a connected, planar graph it follows from Euler's Planar Graph Theorem that the number of edges in G is $E = V + R - 2$.

   Further, as G is planar there is an upper bound on the number of edges in G: $E \leq 3V - 6$.

   If follows then that:

$$\begin{aligned} V + R - 2 &\leq 3V - 6 \\ R &\leq 2V - 4 \end{aligned}$$

   So, the upper limit on the number of regions in G is $2V - 4$, as desired.

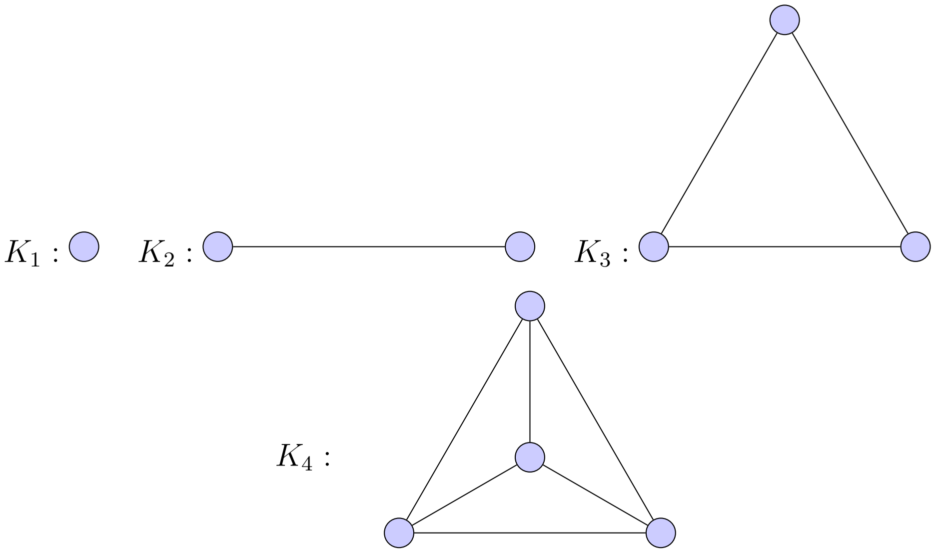
6. First, notice that G cannot be a tree as if G were a tree then $E = V - 1 = 3V - 6$. This would imply that $2V = 5$, a contradiction that the number of vertices in a graph is an integer. This means each region is bounded by at least 3 edges.

   Since G is planar, $V - E + R = 2$. Thus, $3V - 6 = 3E - 3R$ and so $2E = 3R$.

   If any region were bounded by more than 3 edges, then $2E > 3R$. As this is not true, it follows that every region is bounded by three edges. That is, each region is a triangle.

7. It follows from Kuratwoski's Theorem that if $n \geq 5$ then the graph is nonplanar, as for all K_n, with $n \geq 5$ K_5, K_5 will be a subgraph.

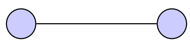
For K_n where $n = 1, 2, 3, 4$ one can verify by hand that each graph is indeed planar, as shown below.



8. Without loss of generality, if $m \leq 2, n \leq 3$ then $K_{m,n}$ is planar.

   If both m and n are greater than or equal to 3, then $K_{3,3}$ will be a subgraph of $K_{m,n}$ and so by Kuratowski's theorem the graph will be nonplanar.

9. First we can draw P_2:

   We can obtain any other P_n from P_2 by subdividing the single edge of P_2 $n-2$ times. This will form P_n and implying P_n and. P_2 to homeomorphic, as desire.

10. As the minimum degree of any vertex in G is 5, it follows from Euler's Theorem that: $5V \leq 2E$.

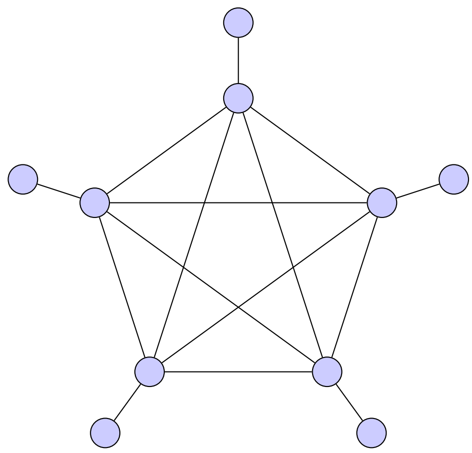
As G is planar, it follows from Euler's Planar Graph Theorem that $E \leq 3V - 6$ as G must have at least 3 vertices.

Thus, by transitivity of order, it follows that $\frac{5}{2}V \leq 3V - 6$. Rearranging this inequality shows that $12 \leq V$, as desired.

11. If G is a planar graph with $V = 22$, then we can apply Euler's Planar Graph Theorem. It follows that, $E \leq 3V - 6 = 3(22) - 6 = 60$, as desired.

12. (a) False.

    If G is planar *then* $E \leq 3V - 6$. The converse of this statement is false however. The graph below has 10 vertices and 15 edges, so $E \leq 3(V) - 6 = 3(10) - 6 = 24$. However, it contains a subgraph that is isomorphic to K_5 and so the graph is nonplanar.

    (b) True.

    If a graph contains a nonplanar subgraph, then the graph must be nonplanar.

    (c) True.

For every planar graph of order 3 or more, $E \leq 3V - 6$. Suppose for a contradiction that there exists a planar graph of order at least 4 where every vertex is of degree 6 or more.

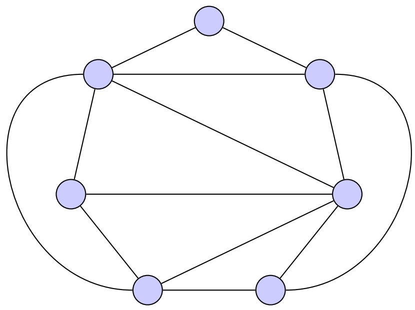
Then, by Euler's Theorem, $6V \leq 2E$. It then follows that $6V \leq 6V - 12$, a contradiction.

(d) True.

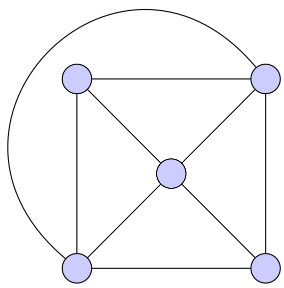
The total number of edges in a complete graph is $\binom{11}{2} = 55$. Hence, if G has x edges then $\overline{G}$ has $55 - x$ edges. If a graph is planar then $E \leq 3V - 6$, so it follows that if a graph has more than $3V - 6$ edges then it is nonplanar.

Notice that $2(11) - 6 = 27$. Thus, if G has 28 or more edges we are done. Let us suppose instead that G has no more than 27 edges. Then $\overline{G}$ at least $55 - 27 = 28$ edges, and so is nonplanar.

13. (a) Such a graph does exist and has exactly 8 regions.



(b) Such a graph does exist and, using Euler's Planar Graph Theorem, will have exactly 9 edges.



(c) There is no such planar graph. We know that any planar graph with more than 3 vertices will have an upper bound on size: $E \leq 3V - 6$. This graph however has $20 > 3(8) - 6 = 18$ and so cannot be planar.

(d) There is no such planar graph. Suppose that there is such a plane graph, then by Euler's Planar Graph Theorem $V - E + R = 2$. It follows that $V = 2 - R + E = 2 - 10 + 5 < 0$, which is impossible.

14. A graph that has 5 regions all bounded by 4 edges then we can sum the total number of edges in the graph to see that $2E = 4R$.

    Using Euler's Planar Graph Theorem, $V - E + R = 2$, it follows that $V = 2 + \frac{E}{2}$.

    Since the graph is planar, we can use the upper bound on the number of edges:

$$\begin{aligned} E &\leq 3V - 6 \\ &\leq 3(2 + \frac{E}{2}) - 6 \\ &\leq \frac{3E}{2} \\ &\leq \frac{3E}{2} \end{aligned}$$

    This is a contradiction, and so no such planar graph exists.

15. Suppose C is a circuit in a planar graph enclosing exactly two regions that each have an even number of boundary edges. As G is planar, these two regions must share exactly one edge. Therefore, C includes every edge of each region, except the one that is shared. So, C includes an odd number of edges from the both regions. The sum of two odd numbers is even hence the length of C is even.

2.8 Colouring Graphs

Solutions:

1. Colouring a graph refers to assigning every vertex of a graph colours such that no adjacent vertices have the same colour.

2. If a graph can be coloured with exactly one colour then it follows that there are no adjacent vertices in the graph. Hence, the graph is edge-less/empty.

3. (a) $\chi(K_n) = n$.

   Every vertex in a complete graph is adjacent to every other vertex, so each vertex must be assigned a unique colour.

   (b) $\chi(K_{m,n}) = 2$.

   Vertices are only adjacent to vertices in the opposite partite set. We can colour every vertex in one partite set one colour, and a different colour for every vertex in the other partite set.

   (c) Chromatic number: 2.

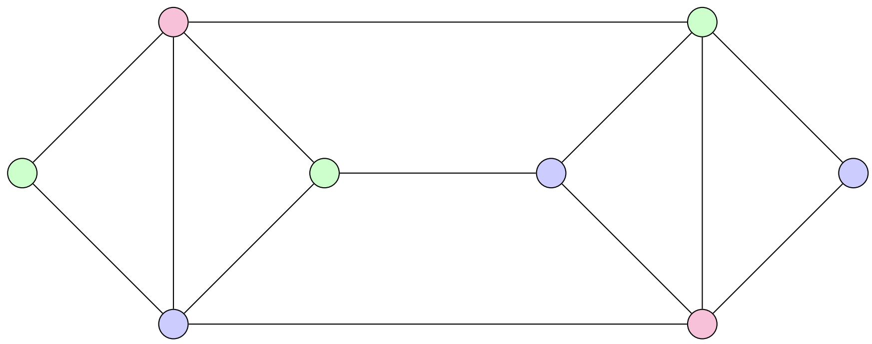
   See explanation for (b), the fact the graph was *complete* bipartite was irrelevant.

   (d) Chromatic number: 2.

   Every tree is bipartite. See (b).

(e) $\chi(G) = 3$.

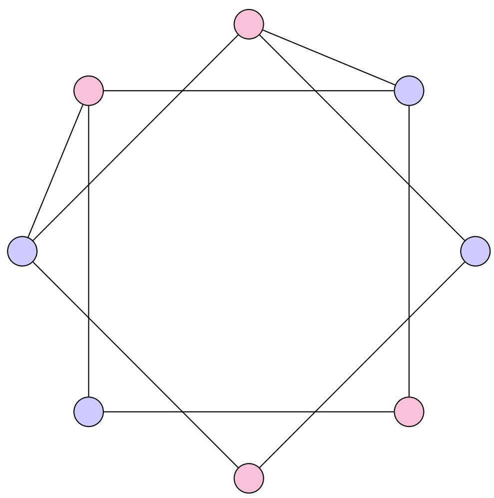
This graph has a triangle, three mutually adjacent vertices, hence $\chi(G) \geq 3$. Consider the following assignment of colours:



We have identified a 3-colouring, so we can conclude that $\chi(G) = 3$.

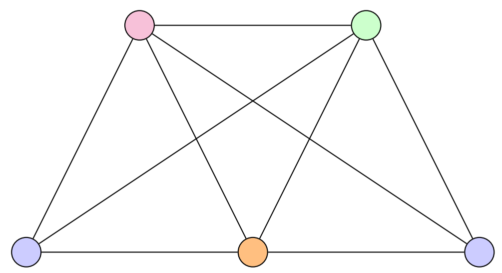
(f) $\chi(G) = 2$.

The given graph is bipartite.



(g) $\chi(G) = 4$.

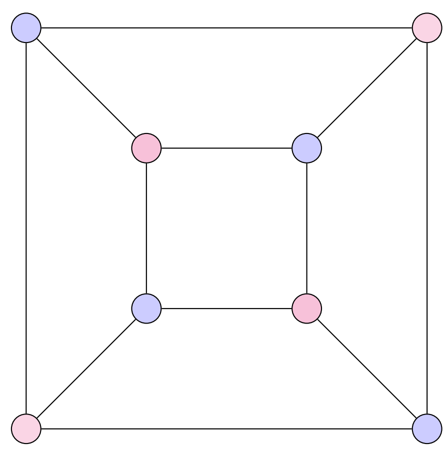
The graph has a K_4 subgraph implying $\chi(G) \geq 4$. Consider the following colouring:



We have found a proper 4-colouring of G hence $\chi(G) = 4$.

(h) $\chi(G) = 2$.

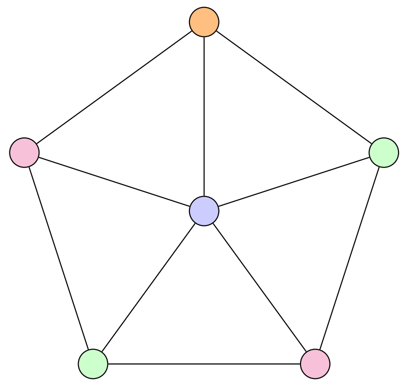
The graph is bipartite.



(i) $\chi(G) = 4$.

The graph has a 5-cycle and triangles hence $\chi(G) \geq 3$. We know colouring the outer cycle will require three colours since 5 is odd. However, the central vertex is adjacent to every vertex in the cycle so, $\chi(G) > 3$.

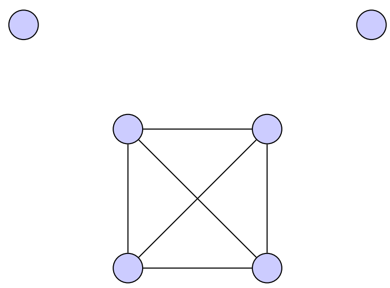
Consider the following colouring:



We have identified a proper 4-colouring, hence $\chi(G) = 4$.

4. False.

Consider the following disconnected graph:



It has a K_4 subgraph and so requires at least four colours, but it has 6 vertices and 6 edges.

5. (a) False.

   Every odd cycle with length $n \geq 3$ has chromatic number 3 and no triangles.

   (b) False.

   K_4 is planar and contains a triangle, but $\chi(K_4) = 4$.

   (c) True.

   An isomorphism preserves adjacencies between vertices and hence will also preserve a proper colouring.

   (d) False.

   C_4 and $K_3 = C_3$ are homeomorphic but $\chi(C_4) = 2$ since it is an even cycle, while $\chi(K_3) = 3$ since it is an odd cycle.

   (e) False.

   Consider $K_{3,3}$ which is Hamiltonian (proved in 2.5 question 9) and bipartite, hence $\chi(K_{3,3}) = 2$, but $K_{3,3}$ is certainly not planar by Kuratowski.

   (f) True.

   A graph is bipartite if and only if it contains no odd cycles. We know that if a graph contains an odd cycle, its chromatic number is at least 3. Hence if the chromatic number is two the graph contains no odd cycles and is bipartite.

(g) False.

Consider $K_{3,3}$. This graph has chromatic number $2 \geq 4$, but is not planar.

(h) False.

Consider any odd cycle, C_n, with $n \geq 5$, such as C_5. These graphs have chromatic number 3, but do not contain K_3 subgraphs.

(i) False.

All graphs with $\chi(G) \leq 3$ and order at least four can be coloured using four colours, this colouring just may not be a minimum colouring.

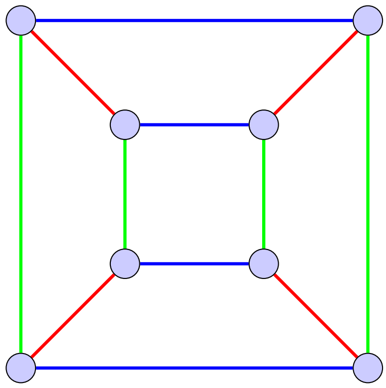
(j) True.

The chromatic number of a graph is at least as large as the chromatic number of all its subgraphs.

(k) False.

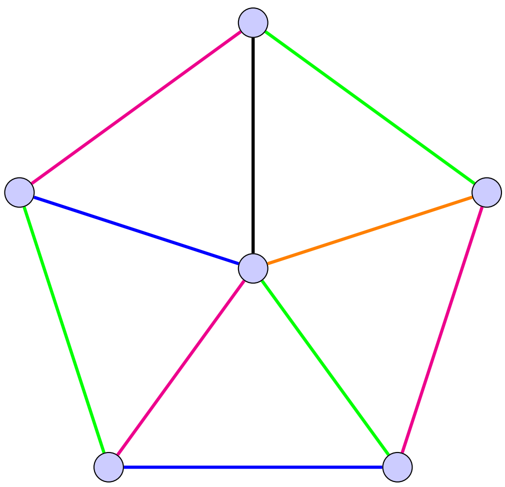
K_5 does not have a 3-colouring, but it also does not have a 4-colouring, hence $\chi(G) \neq 4$.

6. (a) Here is one such minimum edge colouring:



We see a minimum edge colouring requires three colours while a minimum vertex colouring required two.

(b) Consider one such minimum edge colouring:



We see a minimum edge colouring requires five colours while a minimum vertex colouring required four.

7. (a) Every animal represents a vertex. Two vertices are adjacent (i.e. there is an edge) if these animals *cannot* live together peacefully. The vertices assigned the same colour represent the animals that can live in the same enclosure. The zoo is attempting to find the chromatic number of such a graph.

   (b) Let each course be a vertex, with two vertices adjacent if a student indicates that they would like to enroll in both courses. Vertices assigned the same colour represent courses that can run at the same time. Any colouring of this graph will give the department such a schedule, however the most efficient schedule would be indicated by the chromatic number of the graph.

8. We know $\chi(G) > 1$ since there is an edge. We also know a graph is bipartite if and only if $\chi(G) = 2$, so it will suffice to prove that G is not bipartite.

   Suppose for contradiction that G is a bipartite graph with partite sets V and U. Let $|V| = x$ and $|W| = 8 - x$, with $1 \leq x \leq 7$.
   If $x = 1$ then $K_{1,7}$ has 7 edges.
   If $x = 2$ then $K_{2,6}$ has 12 edges.
   If $x = 3$ then $K_{3,5}$ has 15 edges.
   If $x = 4$ then $K_{4,4}$ has 16 edges.

   The graph we are considering has 17 edges and certainly is not bipartite since it has more edges than any possible complete bipartite graph on 8 vertices. Hence $\chi(G) > 2$, as desired.

9. We will proceed by induction on the order of the graph.

Base Case: $n = 1$. Certainly $G = \overline{G}$ with $\chi(G) = 1$, therefore $\chi(G) + \chi(\overline{G}) = 2 \leq 1 + 1$.

Induction Hypothesis: Suppose that $\chi(G) + \chi(\overline{G}) \leq n + 1$ for $n = 1, ..., k$.

Induction Step: Consider a graph, G, with $k + 1$ vertices. Consider some vertex, x, of G. By the induction hypothesis, $\chi(G - x) + \chi(\overline{G - x}) \leq k + 1$. Between G and $\overline{G}$, x is adjacent to at most k neighbours. If x can be assigned a colour in G then in $\overline{G}$ we will need at most one additional colour. Therefore $\chi(G) + \chi(\overline{G}) \leq k + 1 + 1 = k + 2$, as desired.

The result follows by the Principle of Mathematical Induction.

10. The statement is true.

    Consider a planar embedding of such a graph. Identify one region, since it is bounded by an even cycle, we can colour this cycle with two colours. We can do the same for every region, and if we have already coloured one of the vertices of the boundary cycle then colour the adjacent vertices the opposite colour.

11. We must consider three distinct cases.

    Case 1: Suppose G has no odd cycles. Then G is bipartite with $\chi(G) = 2$, so certainly we can colour G with three colours.

    Case 2: Suppose that G contains exactly one odd cycle, C. Then certainly $\chi(G) > 2$ as this graph is not bipartite. Consider removing an arbitrary vertex $u \in C$ from G, this would create a graph with no odd cycles. So, $G - u$ 2-colourable. Adding u back to G would only require one additional colour, so

G is 3-colourable.

Case 3: Suppose G contains exactly two odd cycles, C_1 and C_2, we now consider two subcases:

> **Case 3a:** Suppose that C_1 and C_2 share a common vertex, u. Consider $G-u$, which now has no odd cycles since removing a vertex from a cycle breaks the cycle. Thus, $G-u$ is bipartite and 2-colourable. Adding back u will require at most one additional colour, so G is 3-colourable.
>
> **Case 3b:** Suppose C_1 and C_2 share no common vertices. If every vertex in C_1 is adjacent to every vertex in C_2 then there would be another odd cycle in G which is impossible by assumption. Thus, there exists two vertices, $u \in C_1$ and $v \in C_2$, such that $uv \notin E(G)$. Consider obtaining the graph $G-u-v$, this will break both cycles in G, making $G-u-v$ a graph free of odd cycles, and hence 2-colourable. As u and v are not adjacent in G readding them will require at most one additional colour. Thus, G is 3-colourable.

We have now shown that all graphs with no more than two odd cycles are 3-colourable.

12. There are no adjacent vertices within a set of vertices of the same colour.

3 Counting: Fundamental Topics

3.1 Basic Counting Principles

3.2 The Rules of Sum and Product

Solutions:

1. (a) The sum rule for multiple events is for events that are *independent* (i.e. events/situations that cannot occur at the same time). If one event can occur m ways, and the other event can occur in n ways, where both events are independent, then the two events together can occur in $m + n$ ways.

   (b) The product rule for multiple events is for events that are happening in sequence of one another, thus are *not* independent. For example, if one event can occur in m ways and then another event follows and can occur in n ways, then the sequence of these two events can occur in mn ways.

2. The product and sum rule are used together when a set of sequences of events are occurring independently.

3. (a) *Rule of product*

   There are exactly 10 digits and 26 letters in the alphabet, therefore each character has $10 + 26 = 36$ possibilities. The licence plate has 6 characters so there are $36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6$ licence plates.

   (b) *Rule of product*

   For the first two characters there are 10 digit choices, since the digits may be repeated. For the first letter there are 26 choices, but only 25 for the second since no possible repetition. Similarly, there are 24 options for the third letter and 23 for the fourth letter. Putting all of these options together there are a total of $10 \cdot 10 \cdot 26 \cdot 25 \cdot 24 \cdot 23 = 35\,880\,000$ licence plates.

(c) *Rule of product and rule of sum*

First, consider the case when the plate begins with a letter. There are 26 options for the first letter, 10 options for the first digit, 26 for the second letter, 9 for the second digit, 26 for the last letter and 8 for the last digit (recall that only letters can repeat). Then, multiply the number of options together to see there is a total of $26 \cdot 10 \cdot 26 \cdot 9 \cdot 26 \cdot 8 = 12\ 654\ 720$ licence plates with this specification.

Now we consider the case when the plate begins with a digit and then alternates. The counting is the same as multiplication is commutative.

Since these cases are disjoint there are $12\ 654\ 720 + 12\ 654\ 720 = 25\ 309\ 440$ possible licence plates with this restriction.

(d) *Rule of product and rule of sum.*

To count the number of possible licence plates with at most one of the characters a digit, we consider two cases: plates with 0 digits and 1 digit, then sum these together.

The number of possible licence plates with no digits is $26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$.

The number of possible licence plates with exactly one digit is slightly more complex. There are six possible places that the single digit can appear. We can account for this by applying the six possible places the digit can go to the choice of the digit. So for the digit there will be $6 \cdot 10$ possibilities. The remaining five spots will all have a letter and so, in total, there are $6 \cdot 10 \cdot 26^5$ possible licence plates with this restriction.

Thus, the total number of licence plates with at most one digit is $26^6 + 6 \cdot 10 \cdot 26^5 = 1\ 021\ 798\ 336$.

(e) *Rule of product and sum.*

Based on the restrictions, we know that there are 4 cases.

The plate can begin with "T" and end with "J". There are no further restrictions therefore for each of the four middle characters there are 36 options so there are 36^4 possible plates with this restriction.

Similarly, the plate can begin with "T" and end in "Q" and so there are 36^4 possible plates.

The number of possible plates in the other two remaining cases is the same: the plate begins with 0 and ends with "J" or the plate begins with 0 and ends with "Q".

Therefore there are $36^4 + 36^4 + 36^4 + 36^4 = 6\ 718\ 464$ possible licence places with these restrictions.

4. *Rule of sum.*
   Since the student would like either a book on frogs or on fireflies there are $45 + 13 = 58$ books to choose from.

