Department of Mathematics

MATH 222 - Discrete and Combinatorial Mathematics

Practice Questions

Contributors:

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Acknowledgements

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- Kieka Mynhardt's notes, assignments, and tests for Math 222
- Introduction to Combinatorics and Graph Theory - Custom Edition for the University of Victoria
- Discrete Mathematics: Study Guide for MAT212-S - Dr. Kieka Myndardt
- Discrete Mathematics - Norman L. Biggs
- Applied Combinatorics, fourth edition - Alan Tucker
- Discrete Mathematics, An Introduction to Mathematical Reasoning - Susanna S. Epp
- Discrete Mathematics with Combinatorics - James A. Anderson

Contents

1 Preliminaries

1.1 Sets

Questions:

1. Consider the sets A and B where:

$$A = \{a \in \mathbb{Z} \mid a = 2k, \text{ for some integer k}\},$$

$$B = \{b \in \mathbb{Z} \mid b = 2j - 2, \text{ for some integer j}\}.$$

   Does $A = B$? If yes, prove it. If no, explain why not.

2. Consider the sets $A = \{1, 2, 3\}$, $B = \{x, y\}$, and $C = \{u, v\}$. Let $\mathcal{P}(A)$ denote the *powerset* of A. Find each of the following:

   (a) $\mathcal{P}(A \cup B)$

   (b) $\mathcal{P}(B \times C)$

   (c) $\mathcal{P}(\mathcal{P}(C))$

   (d) $A \times (B \cap C)$

   (e) $(A \times B) \times C$

3. Prove the following: The empty set is a subset of every set.

4. Prove or disprove: For the arbitrary sets A, B, and C, knowing that $A \subseteq B$ and $A \subseteq C$ implies that $A \subseteq B \cap C$.

5. Prove the following identity:

$$A \times (B \cup C) = (A \times B) \cup (A \times C),$$

for arbitrary, nonempty sets, A, B and C.

6. Prove that for any three arbitrary sets, A, B, and C, if $C \subseteq B - A$, then $A \cap C = \emptyset$.

7. If the statement about powersets is true, prove it, or find a counterexample if it is false. For all sets A and B,

   (a) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

   (b) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

   (c) If $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

8. Prove this statement of *De Morgan's Laws*:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

1.2 Relations and Graphs

Questions:

1. Given the relations, R, that are defined on the sets S, determine if R is reflexive, symmetric, transitive, and/or antisymmetric. Explain your reasoning.

   (a) Let S denote the set of all nonempty subsets of $\{a, b, c, d, e\}$ and define $A \ R \ B$ to mean that $A \cap B = \emptyset$, for $A, B \subseteq S$.

   (b) Let S be the set of all residents in Victoria, B.C., and $x \ R \ y$ means that x is a friend of y.
   Note: Assume that friendship goes both ways (i.e. if x is a friend of y, then y is a friend of x).

   (c) Let S be the set of ordered pairs of real numbers with $(x_1, x_2) \ R \ (y_1, y_2)$ if and only if $x_1 = y_1$ and $x_2 \leq y_2$.

   (d) Let Q be any nonempty set with $S = \mathcal{P}(Q)$. For all $X, Y \in S$, $X \ R \ Y$ if and only if $X \subseteq Y$.

2. Prove that the following relations, R, defined on the sets S are equivalence relations. Describe the equivalence class of $z \in S$, and determine the number of total equivalence classes of R.

   (a) Let S be the set of all positive integers. Let $x \ R \ y$ if and only if x and y have the same largest prime divisor. Describe the equivalence class of $z = 11$.

   (b) Let S be the set of ordered pairs of real numbers and define $(x_1, x_2) \ R \ (y_1, y_2)$ if and only if $x_1^2 + x_2^2 = y_1^2 + y_2^2$. Describe the equivalence class of $z = (2, 5)$.

3. Draw a directed graph corresponding to the relation, R, on the set $S = \{1, 2, 3, 4, 5, 6\}$.

   (a) $x \ R \ y$ if and only if y divides x.