5. *Rule of sum and product.*
   Consider first the number of options if Joselyn orders a drink and a cookie. Then, consider the number of options if she orders a sandwich, drink, and chips.

   There are 3 types of cookies and 7 drinks to choose from, so there are $3 \cdot 7 = 21$ possible combinations of a cookie and a drink.

   There are 10 different sandwiches, 7 drink options, and 3 types of cookies. Therefore there are $10 \cdot 7 \cdot 3 = 210$ combinations for this meal.

   In total Joselyn can order $21 + 210 = 231$ possible combinations at the shop.

6. *Rule of product.*
   As each die is a different colour, they are distinct. Let us assume the dice are blue and red. This means rolling a 1 on the blue die and a 2 on the red die is distinct from rolling a 2 on the blue die and a 1 on the red die.

   On each die there are 6 options therefore there are $6 \times 6 = 36$ possible outcomes.

7. *Rule of product.*
   Consider an ordered pair, (X, Y) where $0 \leq X \leq 3$ and $0 \leq Y \leq 5$. Let the ordered pair represent the number of X's and Y's in the given non-empty set.

   There are 4 possibilities for the value of X's and 6 possibilities for the value of Y's in the pair. In total there are $6 \times 4 - 1 = 23$ possible pairs. We must subtract 1 because one of our pairs represents the empty set, $(0, 0)$. Thus, there are 22 possible non-empty sets.

8. (a) *Rule of product.*
   All such integers will end in either 0, 2, 4, 6, or 8, thus there are 5 options for the last digit. For the first digit, any number other than 0 is possible, thus there are 9 options. For the other four remaining digit any digit is possible so there are 10 options. In total there are $5 \cdot 9 \cdot 10^4$ integers x.

   (b) *Rule of product.*
   First, notice that the 0 can occur in any place in x but the first digit. We can build x by first placing the 0. There are 4 different places the 0 can go. The remaining four places can be assigned any non-zero digit. This means each place has 9 options. Thus, in total there are $5 \cdot 9^4$ integers x.

   (c) *Rule of product.*
   We can count the number of five digit numbers with at least one repeated digit by counting the total number of five digit numbers and subtracting

the number of five digit numbers with no repeated digits.

The number of five digit numbers is $9 \cdot 10^4 = 90\ 000$. The number of five digit numbers with no repeated digits is $9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 27\ 216$. Thus, there are $90\ 000 - 27\ 216 = 62\ 784$ five digit numbers that have at least one repeated digit.

9. *Rule of product.*
The first digit in the sequence can be any of 0,1, or 2. Thus there are 3 possibilities. The second digit cannot be the same as the first digit. This means there are 2 possibilities for it. Similarly, for each following digit they cannot be the digit that proceeds them, meaning there are 2 possibilities. Therefore there are $3 \cdot 2^9 = 1\ 536$ ternary strings with the required restriction.

10. *Rule of product.*
An integer is divisible by 5 if and only if the last digit place is either a 5 or 0. This means there are 2 possibilities for the final digit of x.

The first digit of x can be any digit except 0. Thus there are 9 possibilities. The second digit of x is any digit, thus there are 10 possibilities. Using the product rule, there are $5 \cdot 9 \cdot 10$ possible values of x.

11. *Rule of product.*
In a palindrome of length 7 the first and last letter, the second and sixth letter, and the third and fifth letter are the same. The fourth letter can be any letter. So, we can assign any letters to the first four letters but the last three letters are determined by this as well. Therefore there are: $26 \cdot 26 \cdot 26 \cdot 26 \cdot 1 \cdot 1 \cdot 1 = 26^4 = 456\ 976$ palindromes of length 7.

12. *Rule of product.*

(a) $n^{m-1} \cdot 1$

This is because all elements in A have n possible possibilities, except $f(a_2)$ which has only one possibility – whatever was assigned to $f(a_1)$.

(b) $1 \cdot n^{m-2} \cdot n - 1$

This is because all elements in A except $f(a_1), f(a_2)$ are unrestricted. $f(a_1)$ must be b_1 and so has only one possible value, while $f(a_2)$ can be any element in B except b_1 and so has $n-1$ possible values.

(c) $3 \cdot n^{m-1}$

This is because all elements in A, except $f(a_1)$, can be assigned to any element in B and so have n options while $f(a_1)$ can only be one of three possibilities.

(d) $n \cdot n - 1^{m-1}$

This is because whatever element is assigned to $f(a_1)$ cannot be assigned to any other element. This means all elements but $f(a_1)$ have only $n-1$ possible values in B.

(e) $n^{m-1} \cdot n - 1$

This is similar to part b except $f(a_1)$ is not restricted.

13. *Rule of product.*
For each of the five elements in the first set, there are 3 possible values in the second set. Therefore, there are 3^5 possible functions.

14. *Rule of product.*
Each of the 25 question can be answered in one of two ways: true or false. Thus, there are 2^{25} possible ways to answer the test.

15. *Rule of product and sum.*
For the first part of the question we can use the product rule, giving that are

$1500 \cdot 50$ possible pairs. For the second part, we can use the sum rule to see that there are $50 + 1500$ possible people to give a speech.

16. *Rule of product.*
Each tuple will be of the form (a, b, c) where $a \in A$, $b \in B$ and $c \in C$. The number of possible options in the first position is 3, the number of possible options in the second position is 4, and the number of possible options in the last position is 5. Using the rule of product it follows that there are $3 \cdot 4 \cdot 5 = 60$ distinct elements in $A \cdot B \cdot C$.

17. *Rule of product.*
Jamie should purchase the most secure lock, which will be the lock with the highest number of possible combinations.

Lock 1's combination is an ordered sequence of 3 numbers, with 35 options for numbers. The first number can be any of the 35 possible numbers. The second number can also be any of the 35 possible numbers. The last number can be any number between 1 and 35 except for whatever the first number was, so there are 34 possibilities. Hence, this combination lock has $35 \cdot 35 \cdot 34 = 41\ 650$ possible combinations.

Lock 2's combination is an ordered sequence of 4 numbers, with 25 options for number. The first number can be any of the 25 possible number. The second number can be any of the 25 possible number, except for whichever was assigned to the first number, leaving 24 options. Similarly, the third number can be any number except whatever was assigned to the first two numbers, leaving 23 options. The fourth number must be one of the first three numbers, meaning it has 3 options. This means this lock has $25 \cdot 24 \cdot 23 \cdot 3 = 41\ 400$ possible combinations.

Therefore, lock 1 has a higher number of possible combinations and so is the lock Jamie should buy.

18. *Rule of sum and product.*

The different cases that we must consider are words of length 1, length 2, and length 3. There are exactly 26 words of length 1, 26^2 words of length 2, and 26^3 words of length 3 as there are no restrictions on these words. Therefore there are $26 + 26^2 + 26^3 = 18\,278$ words of length 1, 2 and 3.

3.3 Permutations

Solutions:

1. A permutation is a linearly ordered arrangement of distinct objects.

2. If objects could be repeated, two identical arrangements would be counted as different.

3. (a) 120.

   We are arranging, in a particular order, five distinct letters, therefore there are $P(5,5) = 120$ different permutations of MATHS.

   (b) 60.

   We are arranging, in a particular order, three out of five distinct letters therefore there are $P(5,3) = 60$ different 3 letter permutations of MATHS.

   (c) 5^{10}.

   There are ten letter positions available, with 5 options for each letter, so there are 5^{10} such words.

4. 20.

   There are six letters and two pairs of two repeating letters (double E's and double O's), hence there are $\frac{6!}{2!2!} = 20$ possible permutations.

5. (a) 5 040.

   We avoid having potentially the same arrangement in different spots around the table by fixing one person and then arranging those around them, which gives $7! = 5\ 040$ choices.

   (b) 1 440.

   If two people insist on sitting together, we begin by fixing their seats together then arrange the remaining six friends, $6! = 720$. For the two individuals sitting together, we are unsure who will be on the left versus right, so we multiply by two to account for the two possibilities of their positions. All together there are $6! \cdot 2 = 1\ 440$ ways to seat everyone.

6. (a) 6 375 600.

   We are interested in an ordered arrangement of five individuals from twenty-five swimmers, which gives $P(25, 5)$ possible top five medal distributions.

   (b) 2 772

   There are $P(3, 3) = 3!$ ways to place Nia, Andre, and Katie in the top three positions. The remaining 2 positions can be filled in $P(22, 2)$ ways. Thus, using the rule of product there are $P(3, 3) \cdot P(22, 2) = 2772$ ways for the medals to be given out.

7. 30 240.

We begin by sitting the six individuals who are not Andrew or Asiya, there are $P(6, 6)$ ways to do this. To guarantee Andrew and Asiya are not beside each other, they will either sit on an end or between two other friends. There are five seats between friends and two on the ends, which gives seven possible seats for Andrew and Asiya thus $P(7,2)$ choices. By applying the Rule of Product it follows that there are $P(7,2) \cdot P(6,6) = 30\ 240$ ways to seat the group.

8. 1 440.

   We only consider numbers less than 6 000 000, thus the only possible first digits are ≤ 5. There are four options for the first digit, and the remaining digits can go in any order, $6! = 720$ options. We must divide by 2 to account for the two identical 4's in the set. Hence there are $\frac{4 \cdot 6!}{2!} = 2 \cdot 6! = 1\ 440$ possible numbers possible.

9. 27 907 200.

   Leora is giving out $2 \cdot 3 = 6$ of her twenty books, in a specific order, which is $P(20,6) = 27\ 907\ 200$ possible distributions.

10. 100.

    José has ten possibilities for one of the lost digits and another ten for the second lost digit. By the Rule of Product there are $10^2 = 100$ possible phone numbers he will have to call before calling his friend.

11. (a) 3 315 312 000.

    We are arranging, in a particular order, seven letters from 26 possible letters, which is simply $P(26,7) = 3\ 315\ 312\ 000$.

(b) 892 584 000.

We begin by placing the T in one of the 7 spots and arranging the other six letters around it. There are $P(25,6)$ ways to assign the six letters that aren't T, and seven positions the T can be placed. By the Rule of Product there are $7 \cdot P(25,6) = 892\ 584\ 000$ such words.

(c) 127 512 000.

There is only one option in the assignment of the first letter, A, only in the ordered arrangement of the remaining six letters selected from the remaining 25 from the alphabet, which makes $P(25,6) = 127\ 512\ 000$ words.

(d) 1 356 727 680.

We first pick whether the word has X or Y, two choices, and then place it in one of the seven spots, seven options. Now we arrange the remaining six letters which are picked from the 24 possible letters (since *exactly* one of X and Y is used), $P(24,6)$ possibilities. By the Rule of Product there are $7 \cdot 2 \cdot P(24,6) = 1\ 356\ 727\ 680$ such words.

12. $m!(n+1)!$.

Let us consider the m brunette's to be one distinct object. We now must arrange $n+1$ individuals into a line which yields $P(n+1, n+1) = (n+1)!$ possible arrangements. Within the single 'object' of brunettes, there are $P(m,m) = m!$ ways to arrange the brunette individuals. By the Rule of Product there are are $m!(n+1)!$ such arrangements.

13. Applying the definition,

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

14. The permutation function is one specific application of the product rule if you are trying to determine the number of possible arrangements of k out of n distinct objects ($k \leq n$). If this is not the specific case, count using the Rule of Product.

15. Solve your own problem to verify this.

16. We will use the definition of the permutation function to show this result:

$$P(n+1,2) = \frac{(n+1)!}{(n+1-2)!} = \frac{(n+1)(n)(n-1)!}{(n-1)!} = n(n+1),$$

$$P(n,2) = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$$

Subtracting,

$$P(n+1,2) - P(n,2) = n(n+1) - n(n-1) = n(n+1-(n-1)) = 2n$$

We see that,

$$P(n,1) = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n,$$

therefore $2P(n,1) = 2n$, as desired.

17. 5 184.

To start, we consider the posters for the same K-pop group to be a distinct object, which reduces the problem to arranging three distinct objects in a line, $P(3,3) = 3!$ possibilites. Now within each group of posters there are 3! ways to arrange the first group's posters within the line, 4! the second, an 3! the third. By the Rule of Product there are $(3!)^3 \cdot 4! = 5\ 184$ ways to line up these posters.

18. 34 650.

For any arrangement with repeated objects, we know that we must divide by the factorial of the number of repeated objects. In the word *MISSISSIPPI* there are four I's, four S's and two P's, therefore in total there are $\frac{11!}{4! \cdot 4! \cdot 2!} = 34\ 650$ distinct permutations of this word.

19. $|A| = m \leq n = |B|$, otherwise no one-to-one functions can exist between the sets.

    (a) 0.

    Assuming $a_1 \neq a_2$, by definition of a one-to-one function this is impossible.

    (b) $P(n-1, m-1)$.

    We have only one possible assignment of $f(a_1)$. For $f(a_i)$, $i = 1, ..., m$, each must equal a distinct element from $\{b_2, ..., b_n\}$, hence we are picking $m-1$ objects from a set of $n-1$ distinct objects, $P(n-1, m-1)$ such functions.

    (c) $\frac{3(n-1)!}{(n-m)!}$.

    There are three choices for $f(a_1)$, and the remaining $m-1$ a_i's must be uniquely assigned to the $n-1$ remaining b_j's, $P(n-1, m-1)$ options. The

Rule of Product yields, $3 \cdot P(n-1, m-1) = \frac{3(n-1)!}{(n-1-(m-1))!}$ such one-to-one functions.

(d) $P(n, m)$.

The requirement is satisfied by the definition of a one-to-one function, hence we are selecting m objects from a set of n distinct objects, $P(n, m)$.

(e) $P(n, m)$.

Same reasoning as (d).

20. (a) 151 200.

We are permuting 10 letters, $P(10, 10) = 10!$, but O and K repeat twice and E thrice, so we must divide by $2!, 2, !, 3!$. All together there are $\frac{10!}{2!2!3!} =$ 151 200 distinct arrangements.

(b) 70 560.

We begin by placing the letters that are *not* E, there are seven of them so $\frac{7!}{2!2!}$, still accounting for the repeated letters, possible arrangements. We now place the three E's between (or on the outside) of the placed letters to guarantee no E's are consecutive. There are eight possible gaps to place the E's so there are $\frac{P(8,3)}{3!}$ ways to put the E's in accounting for the fact that they are identical. By the Rule of Product there are $\frac{7!}{2!2!} \cdot \frac{P(8,3)}{3!} = 70\ 560$ such arrangements.

(c) 10 080.

We can consider the E's to be one letter since their arrangement is irrelevant as they are identical. Now we arrange the eight objects, dividing by $2! \cdot 2!$ since there are two O's and K's, hence $\frac{8!}{2!2!} = 10\ 080$ possibilities.

(d) 3 600.

We will group all the vowels together and consider them to be one distinct object. Within this object of vowels there are $\frac{5!}{2!\cdot 3!}$ possible arrangements since some vowels are repeated. We now arrange this block within the five consonants, so there are $P(6, 6) = 6!$ arrangements, but we must divide by 2! to account for the identical K's. By the Rule of Product there are $\frac{5!}{3!2!} \cdot \frac{6!}{2!} = 3\ 600$ such arrangements.

3.4 Combinations and the Binomial Theorem

Solutions:

1. The main difference between a permutation and a combination is that for a permutation the order in which the elements are selected matters while for a combination the order does not matter. Both a combination and a permutation count the ways in which an event can occur.

   The formula for permutation, $P(n,r)$, is $\frac{n!}{(n-r)!}$ while the formula for combination, $C(n,r)$ is $\frac{n!}{(n-r)!r!}$. Clearly, $C(n,r) = \frac{P(n,r)}{r!}$ which highlights that order is irrelevant in a combination.

2. There are $\binom{69}{5}\binom{26}{1} = 292\ 201\ 338$ possible tickets. Thus, the cost of buying all the tickets is higher than the prize money and so it is not worth it to buy all the tickets to ensure a win.

3. (a) $\binom{80}{21}$

   The coaches must form a team of 21 from 80 players. The order in which the team is selected is irrelevant. So, there are exactly $\binom{80}{21}$ possible teams.

   (b) $\binom{40}{10}\binom{40}{11}$

   First we choose the 10 first year students that make the team, then we choose the remaining 11 spots from the remaining 40 students. Using the product rule, multiply these together to see that there are $\binom{40}{10}\binom{40}{11}$ possible teams.

   (c) $\binom{78}{19}$

   The two highest scoring players are already guaranteed a position on the team, so the remaining 18 players from the 78 other students who tried out need to be selected. So, there are $\binom{78}{19}$ poossible teams.

   (d) $\binom{65}{10}\binom{15}{11}$

   First, select the 11 returning students for the team and then 10 new students. There are $\binom{65}{10}\binom{15}{11}$ possible teams with this structure.

(e) $\binom{20}{4}\binom{15}{5}\binom{10}{4}\binom{20}{4}\binom{15}{4}$

First select the player who play centre position, then shooting guard, point guard, small forward and finally power forward. Using the product rule, it follows that there are $\binom{20}{4}\binom{15}{5}\binom{10}{4}\binom{20}{4}\binom{15}{4}$ possible teams with this structure.

(f) $\binom{75}{16}\binom{5}{5} + \binom{75}{17}\binom{5}{4} + \binom{75}{18}\binom{5}{3}$

The cases we must consider are whether there are $3, 4$ or, 5 graduating students on the team. First, we select the number of graduating students and then, using the product rule, we select the appropriate number of remaining players.

Using the rule of sum, in total there are $\binom{75}{16}\binom{5}{5} + \binom{75}{17}\binom{5}{4} + \binom{75}{18}\binom{5}{3}$ possible teams with this structure.

4. $n = 91$

$$\begin{aligned}
\binom{n}{4} &= 2\ 672\ 670 \\
\frac{n!}{4!(n-4)!} &= 2\ 672\ 670 \\
2\ 672\ 670 &= \frac{n(n-1)(n-2)(n-3)}{4!} \\
2\ 672\ 670(4!) &= n(n-1)(n-2)(n-3) \\
2\ 672\ 670(4!) &= n^4 - 6n^3 + 11n^2 - 6n \\
0 &= n^4 - 6n^3 + 11n^2 - 6n - 2\ 672\ 670(4!)
\end{aligned}$$

Solving above for n, it follows that $n = 91$.

5. $\binom{13}{3}$

A contains 13 elements and we are looking at unordered, subsets of cardinality 3. Thus, there are $\binom{13}{3}$ subsets.

6. (a) $\binom{12}{6}$

   The order in which the students are chosen is irrelevant thus this is a standard combination. So, there are $\binom{12}{6}$ possible groups of six.

   (b) $\binom{10}{5}\binom{2}{1}$

   First, assign the two students who refuse to work together each to one of the two groups. Then choose the remaining five students in each group from the other ten students in the class. Applying the rule of product, there are $\binom{10}{5}\binom{2}{1}$ possible groups of six, given this restriction.

7. $\binom{10}{3} - \binom{8}{1} = \binom{9}{2} + \binom{9}{2} + \binom{8}{3}$

The total number of smoothies without any restriction is $\binom{10}{3}$. The number of smoothies that include both banana and apple are $\binom{8}{1}$ so, the total possible smoothies are: $\binom{10}{3} - \binom{8}{1}$.

Alternatively, we can find the total number of smoothies using 3 cases:

> **Case 1:** Banana is chosen for the smoothie. The total number of smoothies in this case is: $\binom{8}{2}$, since apple cannot be chosen.
>
> **Case 2:** Apple is chosen for the smoothie. The total number of smoothies in this case is: $\binom{8}{2}$, since banana cannot be chosen.
>
> **Case 3:** Neither banana nor apple is chosen for the smoothie. The total number of smoothies in this case is: $\binom{8}{3}$.

Thus, the total smoothies is the sum of all these cases which is: $\binom{9}{2} + \binom{9}{2} + \binom{8}{3}$.

8. $\binom{15}{7} - \binom{12}{5} = \binom{2}{1}\binom{13}{6} + \binom{13}{7} + \binom{12}{4}$

   Similar to the previous question, this can be done in two ways:

   The total number of possible course offerings with no restrictions is: $\binom{15}{7}$. The restricted cases are those in which both 'Logic and Foundations' and 'Linear Algebra' are offered but 'Calculus 1' is not. There are $\binom{12}{5}$ cases of this type. This gives that the total ways courses can be offered is: $\binom{15}{7} - \binom{12}{5}$.

   Alternatively, we can count the possible course offerings by cases.

   Case 1: Exactly one of 'Logic and Foundations' or 'Linear Algebra is offered. First, we can select one of the two and then the other 6 courses. There are $\binom{2}{1}\binom{13}{6}$ ways to do this.

   Case 2: Neither 'Logic and Foundations' and 'Linear Algebra is offered. There are $\binom{13}{7}$ ways to do this.

   Case 3: Both 'Logic and Foundations' and 'Linear Algebra are offered. This means 'Calculus 1' must also be offered. The other 4 courses can be selected in $\binom{12}{4}$ ways.

   The total ways to offer the courses are: $\binom{2}{1}\binom{13}{6} + \binom{13}{7} + \binom{12}{4}$.

9. (a) $\binom{20}{8}$

   (b) $\binom{10}{4}\binom{10}{4}$

   (c) $\binom{20}{8} - \binom{18}{6} = \binom{18}{8} + \binom{2}{1}\binom{18}{7}$

   (d) $\binom{20}{8}\binom{8}{1} = \binom{20}{1}\binom{19}{7}$. Robert can either choose the group of dancers for the opening act and then select a soloist or he can select a soloist from the class and then select the other 7 dancers.

10. **Algebraic:**

$$\begin{aligned} m\binom{n}{m} &= n\binom{n-1}{m-1} \\ \frac{m(n!)}{(n-m)!m!} &= \frac{n(n-1)!}{(n-1-(m-1))!(m-1)!} \\ \frac{n!}{(n-m)!(m-1)!} &= \frac{n!}{(n-m)!(m-1)!} \end{aligned}$$

Clearly from above, the two side are algebraically equivalent.

Combinatorial: Consider a group of n people who all apply to be on a committee of m people that requires a leader. There are two possible ways we can form the committee. We can either first choose from the larger group of n people our committee of m individuals, and then within that committee chose a leader, m possibilities. This is $m\binom{n}{m}$.

Alternatively, we can pick from the leader from the larger group of n people first, and then from the remaining $n-1$ select the remaining $m-1$ non-leader committee members. This is $n\binom{n-1}{m-1}$.

Since we counted the same scenario in two different ways, these expressions are equivalent.

11. We can look at this problem as counting the number of subsets given a set. We know that 2^n counts the total number of subsets from a set of cardinality n. This is because for each element in the set we are given two 'choices': whether or not to include it in the set. Since there are n elements in a set, we can use the product rule to see there are a total of 2^n possible subsets.

Alternatively, we know that $\binom{n}{k}$ is the total number of subsets of size k from a set of size n. And so, the left side is the sum of all possible subsets of n of size 0 to n. This is all possible sizes of subsets and so this is all possible subsets of a set of size n.

Since we counted the same scenario in two different ways, these expressions are equivalent.

12. Suppose a teacher in a classroom of n students is looking to take k students to a special conference. We know that there are $\binom{n}{k}$ to select these students, which is the left side of our equation.

    We can also look at this problem in two cases. Suppose there is a student Chan Ming in the class. We can look at two cases relating to Chan Ming either attending the conference or not. If Chan Ming does not attend the conference there are $\binom{n-1}{k}$ ways of picking students to go to the conference. If Chan Ming does attend the conference there are $\binom{n-1}{k-1}$ ways of selecting the remaining students to attend the conference. Thus, by the Rule of Sum, the total number of possible groups of students to attend the conference is $\binom{n-1}{k} + \binom{n-1}{k-1}$.

    Since we counted the same scenario in two different ways, these expressions are equivalent.

13. (a) $\sum_{k=0}^{n} x^k y^{n-k} \binom{n}{k}$

    (b) $\sum_{k=0}^{6} (3)^k (-x)^{6-k} \binom{6}{k} = x^6 - 18x^5 + 135x^4 - 549x^3 + 1215x^2 - 1458x + 729$

    (c) $\sum_{k=0}^{7} (2x)^k (-3y)^{7-k} \binom{7}{k} = 128x^7 - 1344x^6y + 6048x^5y^2 - 15120x^4y^3 + 22680x^3y^4 - 20412x^2y^5 + 10206xy^6 - 2187y^7$

    (d) $\sum_{k=0}^{15} (4x)^k (7y)^{n-k} \binom{15}{k}$

14. (a) $\binom{13}{9}$

    Notice first: $(x+y)^{13} = \sum_{k=0}^{13} (x)^k (y)^{13-k} \binom{13}{k}$.

    Thus, the coefficient we are looking for occurs when $k = 9$. This is equal to $\binom{13}{9}$.

    (b) $2^9 \binom{13}{9}$

    Notice first: $(2x+y)^{13} = \sum_{k=0}^{13} (2x)^k (y)^{13-k} \binom{13}{k}$.

    Thus, the coefficient we are looking for occurs when $k = 9$. This is equal to $2^9 \binom{13}{9}$.

    (c) $\binom{13}{9} \cdot 4^9 (-3)^4$

    Notice first: $(4x-3y)^{13} = \sum_{k=0}^{13} (4x)^k (-3y)^{13-k} \binom{13}{k}$.

    Thus, the coefficient we are looking for occurs when $k = 9$. This is equal to $\binom{13}{9} \cdot 4^9 (-3)^4$.

15. (a) $2^7 \cdot 3^4 \cdot \binom{11}{7}$

    We can first expand our binomial using the binomial theorem, that is:
    $(2x-3y)^{11} = \sum_{k=0}^{11} (2x)^k (-3y)^{11-k} \binom{11}{k}$.

    The term $x^7 y^4$ will occur when $k = 7$, so the coefficient of $x^7 y^4$ is $2^7 \cdot 3^4 \cdot \binom{11}{7}$.

    (b) $2^7 \cdot 5^2 \cdot \binom{9}{7}$

    Notice first: $(2x+5y)^9 = \sum_{k=0}^{9} (2x)^k (5y)^{9-k} \binom{9}{k}$.

    Thus, the coefficient we are looking for occurs when $k = 7$. This is equal to $2^7 \cdot 5^2 \cdot \binom{9}{7}$.

    (c) 3^5

    Notice first: $(3x-y)^9 = \sum_{k=0}^{5} (3x)^k (-y)^{5-k} \binom{5}{k}$.

    Thus, the coefficient we are looking for occurs when $k = 5$. This is equal to 3^5.

(d) $(-2)^3 \cdot (2)^9 \cdot \binom{12}{3}$

Notice first: $(-2x+2y)^{12} = \sum_{k=0}^{12} (-2x)^k (2y)^{12-k} \binom{12}{k}$.

Thus, the coefficient we are looking for occurs when $k = 3$. This is equal to $(-2)^3 \cdot (2)^9 \cdot \binom{12}{3}$.

(e) $2 \cdot 4^6 \cdot 7$

Notice first: $(2x-4y)^7 = \sum_{k=0}^{7} (2x)^k (-4y)^{7-k} \binom{7}{k}$.

Thus, the coefficient we are looking for occurs when $k = 1$. This is equal to $2 \cdot 4^6 \cdot 7$.

16. (a) Notice: $3^n = \sum_{k=0}^{n} 2^k 1^{n-k} \binom{n}{k}$ and so:

$$\sum_{k=2}^{n} 2^k \binom{n}{k} = 3^n - \sum_{k=0}^{1} 2^k \binom{n}{k} = 3^n - 1 - 2n$$

(b)

$$\begin{aligned}
\sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} &= \frac{1}{n!} \sum_{k=0}^{n} \frac{(-1)^k (1)! n!}{k!(n-k)!} \\
&= \frac{1}{n!} \sum_{k=0}^{n} (-1)^k (1)^{n-k} \binom{n}{k} \\
&= \frac{1}{n!} (1-1)^n \\
&= 0, \text{as } n \geq 0
\end{aligned}$$

17. By the Binomial Theorem:

$$\begin{aligned}
(1+i)^n + (1-i)^n &= \sum_{k=0}^{n} (i)^k \binom{n}{k} + \sum_{k=0}^{n} (-i)^k \binom{n}{k} \\
&= \sum_{k=0}^{n} ((i)^k + (-i)^k) \binom{n}{k}
\end{aligned}$$

We know that $\binom{n}{k} \in \mathbb{Z}$ and so we can focus on $((i)^k + (-i)^k)$. Let us proceed by cases on the parity of k.

Case 1: Suppose k is even. If k is even, let us write $k = 2t$, for some $t \in \mathbb{Z}$. Then $((i)^k + (-i)^k) = ((i)^{2t} + (-i)^{2t})$. This implies $(i)^{2t} = (i^2)^t = (-1)^t$ and $(-i)^{2t} = ((-i)^2)^t = (1)^t$. Thus, $((i)^k + (-i)^k) = (-1)^t + (1)^t$ which implies that $((i)^k + (-i)^k) \in \mathbb{Z}$, as desired.

Case 2: Suppose k is odd. If k is odd, let us write $k = 2t + 1$, for some $t \in \mathbb{Z}$. Then $((i)^k + (-i)^k) = ((i)^{2t+1} + (-i)^{2t+1})$. This implies that $(i)^{2t+1} = i(i^{2t}) = i(-1)^t$ and $(-i)^{2t+1} = -i(i^{2t}) = -i(-1)^t$. Thus, we have that $((i)^k + (-i)^k) = i(-1)^t - i(-1)^t = 0 \in \mathbb{Z}$, as desired.

18. (a) $\sum_{k=0}^{n} 6^k \binom{n}{k} = (6+1)^n = 7^n$

(b)

$$\begin{aligned}
\sum_{k=0}^{n} 4^n \binom{n}{k} &= 4^n \sum_{k=0}^{n} \binom{n}{k} \\
&= 4^n (1+1)^n \\
&= 4^n \cdot 2^n \\
&= 2^{3n}
\end{aligned}$$

(c) $\sum_{k=0}^{n} (-3)^k (2)^{n-k} = (-3+2)^n = (-1)^n$

(d)

$$\begin{aligned}
\sum_{k=0}^{n} 6 \binom{n}{k} &= 6 \sum_{k=0}^{n} \binom{n}{k} \\
&= 6(1+1)^n \\
&= 6 \cdot 2^n
\end{aligned}$$

(e) $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} = (-1+1)^n = 0$

19. When evaluating a polynomial with more than two terms to some integer power the multinomial theorem is used to determine the coefficients of the terms. The binomial theorem is a version of the multinomial theorem that can be used for binomials.

20. (a) Using the theorem, the coefficient is $\frac{4!}{2!1!1!} = 12$.

    (b) Using the theorem, the coefficient is clearly $\frac{8!}{2!2!2!2!} \cdot 2^2 \cdot (-1)^2 \cdot 3^2 \cdot (-3)^2 = 816480$.

    (c) There is no term with xyz since the sum of the exponents of the variables must add up to the exponent, 10.

3.5 Combinations with Repetitions

Solutions:

1. $i)$ The number of ways r identical elements can be distrbuted into n distinct containers.
   $ii)$ The number of non-negative integer solutions to:
   $$x_1 + x_2 + ... + x_n = r.$$

2. We are selecting r elements, with possible repetition, from a set of n distinct objects.

3. 330.

   We are interested in selecting seven items from the menu, potentially with repetition, from five distinct options, hence there are $\binom{5+7-1}{7} = \binom{11}{7} = 330$ ways these teammates can order.

4. 41 120 525.

   We are interested in selecting 12 teas out of 20 possible teas with potential repetition, hence there are $\binom{20+12-1}{12} = \binom{31}{12} = 141\ 120\ 525$ ways to select twelve teas.

5. (a) $\binom{40}{12}$.

   This is just a combinations problem, picking 12 shirts from 40 gives $\binom{40}{12}$ ways to select them.

(b) $\binom{51}{12}$.

We are interested in selecting 12 shirts, potentially the same, from 40 distinct colours, hence there are $\binom{40+12-1}{12} = \binom{51}{12}$ ways to do this.

6. (a) $\binom{20+m-1}{20}$.

The m children are certainly distinct, so we are interested in distributing the 20 identical toys to the m distinct children, which gives precisely $\binom{20+m-1}{20} = \binom{20+m-1}{m-1}$. different ways to do this.

(b) m^{20}.

This is equivalent to distributing 20 distinct objects into m distinct boxes. Each of the toys has m choices for which box to be assigned to, which yields m^{20} ways to distribute these toys.

7. 1680.

After giving each child one Skittle, we count how to distribute the 3 remaining Skittles and 6 chocolate bars. First we count the ways to distribute the remaining Skittles, then the chocolate, and then we'll apply the Rule of Product.

Skittles: We are arranging three identical objects, potentially with repetition, between four distinct individuals, so there are $\binom{4+3-1}{3} = \binom{6}{3}$ ways to distribute the Skittles.

Chocolate: Arranging six identical chocolate bars between four distinct children gives $\binom{4+6-1}{6} = \binom{9}{6}$ ways to distribute the chocolate.

In total there are $\binom{6}{3} \cdot \binom{9}{6} = 1\ 680$ ways to distribute this candy.

8. There are many possible correct answers, one example is, in how many can you distribute ten treats between three dogs?

9. 42 504.

We can re-express this problem as how many solutions are there to the *equation*

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19,$$

where $x_i \geq 0$ for $i = 1, ..., 5$ and $x_6 \geq 0$.
x_6 is a placeholder variable which accounts for when $\sum_{i=1}^{5} x_i < 20$, since every positive value of x_6 corresponds to,

$$x_1 + x_2 + x_3 + x_4 + x_5 = 19 - x_6 < 20$$

Therefore we are distributing 19 identical objects between 6 distinct containers, hence there are $\binom{6+19-1}{19} = \binom{24}{19} = 42\ 504$ solutions to the original inequality.

10. 56.

First we must rewrite the problem accounting for the x_i's with restrictions larger than one. We first distribute one into x_1 which means our problem is equivalent to asking how many solutions there are to,

$$x_1 + x_2 + x_3 + x_4 = 7$$

where $x_2 > 1$ and $x_1, x_3, x_4 \geq 0$. Since $x_2 \in \mathbb{Z}$, $x_2 > 1$ is equivalent to $x_2 \geq 2$. Putting two into x_2, our problem is equivalent to asking for the number of integer solutions to,

$$x_1 + x_2 + x_3 + x_4 = 5,$$

where $x_i \geq 0$ for $i = 1, 2, 3, 4$. Thus we are arranging 5 identical objects into 4 distinct containers, so there are $\binom{4+5-1}{5} = \binom{8}{5} = 56$ unique integral solutions.

11. 2 925.

We begin by distributing one into x_1, one into x_2, and four into x_4, therefore this problem is equivalent to the number of integer solutions of,

$$x_1 + x_2 + x_3 + x_4 = 24,$$

where $x_i \geq 0$ for all i. We are interested in distributing 24 identical objects into four distinct containers, hence there are $\binom{24+4-1}{4-1} = \binom{27}{3} = 2\ 925$ integer solutions.

12. 26 334.

We first put one into x_4 and three into x_5 which makes the problem equivalent to the number of integer solutions to:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 17,$$

where $x_i \geq 0$, for $i = 1, 2, 3, 4, 5$. Adding a placeholder variable to account for the inequality (as in question 9), this is equivalent to the number of integer solutions to:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 17,$$

where $x_i \geq 0$ for $i = 1, 2, 3, 4, 5, 6$. We are distributing 17 identical objects into six distinct containers, so the number of integer solutions to this is $\binom{6+17-1}{6-1} = \binom{22}{5} = 26\ 334$.

13. 126.

First we will put a towel in each bathroom. Now we are left to distribute the five remaining identical towels into the five bathrooms, with potential repetition. Using the combination with repetition formula there are exactly $\binom{5+5-1}{5}$ $= \binom{9}{5} = 126$ ways to distribute these towels.

14. $n = 7$.

Notice that positive integer solutions implies that each $x_i, y_i \geq 1$ for all i. So we can start by giving one to each x_i, y_i, reducing the equations to:

$$x_1 + x_2 + x_3 + ... + x_{22} = n - 22 \quad \text{and}$$

$$y_1 + y_2 + y_3 + ... + y_{51} = n - 55,$$

where $x_i, y_i \geq 0$.

The number of non-negative solutions to the first equation is $\binom{22+(n-22)-1}{n-22} = \binom{n-1}{n-22}$. The number of non-negative solutions to the second equation is $\binom{55+(n-55)-1}{n-55} = \binom{n-1}{n-55}$.

If we want the same solutions to the two equations then we need $\binom{n-1}{n-22} = \binom{n-1}{n-55}$.

Recall that $\binom{n}{m} = \binom{n}{n-m}$, so we know $\binom{n-1}{n-22} = \binom{n-1}{21}$. Now we have $\binom{n-1}{21} = \binom{n-1}{n-55}$. This tells us that $21 = n - 55$, which gives $n = 76$.

15. $n = 7$.

If we know Hannah has 593 775 different ways of choosing two dozen loaves with repetition from n distinct bread varieties, then we know that,

$$\binom{n+24-1}{24} = \binom{n+23}{24} = 593\ 775$$

With trial and error, we can see that,

$$\binom{30}{24} = 593\ 775$$

Which means that $n + 24 - 1 = 30$, and so $n = 7$.

16. **Algebraically**: Expanding using the definition of the combination function, we know,

$$\tbinom{n+r-1}{r} = \tfrac{(n+r-1)!}{r!(n-1)!},$$

and that,

$$\tbinom{n+r-1}{n-1} = \tfrac{(n+r-1)!}{(n-1)!r!}.$$

It is easy to see that these expansions are equal since multiplication is commutative, so we're done.

Combinatorial proof: We can look at this problem as placing r balls into n boxes.
If we wish to do this, we can line up the r balls and place $n-1$ dividers between them. The balls between either the beginning/end and a divider or two dividers represents the number of balls in a box. Thus, there are $n-1+r$ total positions, where each position is either filled with a ball or a divider.
From these positions we can either choose where to first place the dividers and

then have the balls fill the remaining positions, $\binom{n+r-1}{n-1}$, or we can choose where to place the balls first and have the dividers fill the remaining positions, $\binom{n+r-1}{r}$. The two options are equivalent and are equal to the right and left sides of the equation, respectively.

17. (a) $P(n,r)$

    (b) $C(n,r)$

    (c) n^r

    (d) $C(n+r-1,r) = C(n+r-1,n-1)$

18. 136.

To solve this problem, we would like to find an equivalent equation so that $x_i \geq 0$ for $i = 1,2,3$ and we have a familiar problem to solve. We begin by removing -5 from each x_i, by giving 5 to each x_i. Our problem is now to determine the number of integer solutions to

$$x_1 + x_2 + x_3 = 15,$$

for each $x_i \geq 0$. This gives precisely $\binom{3+15-1}{15} = \binom{17}{15} = 136$ possible solutions.

19. Determine the number of integer solutions to:

    (a)
    $$x_1 + x_2 + x_3 = 7,$$
    where $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$ and $0 \leq x_3 \leq 2$. We have x_1 representing the red marbles, x_2 the blue and x_3 the green marbles.

(b)

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30,$$

where $x_i \geq 0$ for $i = 1, 2, 3, 4, 5$. Each x_i represents how many poker chips each individual has. (We require each x_i to be positive since we are modelling a situation where having negative poker chips is nonsensical.)

(c)

$$x_1 + x_2 + x_3 + x_4 = 12,$$

where $x_i \geq 2$ for $i = 1, 2, 3, 4$. Each x_i represents the number of apples of each variety that have been selected.

(d) $x_1 + x_2 + x_3 + x_4 = 15$, where $x_i \geq 0$ for $i = 1, 2, 3, 4$ with $x_1 = x_2$. Each x_i represents the amount of markers in each box.

20. $\binom{n-m+x-1}{x} \cdot \binom{n}{m}$.

We would like to distribute x identical marbles into $n - m$ distinct boxes. This can be done in $\binom{n-m+x-1}{x}$ ways. However, we do not know which m boxes will be left empty, so we must multiply by $\binom{n}{m}$. Hence there are $\binom{n-m+x-1}{x} \cdot \binom{n}{m}$ ways to distribute these marbles.

21. $\binom{m+s-1}{s} \cdot \binom{n+r-s-1}{r-s}$.

We begin by distributing s shoes into the first m boxes. These s shoes are identical and the m boxes distinct. There are $\binom{m+s-1}{s}$ ways to do this. Next, we must distribute the remaining $r - s$ identical shoes into the n boxes. There are $\binom{n+r-s-1}{r-s}$ ways to do this. So by the Rule of Product, there are $\binom{m+s-1}{s} \cdot \binom{n+r-s-1}{r-s}$ ways to distribute these shoes.

3.6 The Pigeonhole Principle

Solutions:

1. If there are k pigeons that are flying into n pigeonholes where $n < k$, then there must be at least one box with at least two pigeons.

2. A function from one finite set to a smaller finite set cannot be one-to-one. There will be at least two elements from the domain that map to the same image in the co-domain/range. The pigeons represent the domain and the pigeonholes the co-domain. The function is the assignment of pigeons to pigeonholes.

3. There is nothing specific that can be said, unless $m > n$. If $m > n$ we know that at least one pigeonhole will be empty. Beyond that, there are many possible arrangements of the pigeons.

4. First notice if $k = 1$, this is precisely the pigeonhole principle.

   Consider when $k > 1$. Suppose for contradiction that each pigeonhole houses at *most* k pigeons. Then there are, at most, $k \times n = kn$ pigeons, which is a contradiction as there are $kn+1$. Thus, at least one pigeonhole hosts $k+1$ pigeons.

5. (a) There are 366 possible birthdays, including February 29. As $367 > 365$, there will be at least two people who have the same birthday.

   The pigeons are the people and the pigeonholes are the birthdays.

   (b) Every integer can be written in the form $28k + m$, where $k \in \mathbb{Z}$ and $m \in \{0, 1, ..., 27\}$. So, m represents the remainder of the integer when divided by 28. There are 28 possible values of m. Therefore, there must be at least two integers in the set of 29 integers with the same remainder.

The pigeons are the different integers and the pigeonholes represent the possible remainders upon division by 28.

(c) You must take out at least 10 shoes before you are guaranteed to obtain a pair. Any less and it is possible that each shoe is from a different pair.

The pigeons are the shows and the pigeonholes are the pairs of shoes.

(d) To determine this, we must first count the number of distinct three letter words. For each position in the word there are 26 possible letters. Therefore, in total there are $26^3 = 17\,576$ distinct three letter words. This means that it is possible, but not definitive, that all words on this list are distinct as there are more three-letter words possible than there are words on the list.

The pigeons are the number of three-letter words and the pigeonholes are the words on the list.

6. If we prove the statement about subsets of size 6, the result will follow for all larger subsets.

   Let the pigeonholes be: $\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$. Let the pigeons be the integers in the subset. As there are at least 6 distinct integers chosen and only 5 pigeonholes, two pigeons must be in the same pigeonhole. Notice there is only one box whose sum does not add to 10, but this box can only contain at most one pigeon, the number 5. Thus, a box with two pigeons indicates there must be a pair of integers whose sum is 10.

7. Let each flavour a child orders be a pigeon and each flavour option be a pigeonhole. There are eight children who each order two flavours of ice cream and 15 flavour options. This means there are 15 pigeonholes and 16 pigeons. This

means that at least one flavour must be ordered twice.

8. (a) Any even positive integer can be written as $x = 2^k y$ (essentially just factoring out the two's), where $k \in \mathbb{N}$, and y is odd. There are exactly 1000 odd numbers in $A = \{1, ..., 2000\}$.

   Let us define a pigeonhole for each odd integer $y \in A$ as:

   $$PH_y = \{x \in A : x = 2^k y, \text{ where } y \text{ is odd and } k \geq 0\}.$$

   This gives us our 1000 pigeonholes.

   Select any 1001 numbers from A, these are the pigeons. Then, by the pigeonhole principle, there exists a pigeonhole, PH_y, that contains two selected numbers: a and b. Say $a = 2^k y$ and $b = 2^p y$ for some distinct, nonnegative integers k and p. If $k > p$, then b divides a and if $k < p$, then a divides b.

   (b) We can partition the numbers into sets of size two, where the second digit is one less than the first: $\{1, 2\}, \{3, 4\}, ..., \{1997, 1998\}, \{1999, 2000\}$. Then, there are exactly 1000 of these disjoint subsets, which represent our 'pigeonholes'.

   Choose any 1001 integers and let them represent our 'pigeons'. Then by PHP, we will have two integers from the same disjoint subset. Hence, two integers are relatively prime.

9. First determine the number of possible, distinct initials. There are 26 options for each one's first and last initial. Therefore, there are $26^2 = 676$ different possible initials.

   Let the attendees represent the 'pigeons' and the possible initials represent the 'pigeonholes'. To guarantee there are at least two attendees with the same initials, it follows from the Pigeonhole Principle that we need more than 676 attendees. So, we require at least 677 attendees.

10. Notice that Brynn spent $6 \cdot 7 = 42$ days sending out scholarship applications.

    For $1 \leq i \leq 42$, let x_i represent the number of scholarships Brynn has sent out in total as of day i. Since Brynn sends out at least one application per day and no more than 60 total, we know that $1 \leq x_1 < x_2 < ... < x_{42} < 60$.

    Now adding 23 to every term of the inequality we obtain:

    $$1 + 23 = 24 \leq x_1 + 23, x_2 + 23, ..., x_{42} + 23 < 60 + 23 = 83$$

    Note: Recognizing that you need to do this is the heart of the proof. We add 23 because we are trying to prove that there are 23 consecutive days where scholarship applications were sent out, which allows us to conclude this by the PHP.

    Now we have 84 distinct numbers, $\{x_1, x_2, ..., x_{42}, x_1 + 23, x_2 + 23, ..., x_{42} + 23\}$. Let these numbers represent out 'pigeons'. These 84 numbers must all lie between 1 and 83, where the range of integers from 1 to 83 represent out pigeonholes. Thus by the PHP there exists an $x_i = x_j + 23$ for some $i > j \in \{1, ..., 42\}$. This means that from the beginning of day $j + 1$ to the end of day i, Brynn applied for 23 scholarships.

11. It suffices to prove the result for subsets of exactly size three, since that will imply the result for subsets of size larger than three.

    The only way for the sum of two integers to be even is if both of the integers have the same parity, that is both are even or both are odd. Any given integer can be classified as either even or odd, hence any subset of 3 integers will contain at least two with the same parity by the PHP. Thus, there are two integers in the subset with an even sum.

12. There are 12 pigeonholes (computers) and 42 pigeons. In this problem there is a restriction that no pigeonhole can hold more than 6 pigeons.

    We wish to show that there are five computers which are used by three or more people.

    Let us assume for a contradiction that this is not true. This would mean that 8 computers are used by at most 2 people. This would mean that these 8 computers are used by at most 16 people all together.

    There are 42 people who use a computer at the library and so that means the remaining 26 people use 4 computers.

    This means that there are 26 pigeons and 5 pigeon holes where the maximum capacity of each pigeonhole is 6. This however gives that the maximum capacity for the remaining computers is 24, which is a contradiction.

    So, at least 5 computers are to be used by three or more people.

13. Let us assume that if one person speaks to another, the person will respond. That is, assume speaking to someone is a reflective relation.

    If there are n people at the party, each person can speak to between 0 and $n-1$ people, as no person can speak to themselves and speaking to someone is reflective.

    If a person spoke to $n-1$ people, then it is impossible for any person to have spoken to 0 people. In this case every person spoke to between 1 and $n-1$ people. That means there are $n-1$ potential number of people a person could have spoken to.

If a person at the party spoke to 0 people, then it is impossible for someone to have spoken to everyone. In this case every person will have spoken to between 0 and $n-2$ people. That means there are $n-1$ potential number of people a person could have spoken to.

In both of the above cases, there are $n-1$ potential amounts of people a person could have spoken to but n people. Thus, by the PHP two people will have spoken to the same amount of people at the party.

14. For 12 to divide the difference of two numbers, they must have the same remainder upon division by 12. Observe that $12k+m-(12j+m)=12(k-j)$, where $k,j,m \in \mathbb{Z}$ and m represents the remainder of the arbitrary integer when divided by 12.

    Certainly the only possible remainders are $\{0,...,11\}$, of which there are 12 possibilities. Thus by the PHP, as 12 integers have been selected at least two must have the same remainder when divided by 12. Thus, their difference is divisible by 12, as desired.