(b) $R = \{(1,2),(2,3),(5,4),(1,1),(6,1),(2,5),(5,2),(3,3)\}$

4. Write an equivalence relation on the set $S = \{1,2,3,4,5,6\}$ that has the subsets $\{1,3,6\}$, $\{2,5\}$, and $\{4\}$ as the partition of its equivalence classes.

5. Given the set $S = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,5\},\{3,6\},\{4,6\},\{0,3,6\},$ $\{1,5,8\},\{0,3,4,6\}\}$ and the relation, R, being the *is a subset of* relation. Draw a Hasse Diagram for this partial order.

6. Let S be the set of all integers and $x\ R\ y$ if and only if $x \equiv y \pmod 5$. Is R a partial order? Explain why or why not. If R is a partial order, draw its Hasse diagram.

7. If S is a set with $|S| = k$, how many relations on S are:

   (a) symmetric?

   (b) antisymmetric?

   Hint: Consider the $k \times k$ matrix, M. 1 in the i^{th} row and j^{th} column means $m_i\ R\ m_j$, while 0 in the i^{th} row and j^{th} column denotes $m_i \not R\ m_j$. Count the possibilities that will result in the matrix which represents the given relation.

8. Let $S = \mathbb{N}$. If we define $a\ R\ b$ to mean that $\frac{b}{a} \in \mathbb{Z}$, is R antisymmetric?

2 Graph Theory

2.1 Graphing Preliminaries

2.2 Definitions and Basic Properties

Questions:

1. Draw the following graphs and determine how many edges each has.

   (a) K_4

   (b) $K_{3,2}$

   (c) $K_{1,5}$

2. How many edges are in

   (a) K_n?

   (b) $K_{m,n}$?

   For some positive integers m, n.

3. If a graph has five vertices of degree 4 and four vertices of degree 3, how many edges does it have?

4. Draw the following graphs, or explain they cannot exist.

   (a) A graph with an isolated vertex and a universal vertex.

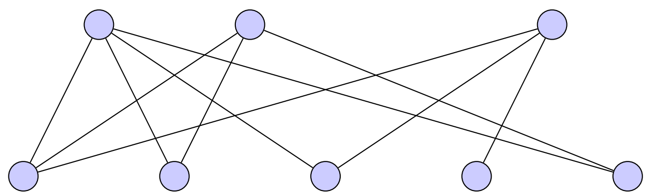
   (b) A cubic graph of order 5.

   (c) A bipartite graph of order 5 and size 7.

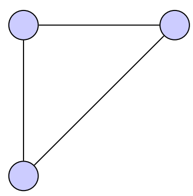
   (d) A bipartite graph of order 8 and size 10.

5. Can a graph have K_3 subgraph and be bipartite? Explain.

6. Let G be the following graph:



   (a) Is the following a subgraph of G?



   (b) Draw an induced subgraph of G with exactly 3 edges.

7. Draw a graph with K_4 as an induced subgraph.

8. A graph has 26 vertices and 58 edges. There are five vertices of degree 4, six vertices of degree 5, and seven vertices of degree 6. If the remaining vertices all have the same degree, what is this degree?

9. A graph has 24 vertices and 30 edges. It has five vertices of degree 4, seven pendant vertices, and seven vertices of degree 2. All other vertices have degree 3 or 4. How many vertices of degree 4 are there?

10. Use graph theory to explain why at any party an even number of people speak to an odd number of people.

11. Can there exist a graph on 13 vertices and 31 edges, with three vertices of degree 1, and seven vertices of degree 4? Explain.

12. Is the subgraph of a bipartite graph also bipartite? Would that change if bipartite graphs required edge sets to be non-empty? Explain.

13. If a graph G has 15 edges and all vertices of the same degree d, what are the possible values of d? Describe briefly each graph.

14. Given the following degree sequences either construct a graph with such a degree sequence, or explain why this would be impossible.