15. Notice that a rectangle with width of 3 metres and height of 4 metres has a diagonal length of 5 metres. If we divide the field into rectangles of this size, we are able to split the field into 30 rectangles. Let the cows represent 'pigeons' and the rectangles represent 'pigeonholes'. Then, by the PHP, at least two cows must be in the same rectangle. The farthest two cows are apart in a rectangle in 5 metres and so the result follows.

16. In A, there are exactly 40 integers divisible by 5. Therefore one must select $200-40+1=161$ integers to guarantee that at least one of them is divisible by 5.

17. There are exactly 30 odd numbers bin X, and 61 numbers to choose from. Therefore at least $61 - 30 + 1 = 32$ numbers must be selected to guarantee that at least one is odd.

4 Inclusion and Exclusion

4.1 The Principle of Inclusion-Exclusion

Solutions:

1. The Principle of Inclusion-Exclusion is a counting method that ensures every possible event is counted, while also taking into account events that can co-occur. This method is a way of ensuring events are not counted twice or "double counted".

2. This mathematical statement is false.

   The left side, $N(\overline{c_1 c_2})$, represents the number of elements where c_1 and c_2 are not simultaneously satisfied. These elements include elements where one of c_1 or c_2 is satisfied.

   The right side, $N(\overline{c_1}\ \overline{c_2})$, represents the elements where neither c_1 nor c_2 are satisfied.

   An example of this situation follows. Suppose you are looking for a movie to watch. You are choosing between action movies, movies that are at least two hours long, and movies in Spanish. Let c_1, c_2, and c_3 represent action movies, movies that are at least two hours long, and movies that are in Spanish.

   If you are looking for pick a movie that satisfies the condition $\overline{c_1 c_2}$. This means you are choosing a movie that is not both an action movies and two hours or longer. So, you may still pick a shorter action movie or a two hour long movie of a different genre.

   If on the other hand you are looking to pick a movie that satisfies the condition $\overline{c_1}\ \overline{c_2}$. This means you are looking for a movie that is neither an action movie nor longer than two hours. This condition excludes movies that are both action and two hours or longer, movies that are only actions movies, and movies that are just two hours or longer.

Thus, $N(\overline{c_1 c_2})$ may be smaller than $N(\overline{c_1}\ \overline{c_2})$ as it excludes fewer elements.

3. Let S be the set of kindergarten students, thus $|S| = 30$. Let c_1 be the condition that students enjoy nap time, and c_2 be the condition that the students enjoy colouring. So, $N(c_1) = 20$ and $N(c_2) = 14$.

   (a) 16 students

   This is the total number of students in the class subtracted by those who enjoy colouring: $30 - 14 = 16$ students do not enjoy colouring.

   (b) 27 students

   From the question, we know $N(c_1c_2) = 7$ as there are 7 students that satisfy both of our conditions. We are looking for the number of students that like one or both of the surveyed activities. So, by PIE: $N(c_1\ \ c_2) = N(c_1) + N(c_2) - N(c_1c_2) = 20 + 14 - 7 = 27$.

   (c) 20 students

   We are looking for the number of students who only enjoy colouring and the number of students who only enjoy nap time.

   The number of students who only enjoy naptime is equal to the number of students who like nap time, subtracted by the number of students who enjoy both colouring and nap time: $N(c_1\overline{c_2}) = N(c_1) - N(c_1c_2) = 20 - 7 = 13$.

   Similarly, $N(\overline{c_1}c_2) = N(c_2) - N(c_1c_2) = 14 - 7 = 7$.

Thus, the total of students who only enjoy one activity is: $N(c_1\overline{c_2}) + N(\overline{c_1}c_2) = 13 + 7 = 20$.

4. 300 homes.

   Let P be the set of families with pets and C the set of families with children. The question asks for $|P \cap C|$. In the question they tell us that $|P \cup C| = 500 - 100 = 400$. By Inclusion-Exclusion,

   $$|P \cup C = |P| + |C| - |P \cap C|.$$

   Rearranging, we get

   $$|P \cap C| = |P| + |C| - |P \cup C| = 300 + 400 - 400 = 300.$$

   Thus, 300 homes have both pets and children in them.

5. (a) 46 cookies.

   Let c_1 represent cookies with chocolate chips and c_2 represent cookies with raisins. From the question we know that $N = 100$, $N(c_1) = 40$, $N(c_2) = 25$, and $N(c_1c_2) = 11$.

   We are looking for $N(\overline{c_1}\ \overline{c_2})$. By PIE, we know that:

   $$N(\overline{c_1}\ \overline{c_2}) = N - N(c_1) - N(c_2) + N(c_1c_2)$$

   We can sub in our given values to see that $N(\overline{c_1}\ \overline{c_2}) = 46$

(b) 35 cookies.

We can introduce an additional variable c_3 to represent cookies with oatmeal. We are given additional information: $N(c_3) = 30$, $N(c_1c_3) = 10$, $N(c_2c_3) = 15$, and $N(c_1c_2c_3) = 6$.

We are interested in $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3})$. By PIE it follows that:

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}) = N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - N(c_1c_2c_3)$$

We can sub in our given values to see that $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) = 35$.

6. (a) This question has no correct answer because it describes an impossible situation. For example, it says that $|A \cup B \cup C| = 150$. It also says that $|A| = |B| = |C|$. This can only occur if A, B and C are disjoint.

However, the question <u>also</u> says that they are NOT disjoint. Hence the question has no correct answer.

(b) 0 families.

We are interested in $N(\overline{c_1}\ \overline{c_2}\ c_3)$. By PIE, we know that:

$$\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ c_3) &= N(c_3) - N(c_2c_3) - N(c_1c_3) + N(c_1c_2c_3)\\
N(\overline{c_1}\ \overline{c_2}\ c_3) &= 50 - 30 - 40 + 20\\
N(\overline{c_1}\ \overline{c_2}\ c_3) &= 0
\end{aligned}$$

So, no families brought only juice.

(c) 135 families.

We are interested in $N(\overline{\overline{c_1}c_2\overline{c_3}})$. By PIE, we know that:

$$\begin{aligned}
N(\overline{\overline{c_1}c_2\overline{c_3}}) &= N - N(\overline{c_1}c_2\overline{c_3})\\
&= N - (N(c_2) - N(c_1c_2) - N(c_2c_3) + N(c_1c_2c_3))\\
&= 150 - (50 - 25 - 30 + 20)\\
&= 135
\end{aligned}$$

So, 135 families did not bring only sandwiches.

7. $11! - \sum_{i=1}^{6}(-2)^i\binom{6}{i}(11-i)!$

Let S be a set of 6 twins. Let us number the twins as 1, 2, 3, 4, 5, and 6. Let c_i represent the condition where twins i sit next to each other for $i = 1, 2, 3, 4, 5, 6$.

To find $N(c_1)$, we can first seat the twins next to each other at the table. We can seat the twins next to each other and then place the other ten people around the table. Since this is a round table, where we seat the twins is irrelevant. There are 10! ways to arrange everyone in this manner. We must also consider which twin is in which seat. There are 2 ways to arrange the twins themselves. Thus, using the product rule, we can see that $N(c_1) = 2 \cdot 10!$ is the total seating arrangements where two twins sit next to each other.

This follows similarly for each $N(c_i)$ for $i = 2, 3, 4, 5, 6$. Thus, we collect these terms together, let $T_1 = \binom{6}{1} 2 \cdot 10!$ where T_1 is the total seating arrangements where two twins are seated next to each other.

To find $N(c_p c_q)$ for $1 \leq p < q \leq 6$, we can treat the seating as the arrangement of 10 distinct objects, where each pair of twins is one object. Since this is a round table, there are 9! ways of seating everyone. We then must account for the arrangement of each twin. There are 2 ways to arrange each twin, and so in total we have $N(c_p c_q) = 2^2 \cdot 9!$.

We can again collect all seatings where two pairs of twins are seating together. Let us use T_2 to represent this total. Then $T_2 = \binom{6}{2} 2^2 \cdot 9!$.

Similarly, $N(c_p c_q c_r) = 2^3 \cdot 8!$, where $1 \leq p < q < r \leq 6$ and so $T_3 = \binom{6}{3} 2^3 \cdot 8!$; $N(c_p c_q c_r c_s) = 2^4 \cdot 7!$, where $1 \leq p < q < r < s \leq 6$ and so $T_4 = \binom{6}{4} 2^4 \cdot 7!$; $N(c_p c_q c_r c_s c_t) = 2^5 \cdot 6!$, where $1 \leq p < q < r < s < t \leq 6$ and so $T_5 = \binom{6}{5} 2^5 \cdot 6!$; and $N(c_1 c_2 c_3 c_4 c_5 c_6) = 2^6 \cdot 5!$ and so $T_6 = \binom{6}{6} 2^6 \cdot 5!$.

The total number of possible seatings is simply $N = 11!$, as there are 12 people to seat at a circular table.

We are looking for $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}\ \overline{c_5}\ \overline{c_6})$, and so we can use PIE and our above values to find this.

$$N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}\ \overline{c_5}\ \overline{c_6}) = N - T_1 + T_2 - T_3 + T_4 - T_5 + T_6$$
$$= 11! - \sum_{i=1}^{6}(-2)^i \binom{6}{i}(11-i)!$$

8. (a) 1714 numbers.

First, let us count how many numbers are divisible by each of 7 and 2. Let us call S_2 the set of numbers in A that divisible by 2, and S_7 the set of numbers in A that are divisible by 7.
The numbers in A that are divisible by 2 are:

$$|S_2| = \left\lfloor \frac{3000}{2} \right\rfloor = 1500$$

The numbers in A that are divisible by 7 are:

$$|S_7| = \left\lfloor \frac{3000}{7} \right\rfloor = 428$$

The numbers in A that are divisible by both 2 and 7 are:

$$|S_2 \cap S_7| = \left\lfloor \frac{3000}{lcm(2,7)} \right\rfloor = \left\lfloor \frac{3000}{14} \right\rfloor = 214$$

Thus, by PIE, the total numbers in A divisible by either 2 or 7 is:

$$\begin{aligned}|S_2 \cup S_7| &= |S_2| + |S_7| - |S_2 \cap S_7| \\ &= \left\lfloor \frac{3000}{2} \right\rfloor + \left\lfloor \frac{3000}{7} \right\rfloor - \left\lfloor \frac{3000}{14} \right\rfloor \\ &= 1500 + 428 - 214 \\ &= 1714\end{aligned}$$

(b) 214 numbers.

We are interested in $|\overline{S_2} \cup S_7|$. We can use the values found in part a to do so.

$$\begin{aligned}|\overline{S_2} \cup S_7| &= |S_7| - |S_2 \cap S_7| \\ &= 428 - 214 \\ |\overline{S_2} \cup S_7| &= 214\end{aligned}$$

(c) 1400 numbers.

Let S_3 and S_5 represent the sets of numbers in A divisible by 3 and 5, respectively. Thus we have that:

$$\begin{aligned}|S_3| &= \left\lfloor \frac{3000}{3} \right\rfloor && = 1000 \\ |S_5| &= \left\lfloor \frac{3000}{5} \right\rfloor && = 600 \\ |S_3 \cap S_5| &= \left\lfloor \frac{3000}{lcm(3,5)} \right\rfloor = \left\lfloor \frac{3000}{15} \right\rfloor && = 200\end{aligned}$$

Thus, by PIE:

$$\begin{aligned}|\overline{S_3} \cup \overline{S_5}| &= 3000 - |S_3| - |S_3| + |S_3 \cap S_5| \\ &= 1000 + 600 - 200 \\ &= 1400\end{aligned}$$

And so, there are 1400 numbers in A divisible by neither 3 nor 5.

(d) 63 numbers.

We are looking for $|S_2 \cap S_3 \cap S_7 \cap \overline{S_{11}}|$.

$$\begin{aligned}|S_2 \cap S_3 \cap S_7| &= \left\lfloor \frac{3000}{lcm(2,3,7)} \right\rfloor \\ &= \left\lfloor \frac{3000}{42} \right\rfloor \\ &= 71 \\ |S_2 \cap S_3 \cap S_7 \cap S_{11}| &= \left\lfloor \frac{3000}{lcm(2,3,7,11)} \right\rfloor \\ &= \left\lfloor \frac{3000}{462} \right\rfloor \\ &= 8\end{aligned}$$

Thus, using PIE, we have that:

$$\begin{aligned}|S_2 \cap S_3 \cap S_7 \cup \overline{S_{11}}| &= |S_2 \cap S_3 \cap S_7| - |S_2 \cap S_3 \cap S_7 \cap 11| \\ &= 71 - 8 \\ &= 63\end{aligned}$$

Thus, there are 63 numbers in A that are divisible by 2,3, and 7 but not 11.

9. $10^9 - 3(9^9) + 3(8^9) - 7^9$ sequences.

Let c_1, c_2, and c_3 be the condition that a 9 digit sequences does **not** include 1, 2, and 3, respectively.

To find $N(c_1)$ we can find the number of sequences we can make with the the other 9 digits, excluding 1. This is 9^9 as there are 9 options for each of of the 9 digits where order does matter. Similarly, $N(c_2) = N(c_3) = 9^9$.

To find $N(c_1c_2)$, we can find the number of sequences we can make with the other 8 digits. This is 8^9, as there are 8 options for each of the 9 digits. Similarly, $N(c_1c_3) = N(c_2c_3) = 8^9$.

To find $N(c_1c_2c_3)$, we can find the number of sequences we can make with the other 7 digits. This is 7^9, as there are 7 options for each of the 9 digits.

The total number of 9 digit sequences in $N = 10^9$.

Thus, by PIE, the number of sequences that include 1, 2, and 3 at least once is equal to $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3})$:

$$\begin{aligned} N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) &= N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) \\ &\quad + N(c_1c_3) + N(c_2c_3) - N(c_1c_2c_3) \\ &= 10^9 - 3(9^9) + 3(8^9) - 7^9 \end{aligned}$$

10. $10! - 2 \cdot 9! - 2 \cdot 9! + 2^2 \cdot 8!$

Let c_1 be the condition that the first digit is less than 2 and c_2 be the condition that the last digit is greater than 7.

The total number of sequences without restrictions is $N = 10!$, as there are 10 digits to permute. In order to use PIE, we will find $N(c_1)$, $N(c_2)$, and $N(c_1c_2)$.

For $N(c_1)$, we are looking at permutations of these 10 digits with a restriction that the first digit can only be 0 or 1. This gives that $N(c_1) = 2 \cdot 9!$, as there are two possibilities for the first digit and the remaining 9 digits are then permuted. Similarly, $N(c_2) = 2 \cdot 9!$.

To find $N(c_1c_2)$, we are looking at permutations of these 10 digits where two positions have restrictions. The first and last digit both must be one of two numbers, as above. The other 8 digits are a permutation. Thus, we have that $N(c_1c_2) = 2^2 \cdot 8!$.

We are looking for $N(\overline{c_1}\ \overline{c_2})$. Thus, by PIE:

$$\begin{aligned} N(\overline{c_1}\ \overline{c_2}) &= N - N(c_1) - N(c_2) + N(c_1c_2) \\ &= 10! - 2 \cdot 9! - 2 \cdot 9! + 2^2 \cdot 8! \end{aligned}$$

11. $\frac{26!}{16!} - \frac{6(21!)}{16!} - \frac{8(23!)}{16!} - \frac{7(22!)}{16!} + \frac{4(19!)}{16!} + \frac{3(18!)}{16!} + \frac{4 \cdot 3 \cdot (19!)}{16!} - 1$

Let $N(c_1)$, $N(c_2)$, and $N(c_3)$ represent the number of words containing GAINS, BUG, and SNAP, respectively. Let N represent the total number of letter sequences of length 10. Thus, $N = P(26, 10) = \frac{26!}{16!}$

To form a sequence that contains GAINS, we need to arrange 5 letters from the remaining 20 letters of the alphabet. This can be done in $P(21, 5) = \frac{21!}{16!}$ ways. Then, choose the position for the word GAINS within this sequence. This can be done in 6 ways. Thus, using the product rule $N(c_1) = 6 \cdot P(21, 5) = \frac{6(21!)}{16!}$.

Similarly for BUG and SNAP, there are $N(c_2) = 8 \cdot P(23, 7) = \frac{8(23!)}{16!}$ and $N(c_3) = 7 \cdot P(22, 6) = \frac{7(22!)}{16!}$.

A sequence containing both BUG and SNAP requires 3 other letters. There are

$P(19,3)$ ways to arrange these letters. There are then 3 ways to place BUG in between these letters and 4 ways to then place SNAP. Thus, using the product rule there are $N(c_2c_3) = 4 \cdot 3 \cdot P(19,3) = \frac{4 \cdot 3 \cdot (19!)}{16!}$.

A sequence that contains both BUG and GAINS must include the sequence BUGAINS, as the sequence does not allow for any repeated letters. Thus, there are $N(c_1c_2) = 4 \cdot P(19,3) = \frac{4(19!)}{16!}$.

A sequence that contains both GAINS and SNAP must contain the sequence the sequence GAINSNAP. Thus, $N(c_1c_3) = 3 \cdot P(18,2) = \frac{3(18!)}{16!}$.

A sequence containing GAINS, SNAP, and BUG must contain the sequence BUGAINSNAP. This sequence is of length 10 and so there is only one sequence containing all three words. Thus, $N(c_1c_2c_3) = 1$

We are looking for the number of sequences not containing any of these words, $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3})$. By PIE, we know that:

$$\begin{aligned} N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) &= N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - N(c_1c_2c_3) \\ &= P(26,10) - 6 \cdot P(21,5) - 8 \cdot P(23,7) - 7 \cdot P(22,6) + 4 \cdot P(19,3) + \\ &\qquad 3 \cdot P(18,2) + 4 \cdot 3 \cdot P(19,3) - 1 \\ &= \frac{26!}{16!} - \frac{6(21!)}{16!} - \frac{8(23!)}{16!} - \frac{7(22!)}{16!} + \frac{4(19!)}{16!} + \frac{3(18!)}{16!} + \frac{4 \cdot 3 \cdot (19!)}{16!} - 1 \end{aligned}$$

12. $\frac{26!}{14!} - \frac{10 \cdot 23!}{14!} - \frac{9 \cdot 22!}{14!} - \frac{8 \cdot 21!}{14!} + \frac{6 \cdot 7 \cdot 19!}{14!} + \frac{4 \cdot 5 \cdot 17!}{14!}$

Let $N(c_1)$, $N(c_2)$, $N(c_3)$ represent the number of words containing DOG, SPUN, and DREAM, respectively.

To count the number of sequences that contain DOG, we can first permute the other 9 letters in the sequence from the remaining 23 letters. There are

$P(23,9) = \frac{23!}{14!}$ ways to do this. We can then place the word DOG somewhere in the sequence. There are 10 different ways to place DOG. Thus, using the product rule we can see that $N(c_1) = \frac{10 \cdot 23!}{14!}$. Similarly, $N(c_2) = \frac{9 \cdot 22!}{14!}$ and $N(c_3) = \frac{8 \cdot 21!}{14!}$.

To count the number of sequences containing both DOG and SPUN, we can first permute the other 5 letters from the remaining 19 letters. There are $P(19,5) = \frac{19!}{14!}$ ways to do this. We can then place DOG and SPUN somewhere in the sequence. There are $6 \cdot 7$ ways to place these words. Thus, using the product rule we can see that $N(c_1c_2) = \frac{6 \cdot 7 \cdot 19!}{14!}$. Similarly, $N(c_2c_3) = \frac{4 \cdot 5 \cdot 17!}{14!}$.

There can be no sequences containing both DOG and DREAM as these sequences of letters are not disjoint. Thus $N(c_1c_3) = 0$. Similarly, there can be no sequences containing all three of the words DOG, SPUN, and DREAM.

The total sequences with 12 letters is $N = P(26,12) = \frac{26!}{14!}$ We are interested in $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3})$. Using PIE, we can see that:

$$
\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) &= N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - N(c_1c_2c_3) \\
&= \frac{26!}{14!} - \frac{10 \cdot 23!}{14!} - \frac{9 \cdot 22!}{14!} - \frac{8 \cdot 21!}{14!} + \frac{6 \cdot 7 \cdot 19!}{14!} + \frac{4 \cdot 5 \cdot 17!}{14!} + 0 - 0
\end{aligned}
$$

13. $\frac{11!}{4!4!2!} - \frac{8!}{4!2!} - \frac{10!}{4!4!} - \frac{8!}{4!2!} + \frac{7!}{4!} + \frac{5!}{2!} + \frac{7!}{4!} - 4!$

We are interested in counting the number of permutations of MISSISSIPPI that satisfy the outlined conditions.

Let c_1, c_2, and c_3 be the conditions that all I's consecutive, all P's are consecutive, and all S's are consecutive, respectively.

First, let us focus on $N(c_1)$. This is the number of sequences in which all I's are consecutive. We can treat the the 4 I's as a single block and arrange the other

7 letters with it. The order of the I's is irrelevant so there is only one way to arrange them. Thus, there are 8! ways to arrange these 8 blocks, but we must account for repetition. The word contains 2 P's and 4 S's, this means we must divide the number of ways to rearrange these letters within our sequence. This gives us $N(c_1) = \frac{8!}{4!2!}$. Similarly, $N(c_2) = \frac{10!}{4!4!}$ and $N(c_3) = \frac{8!}{4!2!}$.

Next, we can look at $N(c_1c_2)$. This is the number of sequences where all the I's are consecutive and all the P's are consecutive. Similar to above, we can look at the I's and P's as blocks where the order is irrelevant. We can arrange these blocks with the remaining 5 letters, giving us a total of 7 elements. Again, as there are 4 S's, we must divide by the number of ways to arrange the S's to account for repetition. This gives us that $N(c_1c_2) = \frac{7!}{4!}$. Similarly, $N(c_1c_3) = \frac{5!}{2!}$ and $N(c_2c_3) = \frac{7!}{4!}$.

Next, we can look at the number of sequences that satisfy c_1, c_2 and c_3. We can look at these consecutive sequences as blocks to arrange where, again, the order within them does not matter. With the remaining letter, there are 4 blocks. That means that $N(c_1c_2c_3) = 4!$.

The total number of permutations without restrictions is $N = \frac{11!}{4!4!2!}$ We can now solve for our desired result using PIE, which is $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3})$.

$$\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) &= N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) \\
&\quad - N(c_1c_2c_3) \\
&= \frac{11!}{4!4!2!} - \frac{8!}{4!2!} - \frac{10!}{4!4!} - \frac{8!}{4!2!} + \frac{7!}{4!} + \frac{5!}{2!} + \frac{7!}{4!} - 4!
\end{aligned}$$

14. Any question that specifies an upper bound on the value of x_i will require the use of the Principle of Inclusion-Exclusion.

 (a) 3 276.

 This is simply the same sort of problem seen in 3.5 (notice that there is no upper bound on x_i's). Thus, there are $\binom{4+25-1}{25} = 3\ 276$ solutions.

 (b) 348 integer solutions.

 Let $N(c_i)$ denote the number of integer solutions that satisfy the given equation where $x_i \geq 10$, for $i = 1, 2, 3, 4$.

 By symmetry, $N(c_1) = N(c_2 = N(c_3) = N(c_4)$, so we need only find $N(c_1)$.

 The value of $N(c_1)$ is the same as the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 15$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. So using the combination with repetition formula we see: $N(c_1) = N(c_i) = \binom{4+15-1}{15} = 816$.

 We then need to find $N(c_i c_j)$ for $i \neq j$ and $i, j = 1, 2, 3, 4$. Similarly, $N(c_i c_j)$ is the same as the number of integer solutions to $x_1+x_2+x_3+x_4 = 5$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. Thus $N(c_i c_j) = \binom{4+5-1}{5} = 56$.

 As we are looking for non-negative integer solutions we see that, $N(c_i c_j c_k) = 0$ for $i \neq j \neq k$ while $i, j, k = 1, 2, 3, 4$ and $N(c_1 c_2 c_3 c_4) = 0$.

 The total number of possible integer solutions without restrictions was found in part (a). We can now solve for our desired result using PIE. Thus,

$$N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) = 3\ 276 - 4(816) + 6(56) - 0 + 0 = 348$$

 (c) 1 509 integer solutions.

 Notice that this is the same as determining the number of solutions to

$x_1 + x_2 + x_3 + x_4 = 20$ where $0 \le x_1 \le 5$, $0 \le x_2 \le 3$, $0 \le x_3 \le 5$, and $0 \le x_4 \le 8$.

Let $N(c_1)$ denote the number of integer solutions that satisfy the given equation with $x_1 \ge 6$.

Let $N(c_2)$ denote the number of integer solutions where $x_2 \ge 4$.

Let $N(c_3)$ denote the number of integer solutions where $x_3 \ge 6$.

Let $N(c_4)$ denote the number of integer solutions where $x_4 \ge 9$.

$N(c_1)$ is equivalent the number of integer solutions to $x_1+x_2+x_3+x_4 = 14$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_1) = \binom{4+14-1}{14} = 680$.

$N(c_2)$ is equivalent the number of integer solutions to $x_1+x_2+x_3+x_4 = 16$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_2) = \binom{4+16-1}{16} = 969$.

$N(c_3)$ is equivalent the number of integer solutions to $x_1+x_2+x_3+x_4 = 14$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_3) = \binom{4+14-1}{14} = 680$.

$N(c_4)$ is equivalent the number of integer solutions to $x_1+x_2+x_3+x_4 = 11$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_4) = \binom{4+11-1}{11} = 364$.

$N(c_1c_2)$ is equivalent the number of integer solutions to $x_1+x_2+x_3+x_4 = 10$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_1c_2) = \binom{4+10-1}{10} = 286$.

$N(c_1c_3)$ equivalent is the number of integer solutions to $x_1+x_2+x_3+x_4 = 8$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_1c_3) = \binom{4+8-1}{8} = 165$.

$N(c_1c_4)$ is the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 5$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_1c_4) = \binom{4+5-1}{5} = 56$.

$N(c_2c_3)$ is the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 10$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_2c_3) = \binom{4+10-1}{10} = 286$.

$N(c_2c_4)$ is the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 7$ where $x_i \ge 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_2c_4) = \binom{4+7-1}{7} = 120$.

$N(c_3c_4)$ is the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 5$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_3c_4) = \binom{4+5-1}{5} = 56$.

$N(c_1c_2c_3)$ is the number of integer solutions to $x_1+x_2+x_3+x_4 = 4$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_1c_2c_3) = \binom{4+4-1}{4} = 35$.

$N(c_1c_2c_4)$ is the number of integer solutions to $x_1+x_2+x_3+x_4 = 1$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_1c_2c_4) = \binom{4+1-1}{1} = 4$.

$N(c_1c_3c_4)$ is the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = -1$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. There are no possible such solutions hence $N(c_1c_3c_4) = 0$.

$N(c_2c_3c_4)$ is the number of integer solutions to $x_1+x_2+x_3+x_4 = 1$ where $x_i \geq 0$ for $i = 1, 2, 3, 4$. Therefore $N(c_2c_3c_4) = \binom{4+1-1}{1} = 4$.

As we are looking for non-negative integer solutions: $N(c_1c_2c_3c_4) = 0$.

Putting this all together and applying the Principle of Inclusion-Exclusion we see that the number of integer solutions to the initial equation is:

$$\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) = {} & [N(c_1 + N(c_2 + N(c_3) + N(c_4)] + [N(c_1c_2) + N(c_1c_3 + N(c_1c_4) \\
& + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)] - [N(c_1c_2c_3) + N(c_1c_3c_4) \\
& + N(c_1c_2c_4) + N(c_2c_3c_4)] + N(c_1c_2c_3c_4)
\end{aligned}$$

$$\begin{aligned}
= {} & 3276 - [680 + 969 + 680 + 364] + [286 + 165 + 56 + 286 + 120 \\
& + 56] - [35 + 4 + 0 + 4] + 0 = 1\ 509.
\end{aligned}$$

15. $\binom{30}{5} - \binom{3}{1} \cdot \binom{23}{17} + \binom{3}{2} \cdot \binom{16}{10} - \binom{9}{3}$.

Let c_i represent the arrangements where box i has *more than* 6 marbles for $i = 1, 2, 3$.

To find $N(c_1)$, we can give the first box 7 marbles and then arrange the other 18 marbles without restrictions. This is an arrangement where order does not matter and repetition is allowed which implies that $N(c_1) = \binom{6+18-1}{18} = \binom{23}{18}$. Similarly, $N(c_2) = N(c_3) = \binom{23}{18}$.

In order to count the number of cases where both the first and second box have more than 6 marbles, we can give the first two boxes 7 marbles each and then arrange the remaining 11 marbles. This is an arrangement of 11 marbles into 6 boxes without restrictions. This implies that $N(c_1c_2) = \binom{6+11-1}{11} = \binom{16}{11}$. Similarly, $N(c_1c_3) = N(c_2c_3) = \binom{16}{11}$.

In order to count the number of cases where all three of the first three boxes contain more than 6 marbles, we can give each box 7 marbles and then distribute the other 4 marbles. This is simply an arrangement of 4 objects into 6 boxes, without restrictions. This implies $N(c_1c_2c_3) = \binom{6+4-1}{4} = \binom{9}{4}$.

The total number of arrangements of marbles without restrictions is the arrangement of 25 objects into 6 boxes. This implies that $N = \binom{30}{25}$. Thus:

$$\begin{aligned}
N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}) &= N - N(c_1) - N(c_2) - N(c_3) + N(c_1c_2) + N(c_1c_3) + N(c_2c_3) \\
&\qquad - N(c_1c_2c_3) \\
&= \binom{30}{5} - \binom{3}{1} \cdot \binom{23}{17} + \binom{3}{2} \cdot \binom{16}{10} - \binom{9}{3}
\end{aligned}$$

16. Let $x \in S$ and let n be the number of conditions, out of c_1, c_2, c_3, c_4, that are satisfied by the element x (i.e. $0 \leq n \leq 4$). We will consider the various possibilities for n and see which of the three terms in the equation x satisfies.

    If $n = 0$ then x satisfies none of the conditions outlined, thus x is included in both $N(\overline{c_2}\ \overline{c_3}\ \overline{c_4})$ and $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})$.

    If $n = 1$, let us assume without loss of generality that x satisfies c_1 and does not satisfy c_i for $i = 2, 3, 4$. Then x will be included in $N(\overline{c_2}\ \overline{c_3}\ \overline{c_4})$ and in $N(c_1\ \overline{c_2}\ \overline{c_3}\ \overline{c_4})$.

If x satisfies c_i for $i = 2, 3, 4$ then x will not be counted in any of the three terms of our equation, so we may ignore these cases.

For $n = 2, 3, 4$ then x is not counted in any of the three terms of the equation.

Thus, all cases have been considered and hence the two sides of our equation are equal.

4.2 Derangements: Nothing in its Right Place

Solutions:

1. Simply, a derangement is a permutation where no element appears is in its original position. Formally, a derangement is a function, f, on a set X such that for all $x \in X$, $f(x) \neq x$.

   One simple example of a derangement would be asking students to mark each other assignments such that no student can mark their own work.

2. Derangements are one specific application of the PIE. In order to find the number of ways to arrange items such that nothing is in its original place, we could use the PIE to exclude all the cases where things *are* in their right place. (Remember that PIE is defined by satisfying *none* of the conditions.)

3. $d(26)$.

   We are interested in permuting 26 elements such that none are in their original position; this is simply a derangement of 26 elements. Thus there are,

$$d(26) = 26! \sum_{k=0}^{26} \frac{(-1)^k}{k!},$$

   ways to derange the alphabet.

4. (a) $d(150)$.

   Every student must give one gift, but they will not be assigned themselves, this is simply a derangement of 150 elements. Thus the number of possible ways to draw names is,

$$d(150) = 150! \sum_{k=0}^{150} \frac{(-1)^k}{k!}$$

(b) $d(50) \cdot d(30)^2 \cdot d(40)$.

The way gift-givers are assigned in each grade is a derangement of a set of size equal to the number of students in that grade. We can then use the Rule of Product to find the total number of possibilities within the school. There are,

$$d(50) \cdot d(30)^2 \cdot d(40) = (50! \sum_{k=0}^{50} \frac{(-1)^k}{k!})(30! \sum_{k=0}^{30} \frac{(-1)^k}{k!})^2(40! \sum_{k=0}^{40} \frac{(-1)^k}{k!}),$$

ways the gift-givers can be assigned.

5. $5! \cdot d(5)$.

For the first round of interviews, Kalil can simply assign one outfit to each interview, giving 5! possible distributions of these outfits. For the second round of interviews, the outfit must be different at that job than worn in the first round. Hence, we would like to derange the five outfits, $d(5)$, from the first round. By the Rule of Product there are $5! \cdot d(5)$ ways for Kalil dress for these interviews.

6. $d(10) + 5 \cdot d(9) + 10 \cdot d(8) + 10 \cdot d(7) + 5 \cdot d(6) + d(5)$.

We solve this problem using cases depending on how many odd numbers remained in their original position.

Case 1: No odd numbers are in their place. Then every number is not in its original place and there are $d(10)$ such permutations.

Case 2: One odd number is in its original place. First we must determine which of the five odd numbers stayed, there are $\binom{5}{1} = 5$ options. The remaining 9 digits will be deranged, $d(9)$. By the Rule of Product there are $5 \cdot d(9)$ such permutations.

Case 3: Two odd numbers are not deranged. There are $\binom{5}{2} = 10$ choices of which odd numbers remained in their initial position. The remaining 8 digits will be deranged, $d(8)$. By the Rule of Product there are $10 \cdot d(8)$ such permutations.

The pattern is clear and we see that,
Case 4: $\binom{5}{3} \cdot d(7) = 10 \cdot d(7)$.

Case 5: $\binom{5}{4} \cdot d(6) = 5 \cdot d(6)$.

Case 6: $\binom{5}{5} \cdot d(5) = d(5)$.

By the Rule of Sum there are,

$$d(10) + 5 \cdot d(9) + 10 \cdot d(8) + 10 \cdot d(7) + 5 \cdot d(6) + d(5),$$

ways to permute the integers one through ten such that no even number is in its original position.

7. $n = 10$.

We begin by deranging the first 6 elements, which will be deranged but remain in the first 6 positions in the list, $d(6)$ ways to do this. We now derange the remaining $n-6$ digits amongst the last $n-6$ positions, which gives $d(n-6)$ ways to do this. By the Rule of Product, $d(6) \cdot d(n-6) = 2385$. Rearranging to solve for n, we obtain,

$$d(n-6) = \frac{2385}{d(6)} = \frac{2385}{265} = 9$$

By trial and error we see that $d(n-6) = d(4) = 9$, hence $n-6 = 4$ which gives $n = 10$.

8. (a) $d(8)$.

   This is a derangement of eight elements, $d(8)$ ways to distribute these meals.

   (b) $8! - d(8)$.

   Any permutation that is not a derangement will result in at least one person receiving the food they ordered. This can be calculated by taking the total number of possible ways the waiter could have delivered the food, $8!$, subtracted by the number of ways that no one gets the right meal, so $8! - d(8)$.

   (c) $28 \cdot d(6)$.

   We are unsure which two people will get the food they ordered, so we must multiply the derangement of the 6 plates, $d(6)$, by $\binom{8}{2} = 28$. Therefore there are $d(6) \cdot 28$ ways exactly two people receive what they ordered.

   (d) This is impossible. Assuming that the kitchen send out the correct orders, if one person gets the wrong meal in the group then someone else must have received a wrong meal as well.

9. (a) $[d(12)]^2$.

   There are $d(12)$ ways the drinks can be redistributed and $d(12)$ ways the meals can be redistributed, so by the Rule of Product there are $[d(12)]^2$ ways to take home the leftovers.

   (b) $[d(6)]^2$.

   There is only one way for vegetarians to take home there own dishes and there are $d(6)$ ways to redistribute their drinks. There is only one way for non-vegetarians to take home their own drinks and there are $d(6)$ ways the leftovers can be given out. So, by the Rule of Product there are $[d(6)]^2$ possible distributions.

   (c) $\sum_{k=0}^{12}(-1)^k[(12-k)!]^2\binom{12}{k}$.

   We will use PIE instead of derangements as adapting the formula of $d(n)$ is more complex.

   Let c_i be the case where i people bring home both their own drink and main dish for $1 \leq i \leq 12$.

   If there were no restrictions, the number of permutations would be $N = (12!)^2$, 12! possible distributions for the drinks, and 12! for the meals.

   If at least one person brings home their own main dish and drink, there are $\binom{12}{1}$ different ways this person could be chosen. There $(11!)^2$ ways to permute the remaining drinks and main dishes. This gives that $N(c_1) =$

$\binom{12}{1} \cdot (11!)^2$.

Similarly, if at least two people bring home their own main dish and drink, there $\binom{12}{2}$ different pairs of people who could take home their own things and $(10!)^2$ ways to permute the remaining drinks and main dishes. Thus, $N(c_2) = \binom{12}{2}(10!)^2$

This patterns continues for each $N(c_i)$.
Since we are looking for $N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}...\overline{c_{12}})$, using PIE we can see that:

$$N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}...\overline{c_{12}}) = (12!)^2 - \binom{12}{1} \cdot (11!)^2 + \binom{12}{2}(10!)^2 - .... + \binom{12}{12}(0!)^2$$
$$= \sum_{k=0}^{12}(-1)^k[(12-k)!]^2\binom{12}{k}$$

10. We will count the number of permutations of the numbers $1, 2, 3, ..., n$, which is certainly $n!$. Alternatively, for every possible permutation we can consider how there are k elements that have been deranged, and hence $n-k$ elements in their original positions for $0 \leq k \leq n$. The $n-k$ fixed elements can be selected in $\binom{n}{n-k} = \binom{n}{k}$ ways, with $d(k)$ ways that the k remaining elements can be deranged. We sum these cases from $k = 0$ to $k = n$ to account for all possible permutations, and the proof is complete since we've counted the same situation in two different ways.

4.3 Onto Functions and Stirling Numbers of the Second Kind

Solutions:

Note: There are two equivalent formulas for Stirling numbers:

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m$$

These formulas are equivalent as:

$$\begin{aligned}
\frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m &= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m + (-1)^n\binom{n}{n}(n-n)^m \\
&= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m + 0 \\
&= \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^m
\end{aligned}$$

Thus, you may use either in your solutions.

1. An onto function is a function $f : A \longrightarrow B$ where for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

2. The number of ways to distribute n different objects into m distinct containers where no container is left empty and $n \geq m$.

3. There are many possible examples, one example is the function $y = x$ where $x \in \mathbb{Z}$.

4. Since every element of B, which is the codomain/range, has been mapped to by f it follows that $|A| \geq |B|$.

5. There exists no surjective functions from C to D. As noted in question 3, the size of the domain no smaller than the codomain/range. In this case, $|D| > |C|$ and so no possible onto functions exist.

6. $\sum_{k=0}^{8}(-1)^k\binom{9}{k}(9-k)^{13}$

   For $f : A \to B$, $|A| = 13$ and $|B| = 9$. Therefore, there are $\sum_{k=0}^{8}(-1)^k\binom{9}{k}(9-k)^{13}$ such onto functions.

7. $2^4 - \sum_{k=0}^{1}(-1)^k\binom{2}{2-k}(2-k)^4 = 2.$

   The number of non-surjective functions g can be found by subtracting the number of surjective g functions from the total number of g functions.

   There are $2^4 = 16$ possible functions. Applying the formula for onto functions, we see that there are $\sum_{k=0}^{1}(-1)^k\binom{2}{2-k}(2-k)^4$ possible surjective functions. Hence there are only $16 - 14 = 2$ non-surjective functions from X to Y.

8. $\sum_{k=0}^{6}(-1)^k \binom{7}{7-k} (7-k)^{27}$

   This is essentially counting the number of onto functions from a set of size 27 to a set of size 7. Therefore, there are $\sum_{k=0}^{6}(-1)^k \binom{7}{7-k} (7-k)^{27}$ possible ways the students can group themselves.

9. $\sum_{k=0}^{2}(-1)^k\binom{3}{3-k}(3-k)^6 + \sum_{k=0}^{3}(-1)^k\binom{4}{4-k}(4-k)^6 = 2\ 100$

   To solve this problem, we must consider two distinct cases and then apply the sum rule.

   Case 1: Suppose Billi is assigned only the most expensive client. Then we are considering how to assign the remaining 6 accounts to the other 3 engineers such that they each get at least one client. This is precisely the number of onto functions from a set of size 6 to size 3, so there are $\sum_{k=0}^{2}(-1)^k\binom{3}{3-k}(3-k)^6 = 540$ ways of doing this.

Case 2: Suppose instead that Billi is assigned the most expensive client as well as other clients. Then we are left to assign the remaining 6 clients between the 4 engineers where each gets at least one client. This is the same as counting the number of onto functions from a set of size 6 to a set of size 4. Therefore there are exactly $\sum_{k=0}^{3}(-1)^k\binom{4}{4-k}(4-k)^6 = 1\ 560$ ways to do this.

Adding these cases together using the sum rule, we see that there are $540 + 1\ 560 = 2\ 100$ ways to assign the engineers to their clients so that Billi is always assigned the most expensive client.

10. A Stirling number of the second kind, denoted $S(m, n)$, is the number of ways to distribute m distinct objects into n identical containers with no container left empty.

    The formula is: $S(m, n) = \frac{1}{n!}\sum_{k=0}^{n-1}(-1)^k \binom{n}{n-k} (n-l)^m$. This is the formula for counting the number of onto functions from a set of size m to a set of size n divided by $n!$. The division by $n!$ is done to account for the identical "container".

11. We know the formula for the the number of onto functions from a set of size k to a set of size j is: $\sum_{n=0}^{j-1}(-1)^n \binom{j}{j-n} (j-n)^k$.

    By definition of Stirling numbers, we know $S(k, j) = \frac{1}{j!}\sum_{n=0}^{j}(-1)^n \binom{j}{j-n} (j-n)^k$.

    It is clear from the formulas that we can express the number of onto functions in terms of Stirling numbers as: $j! \cdot S(k, j)$.

12. (a) $\sum_{k=0}^{4}(-1)^k \binom{5}{5-k} (5-k)^{10} = 5\ 103\ 000$

This is simply the number of onto functions from a set of size 10 to a set of size 5. Thus, there are $\sum_{k=0}^{4}(-1)^k \binom{5}{5-k} (5-k)^{10} = 5\ 103\ 000$ way of distributing the stuffed animals.

(b) $\sum_{k=0}^{3}(-1)^k\binom{4}{4-k}(4-k)^9 + \sum_{k=0}^{4}(-1)^k\binom{5}{5-k}(5-k)^9 = 1\ 020\ 600$
First, put the collectable in the first bin. Now we must consider two cases depending on if we will put any other stuffed animals in the first bin.

Case 1: Suppose the collectable is the only stuffed animal in the first bin. Then we distribute the remaining 9 stuffed animals between the 4 distinct bins. This is the number of onto functions from a set of size 9 to a set of size 4. Thus, there are exactly $\sum_{k=0}^{3}(-1)^k\binom{4}{4-k}(4-k)^9 = 186\ 480$ ways to distribute the stuffed animals in this case.

Case 2: Suppose the first bin has more stuffed animals than just the collectable. Then we distribute other 9 stuffed animals between the 5 containers such that no container is left empty. There are $\sum_{k=0}^{4}(-1)^k\binom{5}{5-k}(5-k)^9 = 834\ 120$ ways to distribute the stuffed animals in this case.

Now apply the rule of sum to see that there are $186\ 480 + 834\ 120 = 1\ 020\ 600$ ways to distribute the stuffed animals such that no bin is left empty and the collectable is put into the first bin.

(c) $\sum_{n=1}^{5} S(10,n) = 86\ 472$

Since the bins are identical we will use Stirling numbers. We must consider multiple cases to account for how many containers are left empty.

Case 1: Suppose no containers are left empty. We can use part a and divide it by 5!, to account for the identical containers. In this case, there are $\frac{5\ 103\ 000}{5!} = 42\ 525 = S(10,5)$ ways to distribute the stuffed animals.

Case 2: Suppose 1 container is left empty. Now we are interested in how many ways we can assign 10 distinct stuffed animals into the 4 remaining identical containers so that none are left empty. There are $S(10, 4) = 34\ 105$ ways to do this.

Case 3: Suppose 2 containers are left empty. Now we are interested in how many ways we can assign 10 distinct stuffed animals into the 3 remaining identical containers so that none are left empty. In this case there are $S(10, 3) = 9\ 330$ ways to distribute the stuffed animals.

Case 4: Suppose 3 containers are left empty. In this case, there are $S(10, 2) = 511$ ways to distribute the stuffed animals.

Case 5: Suppose 4 containers are left empty. Then, there are $S(10, 1) = 1$ way to distribute the stuffed animals.

Case 6: It is not possible for all the containers to be left empty.

We now apply the Rule of Sum and see that there are $\sum_{n=1}^{5} S(10, n) = 42\ 525 + 34\ 105 + 9\ 330 + 511 + 1 = 86\ 472$ ways to distribute these stuffed animals.

13. We recall from number theory that any factor of 55 335 will be the product of some subset of the factors of 55 335. For example. $3 \cdot 5$ and $31 \cdot 3 \cdot 17$ are both factors of 55 335.

    (a) $S(5, 2) = 15$

    We are essentially looking at the number of ways to distribute 5 distinct numbers into 2 identical "containers", the factors, such that no container is left empty. This is because each factor must be greater than 1.

    We know these containers are identical as multiplication is communicative and so the order of the two factors is irrelevant. Thus, there are

$S(5, 2) = 15$ 2-factor factorizations of 55 335.

(b) $\sum_{n=2}^{5} S(5, n) = 15 + 25 + 10 + 1 = 51$

We will consider cases since 55 335 can be written as the product of up to 5 factors. there can be up to 5-factor factorizations of 55 335 since it is made up of 5 distinct prime numbers.

Case 1: How many 2-factor factorizations exist? This follows directly from part (a): 15.

Case 2: How many 3-factor factorizations exist? In other words, in how many ways can we distribute 5 distinct numbers into 3 identical containers/factors such that no container is left empty? $S(5, 3) = 25$.

Case 3: How many 4-factor factorizations exist? $S(5, 4) = 10$.

Case 4: How many 5-factor factoriations exist? $S(5, 5) = 1$.

We can add these cases together using the Rule of Sum to see that there are $\sum_{n=2}^{5} S(5, n) = 15 + 25 + 10 + 1 = 51$ possible factorizations where no factor is 1.

14. (a) $$\begin{aligned} S(n, 1) &= \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \binom{1}{1-k} (1-k)^n \\ &= \binom{1}{1-0} (1-0)^n + \binom{1}{0} (1-1)^n \\ &= 1 \cdot 1 + 0 \cdot 0 \\ &= 1 \end{aligned}$$

(b) $$S(n,2) = \frac{1}{2!}\sum_{k=0}^{2}(-1)^k\binom{2}{2-k}(2-k)^n$$
$$= \frac{1}{2!}\left[(-1)^0\binom{2}{2}(2-0)^n + (-1)^1\binom{2}{1}(1)^n + (-1)^2\binom{2}{0}(0)^n\right]$$
$$= 2^{n-1} - 1$$

(c) $$S(n,n-1) = \frac{1}{(n-1)!}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{n-1-k}(n-1-k)^n$$
$$= \frac{1}{(n-1)!}[(-1)^0\binom{n-1}{n-1}(n-1)^n + (-1)^1\binom{n-1}{n-2}(n-2)^n + ...$$
$$+ (-1)^{n-2}\binom{n-1}{1}(1)^n + (-1)^{n-1}\binom{n-1}{0}(0)^n]$$
$$= \frac{1}{(n-1)!}\left(\frac{n!(n-1)!}{(n-2)!2!}\right)$$
$$= \frac{n!}{(n-2)!2!}$$
$$= \binom{n}{2}$$

5 Generating Functions

5.1 Introductory Examples

5.2 Definition and Examples: Calculating Techniques

Solutions:

1. The pattern for both this question and the one below it is, starting from 0, the i^{th} term in the sequence is the coefficient of x^i.

   (a) $$1 + 2x + 3x^2 + 4x^3 + ... + nx^{n+1} + ... = \sum_{k=0}^{\infty} \binom{k+1}{1} x^k = \frac{1}{(1-x)^2}$$

   (b) $$5 + 4x + 3x^2 = \sum_{k=0}^{2} \binom{5-k}{1} x^k.$$

   (c) $$1 - x + x^2 - x^3 + x^4 - x^5 + ... + (-1)^n x^n + ... = \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1+x}.$$

   (d) $$\binom{10}{10} + \binom{11}{10}x + \binom{12}{10}x^2 + ... = \sum_{k=0}^{\infty} x^k \binom{10+k}{10} = \frac{1}{(1-x)^{11}}$$

   (e) $$\binom{10}{10} - \binom{11}{10}x + \binom{12}{10}x^2 - \binom{13}{10}... = \sum_{k=0}^{\infty} (-x)^k \binom{10+k}{10} = \frac{1}{(1+x)^{11}}$$

   (f) $$1 + x^2 + x^4 + ... = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$$

   (g) $$1 - 2x + 4x^2 - 8x^3 + 16x^4 - 32x^5 = \sum_{k=0}^{5} (-2x)^k = \frac{1-(-2x)^6}{1-(-2x)} = \frac{1-64x^6}{1+2x}$$

2. (a) $0, 0, 0, ...$

   (b) $0, 1, 0, 0, 0...$

   (c) $4, 3, -10, 55.$

(d) $-64, 144, -108, 27.$

Since,

$$(3x-4)^3 = 27x^3 - 108x^2 + 144x - 64$$

(e) $0, 3, 3, 3, 3, ...$

Since,

$$\frac{3x}{1-x} = 3x \cdot \sum_{k=0}^{\infty} x^k = 3x(1) + 3x(x) + 3x(x^2) + ..$$

(f) $1, 6, 27, 108, ...$

Since,

$$\frac{1}{(1-3x)^2} = \sum_{k=0}^{\infty} \binom{k+1}{1} (3x)^k = \sum_{k=0}^{\infty} (k+1)(3x)^k$$

.

3. (a) 1.

We know,

$$\frac{1}{1-x} = \sum_{r=0}^{\infty} x^r$$

Therefore x^3 occurs when $r = 3$ which has a coefficient of 1.

(b) 24.

We know,

$$\frac{1}{(1-x)^n} = \sum_{r=0}^{\infty} \binom{r+n-1}{n-1} x^r$$

So we can determine that,

$$\frac{1}{(1-2x)^3} = \sum_{r=0}^{\infty} \binom{r+3-1}{2} (-2)^r x^r$$

Thus the coefficient of x^2 occurs when $r = 2$ which gives a coefficient of 24.

(c) 1.

We know,

$$\frac{1-x^{n+1}}{1-x} = \sum_{r=0}^{n} x^r$$

This gives $n+1 = 8$ hence $n = 7$. Simply we notice that the coefficient of x^5 in this expansion will be 1.

(d) 57 915.

We know,

$$\frac{1}{(1+x)^n} = \sum_{r=0}^{\infty} (-1)^r \binom{r+n-1}{n-1} x^r$$

Therefore,

$$\frac{1}{(1+3x)^{10}} = \sum_{r=0}^{\infty} (-1)^r \binom{r+10-1}{10-1} (3x)^r$$

Thus the coefficient of x^4 occurs when $r = 4$ and is 57 915.

4. 3 246.

We first notice that thinking of this problem in terms of stacks of pamphlets rather than the pamphlets themselves reduces it to: "In how many ways can $\frac{1000}{50} = 20$ stacks be distributed to five different counselling centers such that each center receives at least $\frac{50}{50} = 1$ but no more than $\frac{500}{50} = 10$ stacks?"

The generating function that represents this set up is,

$$g(x) = (x^1 + x^2 + ... + x^{10})^5,$$

and we are interested in determining the coefficient of x^{20}. Alternatively, we can identify the coefficient of x^{15} in,

$$g'(x) = (1 + x + ... + x^9)^5,$$

which was obtained by factoring out an x.

We now rewrite this using what we know about series,

$$\begin{aligned} g'(x) &= (\frac{1-x^{10}}{1-x})^5 \\ &= (-x^{50} + 5x^{40} - 10x^{30} + 10x^{20} - 5x^{10} + 1)\frac{1}{(1-x)^5} \\ &= (-x^{50} + 5x^{40} - 10x^{30} + 10x^{20} - 5x^{10} + 1) \cdot \sum_{r=0}^{\infty} \binom{r+5-1}{5-1} x^r. \end{aligned}$$

When this expression is expanded, we are interested in the coefficients of x when $r = 15, 5$, which correspond to the coefficients of x^{15}.

When $r = 15$, $\binom{15+5-1}{5-1} = 3\ 876$. When $r = 5$, $\binom{5+5-1}{5-1} = 126$. Therefore the coefficient of x^{15} in $g'(x)$ is $(-5)(126) + (1)3\ 876 = 3\ 246$, which is the number of ways these stacks of pamphlets can be distributed.

5. (a) 120.

Note: We have previously solved this using combinations with repetition, try with a generating function now.

The generating function that represents this problem is,

$$g(x) = (x^2 + x^3 + ...)^3,$$

and we are interested in determining the coefficient of x^{20}. Alternatively, factoring out some x's, we can consider finding the coefficient of x^{14} in,

$$g'(x) = (1 + x + x^2 + ...)^3.$$

Using what we know about this function we can rewrite it as,

$$\begin{aligned} g'(x) &= (1 + x + x^2 + ...)^3 \\ &= [\frac{1}{1-x}]^3 \\ &= \frac{1}{(1-x)^3} \\ &= \sum_{r=0}^{\infty} \binom{r+3-1}{3-1} x^r. \end{aligned}$$

The coefficient of x^{14} occurs precisely when $r = 14$, which gives the coefficient $\binom{14+3-1}{3-1} = 120$.

(b) 48.

This problem boils down to finding the coefficient of x^{20} in the the generating function,

$$g(x) = (x^3 + x^4 + x^5 + ... + x^{10})^3$$

Alternatively, we can factor out some x's and find the coefficient of x^{11} in,

$$g'(x) = (1 + x + x^2 + .. + x^7)^3.$$

Rewriting $g'(x)$,

$$\begin{aligned}
g'(x) &= (1 + x + x^2 + .. + x^7)^3 \\
&= [\frac{1 - x^8}{1 - x}]^3 \\
&= (1 - x^8)^3 \frac{1}{(1 - x)^3} \\
&= (-x^{24} + 3x^{16} - 3x^8 + 1) \frac{1}{(1 - x)^3} \\
&= (-x^{24} + 3x^{16} - 3x^8 + 1) \sum_{r=0}^{\infty} \binom{r + 3 - 1}{3 - 1} x^r.
\end{aligned}$$

The coefficient of x^{11} will occur in the expansion twice, when $r = 11, 3$. For $r = 11$, $\binom{11+3-1}{3-1} = 78$, and $r = 3$, $\binom{3+3-1}{3-1} = 10$. Therefore putting it all together, the coefficient of x^{11} is $(1)(78) + (-3)(10) = 48$.

(c) 21.

As we have already found our generating function for this condition and number of boxes, we are simply looking for the coefficient of x^{25} in $g(x)$, or x^{16} in $g'(x)$.

From (b),

$$g'(x) = (-x^{24} + 3x^{16} - 3x^8 + 1) \sum_{r=0}^{\infty} \binom{r + 3 - 1}{3 - 1} x^r$$

The coefficient of x^{16} will occur thrice when $r = 16, 8, 0$. When $r = 16$,

$\binom{16+3-1}{3-1} = 153$, when $r = 8$, $\binom{8+3-1}{3-1} = 45$, and when $r = 0$, $\binom{0+3-1}{3-1} = 1$. Putting it all together, the coefficient of x^{16}, and hence the number of ways to distribute these 25 identical balls into three distinct boxes while satisfying the conditions is $1(153) + (-3)(45) + 3(1) = 21$.

6. 14.

We represent the eldest child's potential share of the money by $x^4+x^5+x^6+....$ The middle child's, $x^2 + x^3 + x^4 + ...$ The youngest child's, $x^2 + x^3 + x^4 + x^5$.

To determine the number of ways the loonies can be distributed, we are looking for the coefficient of x^{12} in the product,

$$g(x) = (x^4 + x^5 + x^5 + ...)(x^2 + x^3 + x^4 + x^5 + ...)(x^2 + x^3 + x^4 + x^5)$$

We can simplify $g(x)$,

$$\begin{aligned} g(x) &= (x^4 + x^5 + x^5 + ...)(x^2 + x^3 + x^4 + x^5 + ...)(x^2 + x^3 + x^4 + x^5) \\ &= x^8(1 + x + x^2 + ...)(1 + x + x^2 + x^3 + ...)(1 + x + x^2 + x^3) \\ &= x^8(1 + x + x^2 + ...)^2(1 + x + x^2 + x^3). \end{aligned}$$

Alternatively we can reduce this problem to identifying the coefficient of x^4 in,

$$g'(x) = (1 + x + x^2 + ...)^2(1 + x + x^2 + x^3)$$

Using identities and some substitutions we rewrite $g'(x)$ as,

$$\begin{aligned}
g'(x) &= (1+x+x^2+...)^2(1+x+x^2+x^3) \\
&= \frac{1}{(1-x)^2}\cdot\frac{1-x^4}{1-x} \\
&= \frac{(1-x^4)}{(1-x)^3} \\
&= (1-x^4)\sum_{r=0}^{\infty}\binom{r+3-1}{3-1}x^r.
\end{aligned}$$

The coefficient of x^4 will occur when $r=0,4$. When $r=0$, $\binom{0+3-1}{3-1}=1$, when $r=4$, $\binom{4+3-1}{3-1}=15$. Putting this together, there are, $15(1)-1=14$ ways to distribute the loonies.

7. $\frac{1}{8}+\frac{1}{4}\binom{n+1}{1}+\frac{1}{2}\binom{n+2}{2}+\frac{1}{8}(-1)^n$.

The expression, $(1+x+x^2+...)$, will help keep track of the pink and orange balls, while the expression, $(1+x^2+x^4+...)$, will keep track of the even black balls. We are interested in determining the coefficient of x^n in the product,

$$g(x) = (1+x+x^2+x^3+...)^2(1+x^2+x^4+x^6+...)$$

From our identities we see that,

$$\begin{aligned}
g(x) &= [\frac{1}{1-x}]^2\cdot\frac{1}{1-x^2} \\
&= \frac{1}{(1-x)^2}\cdot\frac{1}{1-x^2} \\
&= \frac{1}{(1-x)^2(1-x^2)} \\
&= \frac{1}{-(x-1)^3(x+1)}.
\end{aligned}$$

We need a different, simpler, way to express $g(x)$, so we use a partial fraction expansion,

$$\begin{aligned} g(x) &= \frac{-1}{(x-1)^3(x+1)} \\ &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}. \end{aligned}$$

Multiplying both the left and right hand sides by the common denominator, $(x-1)^3 \cdot (x+1)$, we obtain,

$$-1 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3.$$

Expanding and applying the binomial theorem where necessary we see that,

$$\begin{aligned} -1 &= A(x^3 - x^2 - x + 1) + B(x^2 - 1) + C(x+1) + D(x^3 - 3x^2 + 3x - 1) \\ &= Ax^3 - Ax^2 - Ax + A + Bx^2 - B + Cx + C + Dx^3 - 3Dx^2 + 3Dx - D \\ &= (A+D)x^3 + (-A+B-3D)x^2 + (-A+C+3D)x + (A-B+C-D). \end{aligned}$$

We know the coefficient of x^3, x^2, x are 0, so we match the coefficients with each other and solve for the unknowns:

$$\begin{aligned} A + D &= 0 \\ -A + B - 3D &= 0 \\ -A + C + 3D &= 0 \\ A - B + C - D &= -1. \end{aligned}$$

We must now solve this system of equations. Clearly from the first equation, $D = -A$. Plugging this into the second equation,

$$-A + B - 3(-A) = -A + B + 3A = B + 2A = 0,$$

which implies that $B = -2A$. Plugging these into the *fourth* equation we obtain,

$$A - (-2A) + C - (-A) = A + 2A + C + A = 4A + C = -1,$$

which implies that $C = -1 - 4A$. Finally we can plug everything into the third equation and solve for A,

$$-A + (-1 - 4A) + 3(-A) = -A - 1 - 4A - 3A = -8A - 1 = 0.$$

Certainly from this, $A = \frac{-1}{8}$. We further see that $D = \frac{1}{8}$, $B = -2 \cdot \frac{-1}{8} = \frac{1}{4}$ and $C = -1 - 4 \cdot \frac{-1}{8} = -1 + \frac{1}{2} = \frac{-1}{2}$.

Hence,

$$\begin{aligned} g(x) &= \frac{-1}{8(x-1)} + \frac{1}{4((x-1)^2)} + \frac{-1}{2((x-1)^3)} + \frac{1}{8(x+1)} \\ &= \frac{1}{8(1-x)} + \frac{1}{4(1-x)^2} + \frac{1}{2(x-1)^3} + \frac{1}{8(1+x)}. \end{aligned}$$

We may now use our identities to express $g(x)$ in terms of sums,

$$g(x) = \frac{1}{8}\sum_{r=0}^{\infty} x^r + \frac{1}{4}\sum_{r=0}^{\infty}\binom{r+2-1}{2-1}x^r + \frac{1}{2}\sum_{r=0}^{\infty}\binom{r+3-1}{3-1}x^r + \frac{1}{8}\sum_{r=0}^{\infty}(-1)^r x^r.$$

The coefficient of x^n occurs when $n = r$, hence there are,

$$\frac{1}{8} + \frac{1}{4}\binom{n+1}{1} + \frac{1}{2}\binom{n+2}{2} + \frac{1}{8}(-1)^n,$$

ways to select n balls.

8. 336.

We first note that we will need to apply the Rule of Product here, and use generating functions for the distribution of the fries and then the mini-desserts. In each of the generating functions, the first term in the product will represent the share of the head chef, and the other terms the other employees.

We begin by distributing the fries. We obtain the function,

$$\begin{aligned} g(x) &= (x + x^2 + x^3 + ...)(x^2 + x^3 + x^4)^3 \\ &= x^7(1 + x + x^2 + ...)^4, \end{aligned}$$

and are interested in the coefficient of x^{12}. Alternatively we could determine the coefficient of x^5 in,

$$g'(x) = (1 + x + x^2 + ...)^4.$$

Rewriting,

$$\begin{aligned} g'(x) &= [\frac{1}{1-x}]^4 \\ &= \frac{1}{(1-x)^4} \\ &= \sum_{r=0}^{\infty} \binom{r+4-1}{4-1} x^r. \end{aligned}$$

The term x^5 occurs when $r = 5$, hence there are, $\binom{5+4-1}{3} = 56$ ways to distribute the fries.

We now distribute the desserts and obtain the equation,

$$g(x) = x^3(1 + x + ... + x^5)^3,$$

in attempt to find the coefficient of x^{16}. Alternatively, we may determine the coefficient of x^{13} in,

$$g'(x) = (1 + x + ... + x^5)^3.$$

Rewriting,

$$
\begin{aligned}
g'(x) &= [\frac{1-x^6}{1-x}]^3 \\
&= \frac{(1-x^6)^3}{(1-x)^3} \\
&= (1-x^6)^3 \cdot \frac{1}{(1-x)^3} \\
&= (1-x^6)^3 \cdot \sum_{r=0}^{\infty} \binom{r+3-1}{3-1} x^r \\
&= (-x^{18} + 3x^{12} - 3x^6 + 1) \sum_{r=0}^{\infty} \binom{r+3-1}{3-1} x^r.
\end{aligned}
$$

Certainly the term x^{13} occurs when $r = 13, 7, 1$. When $r = 13$, $\binom{13+3-1}{2} = 105$, when $r = 7$, $\binom{7+3-1}{2} = 36$, and when $r = 1$, $\binom{1+3-1}{2} = 3$. Therefore there are exactly, $1(105) + (-3)(36) + 3(3) = 6$ ways to distribute the mini-desserts.

Now applying the Rule of Product, $56 \cdot 6 = 336$ ways the manager can distribute the leftovers as outlines by the constraints.

9. First we consider one subset which satisfies our condition, say $\{2, 5, 9, 14\}$. Certainly $1 \leq 2 < 5 < 9 < 14 \leq 15$. Now consider the differences between adjacent values in this chain of inequalities,

$$
\begin{aligned}
2 - 1 &= 1, \\
5 - 2 &= 3, \\
9 - 5 &= 4, \\
14 - 9 &= 5, \\
15 - 14 &= 1.
\end{aligned}
$$

Notice how the sum of these differences is 14. Consider any other subset that also satisfies the required conditions and you will see that upon doing the same calculation, the sum of the differences will be 14 too. In other words, the non-negative integers that are the differences that arise from the inequality, are a set of five non-negative integers that sum to 14. This means that there is a $1-1$ correspondence between these four-element subsets of our set, and the non-negative integer solutions to,

$$c_1 + c_2 + c_3 + c_4 + c_5 = 14,$$

where $0 \leq c_1, c_5$ and $2 \leq c_2, c_3, c_4$. The requirement of $c_2, c_3, c_4 \geq 2$ ensures that we have not picked any consecutive integers in the subset as we are now interested in *differences*. Now we can express our generating function with respect to these c_i's. We obtain the function,

$$g(x) = (1 + x + x^2 + ...)^2(x^2 + x^3 + x^4 + ...)^3,$$

and we are interested in the coefficient of x^{14}. Alternatively, by factoring out x^6, we can find the coefficient of x^8 in

$$g'(x) = (1 + x + x^2 + ...)^5.$$

Rewriting,

$$\begin{aligned} g'(x) &= (1 + x + x^2 + ...)^5 \\ &= [\frac{1}{1-x}]^5 \\ &= \frac{1}{(1-x)^5} \\ &= \sum_{r=0}^{\infty} \binom{r+5-1}{5-1} x^r. \end{aligned}$$

The term, x^8 occurs when $r = 8$ and has a coefficient of $\binom{8+5-1}{5-1} = 495$. Therefore there are 495 four-element subsets that contain no consecutive integers.

10. (a)

$$g(x) = (1 + x^2 + x^4 + ...)(x^3 + x^5 + x^7 + ...)(x^4 + x^6).$$

(b) i. 9.

We are looking for the coefficient of x^{15} in $g(x)$. Alternatively we can look for the coefficient of x^8 in

$$g'(x) = (1 + x^2 + x^4 + ...)(1 + x^2 + x^4 + ...)(1 + x^2)$$

.

Rewriting,

$$\begin{aligned} g'(x) &= (1 + x^2 + x^4 + ...)^2(1 + x^2) \\ &= [\frac{1}{1 - x^2}]^2 \cdot (1 + x^2) \\ &= \frac{1}{(1 - x^2)^2} \cdot (1 + x^2) \\ &= \frac{1 + x^2}{(1 - x^2)^2}. \end{aligned}$$

Using a partial fraction expansion we can write (steps omitted),

$$g'(x) = \frac{1}{2(x + 1)^2} + \frac{1}{2(x - 1)^2}.$$

Finally we can write $g'(x)$ in terms of summations,

$$\begin{aligned} g'(x) &= \frac{1}{2(x + 1)^2} + \frac{1}{2(1 - x)^2} \\ &= \frac{1}{2}\sum_{r=0}^{\infty}(-1)^r\binom{r + 2 - 1}{2 - 1}x^r + \frac{1}{2}\sum_{r=0}^{\infty}\binom{r + 2 - 1}{2 - 1}x^r. \end{aligned}$$

The coefficient of x^8 occurs when $r = 8$, hence there are $\frac{1}{2} \cdot (-1)^8 \cdot$

$\binom{8+1}{1} + \frac{1}{2}\binom{8+1}{1} = 9.$

ii. 0.

We can use the generating function from part (i) and this time solve for the coefficient of $x^{22-7} = x^{15}$. This occurs when $r = 15$. The coefficient is, $\frac{1}{2} \cdot (-1)^8 \cdot \binom{15+1}{1} + \frac{1}{2}\binom{15+1}{1} = 0.$

Logically this makes sense since there is no way to sum to 22 from two even numbers and one odd number.

11. In each of these questions, the solution would be the coefficient of the term x^k.

Recall: You were not asked to solve these problems, only determine the generating function that *could* solve them.

(a) The first two terms in the product will represent the possibilities for x_1 and x_2. The third term in the product will stand in for x_3, and the final term x_4.

$$\begin{aligned}
g(x) &= (1 + x + x^2 + ...)(1 + x + x^2 + ...)(x^2 + x^3 + x^4 + x^5)(x^4 + x^5 + ...) \\
&= (1 + x + x^2 + ...)^2 \cdot x^2(1 + x + x^2 + x^3) \cdot x^4(1 + x + x^2 + ...) \\
&= x^6(1 + x + x^2 + ...)^3(1 + x + x^2 + x^3) \\
&= x^6 \cdot [\frac{1}{1-x}]^3 \cdot \frac{1-x^4}{1-x} \\
&= \frac{x^6(1-x^4)}{(1-x)^4}.
\end{aligned}$$

$$
\begin{aligned}
\text{(b)}\ \ g(x) &= (1+x^2+x^4+...)(1+x^2+x^4+...)(1+x+...+x^5)(1+x+x^2) \\
&= (1+x^2+x^4+...)^2(1+x+...+x^5)(1+x+x^2) \\
&= [\frac{1}{1-x^2}]^2 \cdot \frac{1-x^6}{1-x} \cdot \frac{1-x^3}{1-x} \\
&= \frac{(1-x^6)(1-x^3)}{(1-x)^2(1-x^2)^2}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(c)}\ \ g(x) &= (x+x^2+x^3+...)(x^2+x^3+x^4+...)(x^3+x^4+x^5+...)(x^4+x^5+x^6+...) \\
&= x^{10}(1+x+x^2+...)^4 \\
&= x^{10}\frac{1}{(1-x)^4}.
\end{aligned}
$$

$$
\begin{aligned}
\text{(d)}\ \ g(x) &= (1+x^2+x^4+...)(1+x^3+x^6+...)(1+x+x^2+...)(1+x^3+x^6+...) \\
&= (1+x^2+x^4+...)(1+x^3+x^6+...)^2(1+x+x^2+...) \\
&= \frac{1}{1-x^2} \cdot [\frac{1}{1-x^3}]^2 \cdot \frac{1}{1-x} \\
&= \frac{1}{1-x^2} \cdot \frac{1}{(1-x^3)^2} \cdot \frac{1}{1-x} \\
&= \frac{1}{(1-x^2)(1-x^3)^2(1-x)}.
\end{aligned}
$$

12. We are interested in determining the coefficient of x^{50} in the product,

$$(1+x+x^2+...)(1+x^5+x^{10}+x^{15}+...)(1+x^{10}+x^{20}+...)(1+x^{25}+x^{50}+...).$$

The first term of the product represents the pennies used, the second term the nickels used, third the dimes used, and the last term stands in for the quarters.

We can rewrite this product as,

$$\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{25}},$$

which is our desired generating function.

13. (a) 8.

We will first set up the generating function. The first term of the product represents the even black cards, and the second term the odd red cards.

$$\begin{aligned} g(x) &= (1 + x^2 + x^4 + ...)(x + x^3 + x^5 + ...) \\ &= x(1 + x^2 + x^4 + ...)^2 \\ &= x \cdot \frac{1}{(1 - x^2)^2}. \end{aligned}$$

Using a partial fraction expansion (steps omitted) we see that,

$$g(x) = \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2}$$

Finally we write this as a series,

$$\begin{aligned} g(x) &= \frac{1}{4(x-1)^2} - \frac{1}{4(x+1)^2} \\ &= \frac{1}{4}\sum_{r=0}^{\infty}\binom{r+2-1}{2-1}x^r - \frac{1}{4}\sum_{r=0}^{\infty}(-1)^r \cdot \binom{r+2-1}{2-1}x^r. \end{aligned}$$

The coefficient of x^{15} occurs when $r = 15$. Therefore there are exactly $\frac{1}{4} \cdot \binom{16}{1} - \frac{1}{4} \cdot (-1)^{15} \cdot \binom{16}{1} = 8$ ways to pick 15 cards such that an even number of black cards and an odd number of red cards from a standard deck.

(b) 90.

First we note that there are $\frac{52}{4} = 13$ cards of each suit (the suit is the symbol, either hearts, spades, clubs or diamonds), so there may never be more than 13 of each symbol in the 15 cards.

The first term of the function will represent the clubs, the second the diamonds, the third the hearts and the fourth the spades. Our generating function is,

$$\begin{aligned} g(x) &= (x^2+x^3+...+x^{15})(x^2+x^3+...+x^{15})(x^2+x^3+x^4+x^5)(x^2+x^3+...x^6) \\ &= (x^2+x^3+...+x^{15})^2(x^2+x^3+x^4+x^5)(x^2+x^3+...x^6) \\ &= x^8(1+x+x^2+...+x^{13})(1+x+x^2+x^3)(1+x+x^2+x^3+x^4). \end{aligned}$$

We are interested in the coefficient of x^{15} in $g(x)$, or we may determine the coefficient of x^7 in,

$$g'(x) = (1+x+x^2+...+x^{13})(1+x+x^2+x^3)(1+x+x^2+x^3+x^4).$$

We now rewrite this as,

$$\begin{aligned} g(x) &= [\frac{1-x^{12}}{1-x}]^2 \cdot \frac{1-x^4}{1-x} \cdot \frac{1-x^5}{1-x} \\ &= \frac{1}{(1-x)^4} \cdot (x^{33}-x^{29}-x^{28}+x^{24}-2x^{21}+2x^{17}+2x^{16}-2x^{12}+x^9 \\ &\qquad -x^5-x^4+1) \\ &= \sum_{r=0}^{\infty} \binom{r+4-1}{4-1} x^r \cdot (x^{33}-x^{29}-x^{28}+x^{24}-2x^{21}+2x^{17}+2x^{16} \\ &\qquad -2x^{12}+x^9-x^5-x^4+1) \end{aligned}$$

Clearly when expanding this and attempting to find the coefficient of x^7, we will be interested in $r = 2, 3, 7$. When $r = 2$, $\binom{2+4-1}{4-1} = 10$. When $r = 3$, $\binom{3+4-1}{3} = 20$. When $r = 7$, $\binom{7+4-1}{3} = 120$. Therefore the coefficient of x^7 is $(-1)(10) + (-1)(20) + 1(120) = 90$, so there are 90 ways to pick up 15 playing cards in this way.

14. (a) Each student can receive any number of votes, so the generating function would be as follows.

$$(1 + x + x^2 + x^3 + ...)^3$$

To find the number of distributions of n votes, the above generating function would be used to find the coefficient of x^n.

(b) If every student votes for themselves then we know that each student receives at least one vote. Thus the generating function is:

$$(x + x^2 + x^3 + ...)^3 = x^3(1 + x + x^2 + x^3 + ...)^3$$

Similar to in (a), to find the number of distributions of n votes, the above generating function would be used to find the coefficient of x^n.

15. If two distinct dice are rolled, then we can form the following generating function:

$$g(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$$

If we are looking to obtain a sum of 7, then we are looking to find the coefficient of x^7 in $g(x)$.

We can do so by expanding $g(x)$:

$$\begin{aligned} g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \\ &= x^{12} + 2x^{11} + 3x^{10} + 4x^9 + 5x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 \end{aligned}$$

The coefficient of x^7 is 6 and thus there are 6 possible ways to throw two dice in order to obtain a sum of 7.

5.3 Partitions of Integers

Solutions:

1. A partition on a positive integer n is a collection of unordered positive integers that all sum to n.

2. Generating functions are helpful in determining the number of possible partitions of integers as we can use generating functions to represent the number of summands of each possible size. This makes finding the total number of possible summands much easier.

3. A Ferrers diagram is a visual representation of a partition of some integer n where each summand is represented by a vertical row of dots, and the rows are organized from largest to smallest moving from left to right. In a Ferrers diagram for an integer n, there will be n total dots.

4. There are many possible solutions. One example is:

$$54 = 1 + 10 + 20 + 5 + 8 + 7 + 3$$

5. There are 7 partitions of 5, 3 of which contain only unique summands. The partitions are:

$$\begin{aligned}
5 &= 1 + 1 + 1 + 1 + 1 \\
5 &= 1 + 1 + 1 + 2 \\
5 &= 1 + 2 + 2 \\
5 &= 1 + 1 + 3 \\
5 &= 1 + 4 \\
5 &= 2 + 3 \\
5 &= 5
\end{aligned}$$

6. We can use the following generating function:

$$g(x) = (1 + x + x^2 + ...)^5.$$

To solve this problem when 10 balls are distributed, we would determine the coefficient of x^{10}.

7. (a) $$\begin{aligned} g(x) &= (1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)...(x^k + x^{2k} + ...) \\ &= x^k(1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)...(1 + x^k + x^{2k} + ...) \\ &= x^k \frac{1}{(1-x)} \cdot \frac{1}{(1-x^2)} \cdot \frac{1}{(1-x^3)} \cdots \frac{1}{(1-x^k)} \\ &= x^k \prod_{i=1}^{k} \frac{1}{(1-x^i)} \end{aligned}$$

(b) $$\begin{aligned} g(x) &= (1 + x + x^2 + ...)(1 + x^3 + x^6 + ...)(1 + x^5 + x^{10} + ...)...(x^{2k+1} + x^{2(2k+1)} + ...) \\ &= x^{2k+1}(1 + x + x^2 + ...)(1 + x^3 + x^6 + ...)(1 + x^5 + x^{10} + ...)...(1 + x^{2k+1} + ...) \\ &= x^{2k+1} \prod_{i=1}^{k+1} \frac{1}{1-x^{2i-1}} \end{aligned}$$

(c) $$\begin{aligned} g(x) &= (1 + x)(1 + x^3)(1 + x^5)... \\ &= \prod_{i=0}^{\infty} (1 + x)^{(2i+1)} \end{aligned}$$

(d) $$\begin{aligned} g(x) &= (1 + x + x^2 + ...)(x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)... \\ &= x^2(1 + x + x^2 + ...)(1 + x^2 + x^4 + ...)(1 + x^3 + x^6 + ...)... \\ &= x^2 \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot ... \\ &= x^2 \prod_{i=1}^{\infty} \frac{1}{1-x^i} \end{aligned}$$

(e) $$\begin{aligned} g(x) &= (1+x+x^2+...)(1+x^2)(1+x^3+x^6+...)(1+x^4)... \\ &= \frac{1}{(1-x)}\cdot\frac{(1-x^4)}{(1-x^2)}\cdot\frac{1}{(1-x^3)}\cdot\frac{(1-x^8)}{(1-x^4)}\frac{1}{(1-x^5)}\cdot\frac{(1-x^{12})}{(1-x^6)}... \\ &= \frac{1}{(1-x)}\cdot\frac{\cancel{(1-x^4)}}{(1-x^2)}\cdot\frac{1}{(1-x^3)}\cdot\frac{\cancel{(1-x^8)}}{\cancel{(1-x^4)}}\frac{1}{(1-x^5)}\cdot\frac{\cancel{(1-x^{12})}}{(1-x^6)}... \\ &= \prod_{i=1}^{\infty}\frac{1}{1-x^i}, \text{where } i \not\equiv 0 \mod 4 \end{aligned}$$

(f) $$\begin{aligned} g(x) &= (1+x)(1+x^2)(1+x^3)... \\ &= \prod_{i=1}^{\infty}(1+x^i) \end{aligned}$$

(g) $$\begin{aligned} g(x) &= (1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4+x^6+x^8+x^{10})(1+x^3+...+x^{15})... \\ &= \prod_{i=1}^{\infty}(1+x^i+x^{2i}+x^{3i}+x^{4i}+x^{5i}) \end{aligned}$$

(h) $$\begin{aligned} g(x) &= (1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4+...+x^{10})...(1+x^{12}+x^{24}+...+x^{60}) \\ &= \prod_{i=1}^{12}(1+x^i+x^{2i}+x^{3i}+x^{4i}+x^{5i}) \end{aligned}$$

8. A distinct partition is one in which all the summands are distinct. For example, the summands of $3 = 1+1+1$ are not distinct, but the summands of $3 = 1+2$ are.

We begin by finding the generating function for the partitioning of n into distinct partitions. Let us denote this generating function by P_d.

$$P_d(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdot ....$$

We rewrite this as:

$$P_d(x) = \frac{(1-x^2)}{(1-x)}\cdot\frac{(1-x^4)}{(1-x^2)}\cdot\frac{(1-x^6)}{(1-x^3)}\frac{(1-x^8)}{(1-x^4)}\cdot ..$$

We can then cancel common factors:

$$
\begin{aligned}
P_d(x) &= \frac{(1-x^2)}{(1-x)} \cdot \frac{(1-x^4)}{(1-x^2)} \cdot \frac{(1-x^6)}{(1-x^3)} \cdot \frac{(1-x^8)}{(1-x^4)} \cdot ... \\
&= \frac{\cancel{(1-x^2)}}{(1-x)} \cdot \frac{\cancel{(1-x^4)}}{\cancel{(1-x^2)}} \cdot \frac{\cancel{(1-x^6)}}{(1-x^3)} \cdot \frac{\cancel{(1-x^8)}}{\cancel{(1-x^4)}} \cdot ... \\
&= \frac{1}{(1-x)} \cdot \frac{1}{(1-x^3)} \cdot \frac{1}{(1-x^5)} ...
\end{aligned}
$$

Next we will find the generating function corresponding to the number of odd partitions of n. Let us denote this generating function by P_o.

$$P_o(n) = (1 + x + x^2 + x^3 + ...)(1 + x^3 + x^6 + ...)(1 + x^5 + x^{10} + ...)....$$

We can rewrite this as:

$$P_o(n) = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdot ....$$

Certainly, we see that $P_o(n) = P_d(n)$. Thus, since the generating functions are equal, the coefficient of x^n will be equal as well implying our desired result.

9. We start by obtaining the generating function for the number of partitions of n in which no even summand is repeated, denoted $P_e(n)$:

$$P_e(n) = (1+x+x^2+x^3+...)(1+x^2)(1+x^3+x^6+...)(1+x^4)(1+x^5+x^{10}+...)...$$

We can rewrite this as:

$$P_e(n) = \frac{1}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdot ....$$

We can now cancel out some terms to simplify the formula:

$$\begin{aligned} P_e &= \frac{1}{(1-x)} \cdot \frac{\cancel{(1-x^4)}}{(1-x^2)} \cdot \frac{1}{(1-x^3)} \cdot \frac{\cancel{(1-x^8)}}{\cancel{(1-x^4)}} \frac{1}{(1-x^5)} \cdot \frac{\cancel{(1-x^{12})}}{(1-x^6)} \cdots \\ &= \prod_{i=1}^{\infty} \frac{1}{1-x^i}, \text{where } i \not\equiv 0 \mod 4 \end{aligned}$$

Next, we find a generating function to represent the partitions of n where no summand appears more than three times, denoted $P_3(n)$:

$$P_3(n) = (1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^9)(1+x^4+x^8+x^{12}) \cdot .....$$

Which we rewrite as,

$$P_3(n) = \frac{(1-x^4)}{(1-x)} \cdot \frac{(1-x^8)}{(1-x^2)} \cdot \frac{(1-x^{12})}{(1-x^3)} \cdot \frac{(1-x^{16})}{(1-x^4)} \cdot ....$$

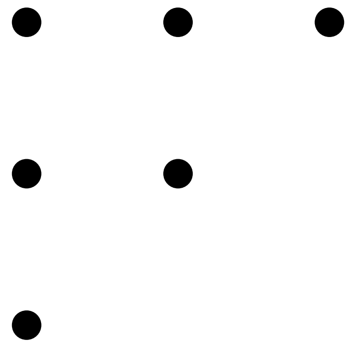
We can again some out some terms to simply our function:

$$\begin{aligned} P_3 &= \frac{\cancel{(1-x^4)}}{(1-x)} \cdot \frac{\cancel{(1-x^8)}}{(1-x^2)} \cdot \frac{\cancel{(1-x^{12})}}{(1-x^3)} \cdot \frac{\cancel{(1-x^{16})}}{\cancel{(1-x^4)}} \cdot \cdots \\ &= \prod_{i=1}^{\infty} \frac{1}{1-x^i}, \text{where } i \not\equiv 0 \mod 4 \end{aligned}$$

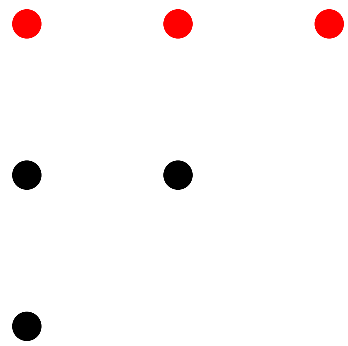
Thus, clearly $P_e = P_3$. As both generating function are equal the coefficient of x^n will be equal as well which implies our desired result.

10. This can be done by creating a one-to-one correspondence between any partitioning of n and any partitioning of $2n$ into n parts.

Let us take some partition of $2n$, where (arbitrarily) $n = 3$.

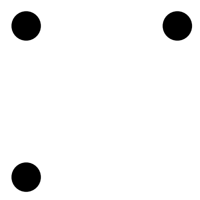


Notice that the top row in this Ferrers diagram has exactly n dots:

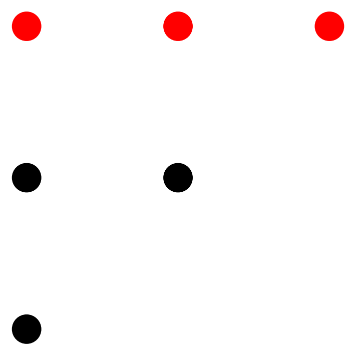


Deleting this row would create a Ferrers diagram for a partition of n. Each unique partition of $2n$ manipulated in this way will create a unique partition of n, as there is no way to remove only the top row and have two equivalent diagrams as then the top row would have to be what distinguishes two partitions of $2n$. The top row must always be n dots by the way the partitioning of $2n$ is defined so this is not possible.

Similarly, we can take an arbitrary diagram of a partition of n, let us again use $n = 3$ for simplicity sake.

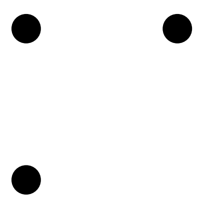


We can add a row to the top. This will add n dots and ensure there are eaxactly n partitions. This converts this diagram into a partition of $2n$ with exactly n partitions.

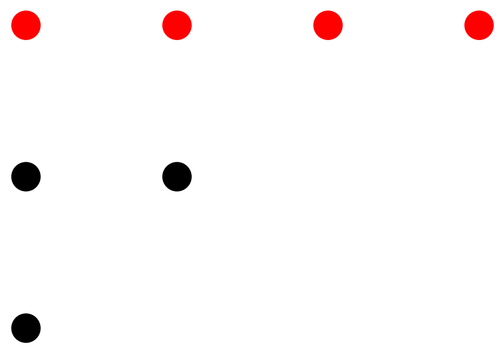


Thus, a one-to-one corresdence exists between the two sets of diagrams and so the two set sizes are equal.

11. If $n \leq m$, then it follows that the number of summands cannot exceed m. Let us take an arbitrary Ferrers diagram for the partitions of n. Let us take $n = 3$, arbitrarily.

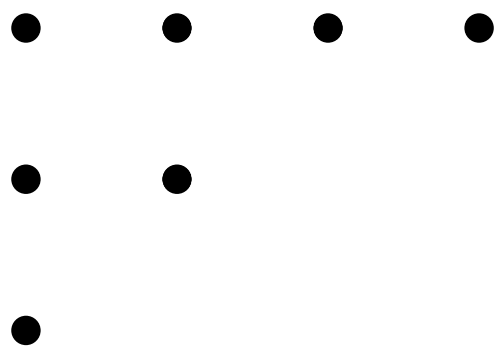


We can simply add a row to the top of the diagram with m dots. This will transform this Ferrers diagram into a partition of $n + m$ with exactly m partitions. Let us suppose $m = 4$.

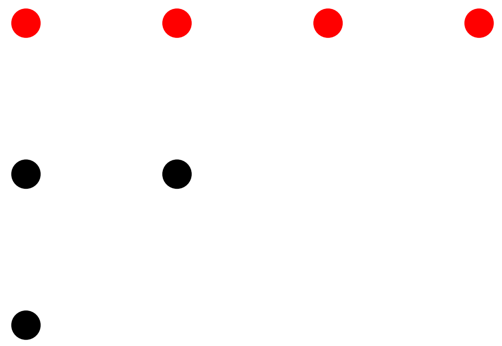


Thus every unique Ferrers diagram of a partition of n is equivalent to one unique partition of $m+n$ into m parts. The partition of $m+n$ must be unique as else the two partitions of n could not be unique.

Conversely, let us take some diagram of a partition of $m+n$ with exactly m summands. Let us again use that $n = 3, m = 4$.



Notice that the top row has exactly m dots.



If we remove these dots we are left with a partition of n. Thus we have a relation between any $m+n$ Ferrers diagram and a unique Ferrers diagram for n. The n diagram must be unique as if two Ferrers diagrams of partitions of $m+n$ create two identical partitions of n by removing the top row it must mean that the top row is what is distinct between them. That is not possible as these Ferrers diagrams have exactly m summands and so must have the same top row.

Thus there is a one-to-one correspondence between elements in both sets of partitions and so the size of the two sets must be equal.

6 Recurrence Relations

6.1 First-Order Linear Recurrence Relations

Solutions:

1. A recurrence relation is an expression for a function $f(n)$ that is defined in terms of previous terms, such as $f(n-1)$, with one or more initial values for $f(k)$ stated.

2. Solving a recurrence relation means determining a function, whose domain is the set of non-negative integers, that describes the recurrence relation for all $n \geq 0$ without solving for previous terms.

3. $a_6 = -(177) + 5(-17) = -262$

   As we are not solving the recurrence relation, we will use the provided recurrence formula to find the terms for $n = 2, 3, 4, 5$ to then find the value for $n = 6$.

$$\begin{aligned}
a_0 &= 2 \\
a_1 &= 7 \\
a_2 &= -7 + 5(2) = 3 \\
a_3 &= -3 + 5(7) = 32 \\
a_4 &= -32 + 5(3) = -17 \\
a_5 &= -(-17) + 5(32) = 177 \\
a_6 &= -(177) + 5(-17) = -262
\end{aligned}$$

   Solving a recurrence relation is better as you do not need to first find all the preceding terms in order to find the n^{th} term. As n gets large, computing each term by hand will become extremely difficult and tedious.

4. $a_n = 5 \cdot (-2)^n$

Determining the first few terms we see that:

$$\begin{aligned} a_1 &= -2(5) \\ a_2 &= -2[-2(5)] = (-2)^2 \cdot 5 \\ a_3 &= -2[(-2)^2 \cdot 5] = (-2)^3 \cdot 5. \end{aligned}$$

We can identify the pattern and see that $a_n = 5 \cdot (-2)^n$ for $n \geq 0$.

5. $a_n = 909(\frac{1}{3})^n$

We can reindex the function such that $a_{n+1} = \frac{1}{3}a_n$ for $n \geq 0$ and $a_2 = 101$. Since this recurrence relation is a *geometric progression* we know that the solution to this is $a_n = a_0(\frac{1}{3})^n$ so we have to determine is a_0.

We know $a_2 = 101 = a_0(\frac{1}{3})^2 = \frac{a_0}{9}$. So rearranging we obtain $a_0 = 909$. Thus the unique solution is $a_n = 909(\frac{1}{3})^n$ for $n \geq 0$.

6. $a_n = k \cdot (\frac{-6}{5})^n$

First we will rearrange the formula: $a_n = -\frac{6}{5}a_{n-1}$ for $n \geq 1$. We do not have an initial condition so we will let $a_0 = k$. Therefore the solution to this recurrence relation is $a_n = k \cdot (\frac{-6}{5})^n$.

7. $a_n = \frac{1296}{2401}(\frac{7}{2})^n$

Let $a_0 = k$. The solution will be of the form $a_n = k \cdot (\frac{7}{2})^n$. We know a_4, so we can find the unique solution by solving for a_0.

As $a_4 = 81 = a_0(\frac{7}{2})^4 = \frac{a_0 \cdot 2401}{16}$, it follows that $a_0 = 81 \cdot \frac{16}{2401} = \frac{1296}{2401}$. Therefore the solution is $a_n = \frac{1296}{2401}(\frac{7}{2})^n$.

8. $b_n = 25 \cdot 3^n$

If the annual interest rate is 8% then the monthly rate will be $\frac{8\%}{12} = 0.\overline{6}\% = 0.00\overline{6}$. For $0 \leq n \leq 16$, let a_n denote the money in the savings account at the end of the n^{th} month. Certainly $a_0 = \$1500$ and we can express $a_{n+1} = a_n + 0.00\overline{6}a_n = a_n(1.00\overline{6})$ for $n \geq 1$.

Solving this recurrence relation we see that $a_n = (\$1500)(1.00\overline{6})^n$ for $n \geq 0$. We can now simply solve for the 16^{th} month by plugging in $n = 16$, and we obtain $a_{16} = (\$1500)(1.00\overline{6})^{16} = \$1668.25$.

9. Make the substitution $b_n = a_n^2$. The recurrence relation now becomes $b_{n+1} = 3b_n$ where $n > 0$ and $b_0 = a_0^2 = 5^2 = 25$.

This is now a first order linear homogeneous recurrence relation. Thus, the solved recurrence relation is $b_n = 25 \cdot 3^n$ for $n \geq 0$.

10. (a) $a_0 = 0$, and $a_{n+1} = a_n + 2n$ for $n \geq 1$.

    (b) $a_0 = 7$, and $a_{n+1} = \frac{2 \cdot a_n}{5}$ for $n \geq 1$.

11. $d = \frac{2}{7}$

We can first rearrange the given recurrence function: $a_n = a_0 d^n$ for $n \geq 0$. Thus, we know that $a_3 = a_0 d^3 = \frac{-8}{343}$ and $a_5 = a_0 d^5 = \frac{-32}{16807}$.

We can solve for a_0 in both equations:

$$a_3 = a_0 d^5 = \frac{-8}{343}$$
$$a_0 = \frac{-8}{343 \cdot d}$$

$$a_5 = a_0 d^5 = \frac{-32}{16807}$$
$$a_0 = \frac{-32}{16807 \cdot d}$$

By equating these equations we have:

$$\begin{aligned}
\frac{-8}{343 \cdot d^3} &= \frac{-32}{16807 \cdot d^5} \\
-8 \cdot 16807 \cdot d^5 &= -32 \cdot 343 \cdot d^3 \\
134456 \cdot d^2 &= 10976 \\
d^2 &= \frac{10976}{134456} \\
&= \frac{4}{49} \\
d &= \sqrt{\frac{4}{49}} \\
&= \frac{2}{7}
\end{aligned}$$

Thus, $d = \frac{2}{7}$ and so our recurrence relation is $a_n = a_0 \cdot \frac{2}{7}$.

12. $5 \cdot (3^{36})$

Let n represent the number of hours the bacteria has been in the container, so $a_0 = 5$. Thus, we can use the recurrence relation $a_{n+1} = 3a_n$ where $n \geq 1$ to represent the bacteria growth. The unique solution to this relation is $a_n = 5 \cdot 3^n$ where $n \geq 0$.

One and a half days is equal to 36 hours, so we compute a_n for $n = 36$. Hence, there are $a_{36} = 5 \cdot (3^{36})$ bacteria after a day and a half.

13. (a) Determining the first few terms we see that:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= -5 \\ a_2 &= 25 \\ a_3 &= -125 \end{aligned}$$

Thus, we can see that $a_n = (-5)^n$, $n \geq 0$.

(b) Determing the first few terms we see that:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 16 \end{aligned}$$

Thus we see that $a_n = (4)^{n-1}$, $n \geq 1$

6.2 Second Order Linear Homogeneous Recurrence Relations with Constant Coefficients

Solutions:

1. The solution will be in the form:

$$a_n = c_1(r_1)^n + c_2(r_2)^n,$$

for $n \geq 0$ and r_1, r_2 come from factoring the initial function. Since the unique solution requires knowing c_1 and c_2, two unknowns, we will need at least two initial values to solve for these two unknowns. Thus, two initial values are needed.

2. To determine the characteristic equation of this second order homogeneous recurrence relation, first we isolate for 0 on one side:

$$C_0 a_n + C_1 a_{n-1} - C_2 a_{n-2} = 0$$

Next, we substitute $a_n = cr^n$ to obtain:

$$C_0 \cdot cr^n + C_1 \cdot cr^{n-1} - C_2 \cdot cr^{n-2} = 0$$

And so, by dividing every term by $c \cdot r^{n-2}$, we arrive at our characteristic equation:

$$C_0 r^2 + C_1 r - C_2 = 0$$

3. (a) $a_n = 19(4^n) - 14(5^n)$, for $n \geq 0$

The characteristic equation is

$$r^2 - 9r + 20 = 0$$

Factoring and solving for r, we obtain $r_1 = 4$ and, $r_2 = 5$. The two characteristic roots are real and distinct, hence 4^n and 5^n are both solutions. Thus our general solution is:

$$a_n = c_1(4^n) + c_2(5^n)$$

where c_1 and c_2 are arbitrary constants. We now use our initial conditions to solve for these constants.

$$a_0 = c_1(4^0) + c_2(5^0) = c_1 + c_2 = 5$$
$$a_1 = c_1(4^1) + c_2(5^1) = 4c_1 + 5c_2 = 6$$

We now solve this system of equations by substituting $c_1 = 5 - c_2$ into the second equation:

$$\begin{aligned} 6 &= 4(5 - c_2) + 5c_2 = 20 - 4c_2 + 5c_2 \\ -14 &= c_2 \end{aligned}$$

We can now solve for c_1:

$$c_1 = 5 - (-14) = 19$$

Therefore the unique solution to this recurrence relation is:

$$a_n = 19(4^n) - 14(5^n), \text{ for } n \geq 0$$

(b) $a_n = -2sin(\frac{\pi \cdot n}{2}), \text{ for } n \geq 0$

The corresponding characteristic equation is:

$$r^2 + 1 = 0,$$

with roots $r_1 = i, r_2 = -i$

Since the characteristic roots are complex conjugates, we know the general solution of this recurrence relation is given by,

$$a_n = c_1(i)^n + c_2(-i)^n,$$

for $n \geq 0$

We now use our initial conditions to solve for constants c_1 and c_2.

$$a_0 = 0 = c_1 + c_2,$$

$$a_1 = -2 = c_1 i - c_2 i = i(c_1 - c_2).$$

From equation one, $c_1 = -c_2$, and plugging this into the second equation we obtain:

$$-2 = i(-c_2 - c_2) = -2c_2$$

Therefore $c_2 = \frac{1}{i} = -i$ and $c_1 = i$. Thus:

$$a_n = i(i)^n - i(-i)^n = i^{n+1} + (-i)^{n+1} = (-i)((-i)^n - i^n)$$

We rewrite this in exponential form, to find the solution:

$$a_n = ie^{\frac{i\cdot\pi\cdot n}{2}} - ie^{\frac{-i\cdot\pi\cdot n}{2}} = -2sin(\frac{\pi \cdot n}{2}), \text{ for } n \geq 0$$

(c) $a_n = 7(-3)^n + 41n(-3)^{n-1}$

The characteristic equation for this recurrence relation is:

$$r^2 + 6r + 9 = 0.$$

Factoring this we find that $r = -3$. Thus, there is only one repeated root and so the general solution to this recurrence relation is:

$$a_n = c_1(-3)^n + c_2 n(-3)^n, \text{ for } n \geq 0$$

Using the initial conditions we see that:

$$a_0 = 7 = c_1 + 0$$

We can substitute c_1 into our equation now:

$$a_1 = 20 = 7(-3) + c_2(-3),$$

It follows that $c_2 = \frac{41}{-3}$. Thus our unique solution for $n \geq 0$ is:

$$a_n = 7(-3)^n + \frac{41n}{-3}(-3)^n = 7(-3)^n + 41n(-3)^{n-1}$$

(d) $a_n = (1+i)^n + (1-i)^n$ for $n \geq 0$

The characteristic equation for this recurrence relation is:

$$r^2 - 2r + 2 = 0$$

Using the quadratic formula we identify that the characteristic roots are complex conjugates, specifically, $r_1 = 1 - i, r_2 = 1 + i$.

As we have complex roots, the general solution is of the form:

$$a_n = c_1(1+i)^n + c_2(1-i)^n$$

Using our two initial conditions we can solve for c_1, c_2:

$$\begin{aligned}
a_0 &= 2 = c_1 + c_2 \\
c_2 &= 2 - c_1 \\
a_1 &= 2 = c_1(1+i) + c_2(1-i) \\
a_1 &= 2 = c_1(1+i) + (2-c_1)(1-i)2 && = c_1 + ic_1 + 2 - 2i - c_1 + ic_1 \\
2 &= 2ic_1 + 2 - 2i1 && = c_1c_2 = 1
\end{aligned}$$

Hence, the particular solution to this recurrence relation is:

$$a_n = (1+i)^n + (1-i)^n \text{ for } n \geq 0$$

. *Note:* An equivalent solution is:

$$a_n = (\sqrt{2}^n[cos(\frac{n\pi}{4})+i\cdot sin(\frac{n\pi}{4})])+(\sqrt{2}^n[cos(-n\frac{\pi}{4})+i\cdot sin(-n\frac{\pi}{4})]) \text{ for } n \geq 0$$

(e) $a_n = -\sqrt{3}^n + \frac{n(3+5\sqrt{3})}{3}\sqrt{3}^n$, for $n \geq 0$

The characteristic equation is:

$$r^2 - 2\sqrt{3}r + 3 = 0,$$

which can be factored as:

$$(r-\sqrt{3})^2 = 0.$$

Therefore the characteristic roots are $r_1 = r_2 = \sqrt{3}$. So our function has repeated real roots. Thus, the general solution is:

$$a_n = c_1(\sqrt{3})^n + c_2 n(\sqrt{3})^n \text{ for } n \geq$$

We can solve for our two unknowns by using the given initial conditions:

$$a_0 = -1 = c_1$$

We can now substitute. $c_1 = -1$ into the equation for a_1:

$$a_1 = 5 = -\sqrt{3} + c_2\sqrt{3}$$
$$c_2 = \frac{5+\sqrt{3}}{\sqrt{3}} = \frac{3+5\sqrt{3}}{3}$$

Hence, the particular solution to this recurrence relation is:

$$a_n = -\sqrt{3}^n + \frac{n(3+5\sqrt{3})}{3}\sqrt{3}^n, \text{ for } n \geq 0$$

(f) $a_n = -2(3^n) + 4^{n+1}$ for $n \geq 0$

The characteristic equation for this relation is:

$$r^2 - 7r + 12 = r$$

Factoring this equation we have that:

$$(r-3)(r-4) = 0$$

So the characteristic roots are $r_1 = 3$ and $r_2 = 4$. The two characteristic roots are real and distinct so our general solution is:

$$a_n = c_1(3^n) + c_2(4^n)$$

where c_1 and c_2 are constants. We can solve for c_1 and c_2 using our initial values:

$$\begin{aligned} a_0 = 2 &= c_1 + c_2 \\ a_1 = 6 &= 3c_1 + 4c_2 \\ 4 &= c_2 \\ -2 &= c_1 \end{aligned}$$

Thus, our recurrence relation is equal to:

$$a_n = -2(3^n) + 4^{n+1} \text{ for } n \geq 0$$

(g) $a_n = \frac{-3i}{2}(3+i)^n + \frac{3i}{2}(3-i)^n$ for $n \geq 0$

The characteristic equation for this relation is:

$$r^2 - 6r + 10 = 0$$

Thus, we can solve for the characteristic roots using the quadratic equation:

$$r_1 = 3 + i \text{ and } r_2 = 3 - i$$

Thus the roots are distinct and complex. The general solution then is:

$$a_n = c_1(3+i)^n + c_2(3-i)^n$$

where c_1 and c_2 are constants. Using the initial values we can solve for c_1 and c_2:

$$\begin{aligned}
a_1 &= 0 = c_1 + c_2 \\
&= -c_1 = c_2 \\
a_2 &= 3 = c_1(3+i) - c_1(3-i) \\
c_1 &= \frac{-3i}{2} \\
c_2 &= \frac{3i}{2}
\end{aligned}$$

Thus, our recurrence relation is equal to:

$$a_n = \frac{-3i}{2}(3+i)^n + \frac{3i}{2}(3-i)^n \text{ for } n \geq 0$$

(h) $a_n = -3(2^n) + 7n(2^n)$ for $n \geq 0$

The characteristic equation is:

$$r^2 - 4r + 4r = 0$$

Factoring this equation we can see that:

$$(r-2)^2 = 0$$

And so, $r_1 = r_2 = 2$. Thus, we have two repeated roots. This gives that our general solution is:

$$a_n = c_1(2^n) + c_2 n(2^n)$$

We can find the particular solution using the initial values given:

$$\begin{aligned} a_0 &= -3 && = c_1 \\ a_1 &= 1 && = -6 + 2c_2 \\ c_2 &= 7 \end{aligned}$$

Thus, the recurrence relation is equal to:

$$a_n = -3(2^n) + 7n(2^n) \text{ for } n \geq 0$$

(i) $a_n = \frac{5}{2}(2^n) + \frac{1}{2}(-2)^n$ for $n \geq 0$

The characteristic equation is equal to:

$$r^2 - 4 = 0$$

Thus, solving for the characteristic roots we can see that we have two distinct roots $r_1 = 2$ and $r_2 = -2$. Thus, the general solution is: $a_n = c_1(2^n) + c_2(-2)^n$ We can solve for the particular solution using our initial values:

$$\begin{aligned} a_0 = 3 &\qquad = c_1 + c_2 \\ a_1 = 4 &\qquad = 2c_1 - 2c_2 \\ 2 &\qquad = c_1 - c_2 \\ 5 &\qquad = 2c_1 \\ \frac{5}{2} = c_1 &\qquad \frac{1}{2} = c_2 \end{aligned}$$

Thus, the recurrence relation is equal to:

$$a_n = \frac{5}{2}(2^n) + \frac{1}{2}(-2)^n \text{ for } n \geq 0$$

4. $a_n = (-1 - \frac{412}{2\sqrt{5423}})(\frac{74-\sqrt{5423}}{53})^n + (\frac{413}{2\sqrt{5423}} - 1)(\frac{74+\sqrt{5423}}{53})^n$ for $n \geq 0$

We start by using the 4 initial conditions to solve for b, c. We obtain the system of equations:

$$14 + b(5) + c(-2) = 0$$
$$39 + b(14) + c(5) = 0$$

Thus: $b = \frac{-148}{53}$ and $c = 153$.

This means that the recurrence relation is:

$$a_n + \frac{-148}{53}a_{n-1} + \frac{1}{53}a_{n-2} = 0 \text{ for } n \geq 2$$

The corresponding characteristic equation is:

$$r^2 - \frac{148}{53}r + \frac{1}{53} = 0$$

Using the quadratic equation we find that the characteristic roots are, $x_1 = \frac{74-\sqrt{5423}}{53}$ and $x_2 = \frac{74+\sqrt{5423}}{53}$.

These are two distinct real roots, hence the general solution to this recurrence relation is:

$$a_n = c_1(\frac{74-\sqrt{5423}}{53})^n + c_2(\frac{74+\sqrt{5423}}{53})^n \text{ for } n \geq 0$$

We solve for the constants by using our initial values:

$$a_0 = -2 = c_1 + c_2$$
$$c_2 = -2 - c_1$$

We substitute this into our other equation:

$$a_1 = 5 = c_1(\frac{74-\sqrt{5423}}{53}) + (-2 - c_1)(\frac{74+\sqrt{5423}}{53})$$

rearranging and solving we obtain $c_1 = -1 - \frac{412}{2\sqrt{5423}}$ and hence $c_2 = \frac{413}{2\sqrt{5423}} - 1$. Thus our unique solution is,

$$a_n = (-1 - \frac{412}{2\sqrt{5423}})(\frac{74-\sqrt{5423}}{53})^n + (\frac{413}{2\sqrt{5423}} - 1)(\frac{74+\sqrt{5423}}{53})^n \text{ for } n \geq 0$$

5. $3a_n = 5a_{n-1} - 11a_{n-2}$ for $n \geq 2$. The initial conditions are $a_0 = a, a_1 = b$ for any $a, b \in \mathbb{R}$.

6. $a_n = \frac{\sqrt{5}}{5}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n]$ for $n \geq 0$

First, we must determine the recurrence relation. We can do so by trial and error and examining the sequence. We find that this sequence can be represented by:

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2, \text{ with } a_0 = 0, a_1 = 1$$

Now we must solve this, so we obtain the characteristic equation:

$$r^2 - r - 1 = 0$$

The characteristic roots are $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$, which can be seen using the quadratic equation. These roots are real and distinct, so the general solution to this relation is,

$$a_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n, \text{ for } n \geq 0$$

Finally, we use our initial conditions to find c_1, c_2:

$$\begin{aligned} a_0 = 0 &= c_1 + c_2 \\ c_1 &= -c_2 \\ a_1 = 1 &= c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) \\ 2 &= c_1 + \sqrt{5}c_1 + c_2 - \sqrt{5}c_2 \\ 2 &= -c_2 - \sqrt{5}c_2 + c_2 - \sqrt{5}c_2 = -2\sqrt{5}c_2 \\ c_2 &= \frac{-\sqrt{5}}{5} \end{aligned}$$

Thus, $c_2 = \frac{-\sqrt{5}}{5}$ and $c_1 = \frac{\sqrt{5}}{5}$.

Therefore the unique solution that describes the Fibonacci sequence is:

$$a_n = \frac{\sqrt{5}}{5}[(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n] \text{ for } n \geq 0$$

7. $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ with $a_1 = 2$ and $a_2 = 3$

Let a_n be the number of binary sequences of length n that have no consecutive 0's. We will split up a_n even further into the sequences that end in 0, a_n^0, and those that end in 1, a_n^1. Then certainly $a_n = a_n^0 + a_n^1$, for $n \geq 1$.

First, we notice that $a_1 = 2$, as the only possible sequences are "0" and "1".

We can build each sequence of length $n+1$ from sequences of length n by adding one addition term at the end as since the sequences of length n already satisfy that there are no consecutive 0's. Thus, we can see that

$$a_n = 2 \cdot a_{n-1}^1 + 1 \cdot a_{n-1}^0$$

This is because for any sequence of length $n-1$ that ends in "1", either "1" or "0" can be added to the end to form a sequence of length n while a sequence of length $n-1$ that ends in "0" can only have "1" added to the end to form a sequence of length n, else there would be consecutive "0"'s.

Now consider some sequences that belong to a_{n-2}. If the sequence $y \in a_{n-2}$ then $y1 \in a_{n-1}^1$, and vice versa. Thus, there is a one-to-one correspondence between the sets and so it follows that $a_{n-2} = a_{n-1}^1$.

We can now find our recurrence relation by putting all of this together:

$$\begin{aligned} a_n &= 2 \cdot a_{n-1}^1 + a_{n-1}^0 \\ &= a_{n-1}^1 + a_{n-1}^1 + a_{n-1}^0 \\ &= a_{n-1}^1 + a_{n-1} \\ &= a_{n-1} + a_{n-2} \end{aligned}$$

Thus, our recurrence relation is:

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3 \text{ with } a_1 = 2 \text{ and } a_2 = 3$$

8. $c_1 = 9$ and $c_2 = -18$

We know from the general solution that the characteristic roots of the characteristic equation must be $r_1 = 3, r_2 = 6$.

The characteristic equation, in terms of c_1, c_2 is $r^2 - c_1 r - c_2 = 0$. Since we know that r_1, r_2 must solve this equation, we can solve for our two unknowns:

$$r_1\colon\ 3^2 - 3c_1 - c_2 = 0,$$
$$r_2\colon\ 6^2 - 6c_1 - c_2 = 0.$$

The first equation implies that $c_2 = 9 - 3c_1$, so we can substitute this into the second equation: $36 - 6c_1 - 9 + 3c_1 = 27 - 3c_1 = 0$.

Hence, $c_1 = 9$ and $c_2 = -18$.

9. $a_n = c_1 + c_2 2^n,\ \text{for } n \geq 0$

This situation can be represented by the following equation:

$$a_n - a_{n-1} = 2(a_{n-1} - a_{n-2}),\ \text{for } n \geq 2$$

We can rearrange this equation to solve for a_n:

$$a_n = 3a_{n-1} - 2a_{n-2}$$

The characteristic equation for this is:

$$r^2 - 3r + 2 = 0$$

which has the roots $r_1 = 1, r_2 = 2$. Therefore the general solution to this relation is:

$$a_n = c_1 + c_2 2^n,\ \text{for } n \geq 0$$

Note: we are not able to find the particular/unique solution since we were not provided with initial conditions.

10. $o_n = 3^n - 2^n$

For $n = 0$, there are no possible words as there is no way to include the letter 'O' in a word of length 0. Thus $o_0 = 0$.

For $n = 1$, the only possible word is the letter 'O' itself. Thus, $o_1 = 1$.

For $n = 2$, the possible words are: OO, OW, ON, NO, WO. Thus, $o_2 = 5$.

To create a word of length n with with at least one O there are two cases:

(i) The first letter of the word is 'O'. Then the following $n-1$ letters can be any other letters. Thus, there are 3^{n-1} words in this case.

(ii) The first letter is either 'N' or 'W'. So, the following $n-1$ letters must include the letter 'O'. This means the following $n-1$ letters must be a valid word of length $n-1$. Thus, there are $2 \cdot o_{n-1}$ words in this case.

As these cases are disjoint, the recurrence equation is: $o_n = 3^{n-1} + 2w_{n-1}$, $o_0 = 0$.

Let the generating function for $o_0, o_1, o_2, \ldots$ be:

$$g(x) = \sum_{n=0}^{\infty} o_n x^n$$

Multiply the equation $o_n = 3^{n-1} + 2w_{n-1}$ by x^n and sum from $n = 1$:

$$\begin{aligned}
\sum_{n=1}^{\infty} o_n x^n &= 2\sum_{n=1}^{\infty} o_{n-1} x^n + \sum_{n=1}^{\infty} 3^{n-1} x^n \\
&= 2x \sum_{n=1}^{\infty} o_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 3^{n-1} x^{n-1} \\
&= 2x \sum_{n=0}^{\infty} o_n x^n + x \sum_{n=0}^{\infty} 3^n x^n
\end{aligned}$$

And so:

$$\begin{aligned} g(x) - w_0 = g(x) &= 2x\sum_{n=0}^{\infty} o_n x^n + x\sum_{n=0}^{\infty} 3^n x^n \\ &= 2xg(x) + \frac{x}{1-3x} \end{aligned}$$

And so:

$$\begin{aligned} g(x) &= \frac{x}{1-3x} \\ g(x) &= \frac{x}{(1-3x)(1-2x)} \end{aligned}$$

And so, using partial fractions, we have that:

$$\begin{aligned} g(x) &= \frac{x}{(1-3x)(1-2x)} \\ &= \frac{1}{(1-3x)} - \frac{1}{(1-2x)} \\ &= \sum_{n=0}^{\infty} 3^n x^n - \sum_{n=0}^{\infty} 2^n x^n \end{aligned}$$

Thus o_n is the coefficient of x^n and so: $o_n = 3^n - 2^n$.

We can also solve this problem using a simple counting argument. All possible words of length n, without restrictions, made with the letter 'O, W, N' is 3^n. All words with n letters that do not include 'O' are 2^n. Thus, the total words that do contain 'O' using PIE is: $3^n - 2^n$, which is the same result we found using generating functions.

11. $a_n = \frac{7}{4}3^n - \frac{3}{4} - \frac{n}{2}$

Let the generating function for $a_0, a_1, a_2, ...$ be:

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

.

Let us multiply the equation for a_n by x^n:

$$a_n x^n = 3x^n a_{n-1} + x^n n$$

We can now sum this equation from $n = 1$, as this is when the equation given begins:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} 3x^n a_{n-1} + \sum_{n=1}^{\infty} x^n n \\
&= x \sum_{n=1}^{\infty} 3x^{n-1} a_{n-1} + \sum_{n=1}^{\infty} x^n n \\
&= 3x \sum_{n=0}^{\infty} x^n a_n + \sum_{n=1}^{\infty} x^n n \\
&= 3xg(x) + \sum_{n=1}^{\infty} x^n n
\end{aligned}$$

We can refer to the table of generating functions to rewrite $\sum_{n=1}^{\infty} x^n n$:

$$\sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} \binom{n+2-1}{1} x^n = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} x^{n-1} n$$

Thus:

$$\begin{aligned}
\sum_{n=1}^{\infty} x^n \cdot n &= x \cdot \sum_{n=1}^{\infty} x^{n-1} n \\
&= \frac{x}{(1-x)^2}
\end{aligned}$$

Thus, we can substitute our values into our summation equation:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} 3x^n a_{n-1} + \sum_{n=1}^{\infty} x^n n \\
g(x) - g(0) &= 3x \cdot g(x) + \frac{x}{(1-x)^2} \\
g(x) - 1 &= 3x \cdot g(x) + \frac{x}{(1-x)^2} \\
g(x) - 3x \cdot g(x) &= 1 + \frac{x}{(1-x)^2} \\
g(x)(1-3x) &= 1 + \frac{x}{(1-x)^2} \\
g(x) &= \frac{1}{(1-3x)} + \frac{x}{(1-x)^2(1-3x)}
\end{aligned}$$

Now we can rewrite $\frac{x}{(1-x)^2(1-3x)}$ using partial fractions:

$$\frac{x}{(1-x)^2(1-3x)} = \frac{A}{(1-3x)} + \frac{B}{(1-x)} + \frac{C}{(1-x)^2}$$

And so:

$$x = A(1-x)^2 + B(1-x)(1-3x) + C(1-3x)$$

Thus:

$$\begin{aligned}
0 &= A + 3B \\
1 &= -2A - 4B - 3C \\
0 &= A + B + C
\end{aligned}$$

And so, solving for A, B, C shows:

$$\frac{x}{(1-x)^2(1-3x)} = \frac{3}{4(1-3x)} - \frac{1}{4(1-x)} - \frac{1}{2(1-x)^2}$$

Thus:

$$\begin{aligned}
g(x) &= \frac{1}{(1-3x)} + \frac{3}{4(1-3x)} - \frac{1}{4(1-x)} - \frac{1}{2(1-x)^2} \\
&= \frac{7}{4(1-3x)} - \frac{1}{4(1-x)} - \frac{1}{2(1-x)^2} \\
&= \frac{7}{4}\sum_{n=0}^{\infty} 3^n x^n - \frac{1}{4}\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n
\end{aligned}$$

Hence a_n which is the coefficient of x^n, is:

$$\begin{aligned}
a_n &= \frac{7}{4}3^n - \frac{1}{4} - \frac{1}{2}(n-1) \\
&= \frac{7}{4}3^n - \frac{1}{4} - \frac{n}{2} - \frac{1}{2} \\
&= \frac{7}{4}3^n - \frac{3}{4} - \frac{n}{2}
\end{aligned}$$

12. (a) $u_n = 2u_{n-1} + 2u_{n-2}$, where $u_0 = 1$

There is one valid word for $n = 0$, namely the empty word. So $u_0 = 1$

For $n = 1$, any single letter is a valid word (ie 'B', 'A', 'R'). So $u_1 = 3$.

There are two possible cases for words of length n:

(i) The word begins with 'B' or 'R'. Then the following $n-1$ letters are a valid word of length $n-1$. Thus, there are a total of $2u_{n-1}$ words of this case.

(ii) The word begins with ‘A’. Then the following letter must be ‘B’ or ‘R’. The remaining $n-2$ letters are any valid word of that length. Thus, there are a total of $2u_{n-2}$ in this case.

Thus: $u_n = 2u_{n-1} + 2u_{n-2}$, where $u_0 = 1$.

(b) $u_n = 2u_{n-2} + u_{n-1}$, where $u_0 = 1$

There is one valid word for $n = 0$, namely the empty word. So $u_0 = 1$.

For $n = 1$, any single letter is a valid word (ie ‘B’, ‘A’, ‘R’). So $u_1 = 3$.

There are two possible cases for words of length n:

(i) The word begins with ‘B’ or ‘R’. Then the following letter must be ‘R’. The remaining $n-2$ letters are any valid word of that length. Thus, there are a total of $2u_{n-2}$ in this case.

(ii) The word begins with ‘R’. Then the following $n-1$ letters must simply be a valid word of length $n-1$. Thus, there are a total of u_{n-1} words in this case.

Thus: $u_n = 2u_{n-2} + u_{n-1}$, where $u_0 = 1$.