    (a) $1, 1, 1, 1, 1, 1$

    (b) $5, 4, 3, 2, 1$

    (c) $6, 6, 4, 2, 2, 2, 2, 1$

15. How many (simple) graphs are there with exactly n vertices?

16. What is the maximum number of vertices on a graph that has 35 edges and every vertex has degree ≥ 3?

17. Suppose all vertices in a graph, G, have odd degree, k. Prove k divides $|E(G)|$.

18. The compliment of a graph, G, of order n, denoted $\overline{G}$, has the same vertex set as G with $E(\overline{G}) = E(K_n) - E(G)$. If every vertex of G has an odd degree, except for one, how many vertices have odd degree in $\overline{G}$?

19. Construct a graph on five vertices with six edges such that there are no three pairwise adjacent vertices (i.e. no triangles).

20. Using graph theory, explain whether or not it is possible for each person, in a group of 15 individuals, to have exactly three friends. (Assume that friendship is a symmetric relation, i.e. friendship goes both ways.)

21. Does there exist a graph where the degree of each vertex is even? Explain

22. Prove that it is impossible for every vertex of a graph to have a different degree.

23. Prove that if G is a graph with n vertices and n edges with no vertices of degree 0 or 1, then the degree of every vertex is 2.

2.3 Isomorphisms

Questions:

1. In your own words, what does it mean for two graphs to be isomorphic?

2. If G_1 and G_2 are isomorphic graphs then they have the same number of vertices, the same number of edges, and the same degree sequence. What is the converse of this statement, and is it true or false? If true, prove it. If false, find a counterexample.

3. Draw all non-isomorphic graphs with n vertices for

   (a) $n = 3$
   Hint: there are four such graphs.

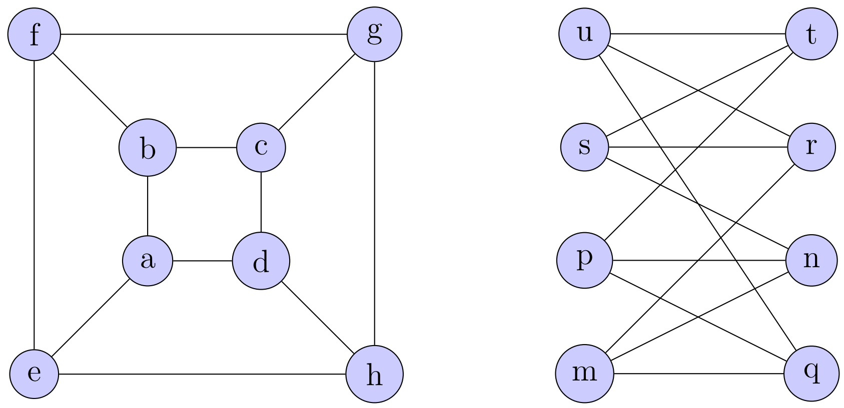
   (b) $n = 4$
   Hint: there are 11 such graphs.

   (c) $n = 5$ *and* connected
   Hint: There are 21 such graphs

4. Show that every graph is isomorphic to the subgraph of some complete graph. What is a necessary lower bound for the order of these complete graphs?

5. Prove that if two graphs are isomorphic, they must contain the same number of triangles.

6. Given the following two graphs, write an explicit isomorphism between them.



7. Let the vertex set of a graph be the set of binary strings of length three. Edges occur between vertices whose binary strings differ by exactly one digit. Show that this graph is isomorphic to the graph formed by the corners and edges of a cube.

8. True or False? If true, provide a brief proof. If false, provide a counterexample.

   (a) If two graph have the same number of vertices with the same quantity and of order cycles, then they are isomorphic.

   (b) Two isomorphic graphs must have the same number of edges and vertices.

   (c) Two isomorphic graphs always look exactly the same.

   (d) Isomorphism is an equivalence relation on all graphs.

   (e) The degree sequence of two isomorphic graphs must be the same.

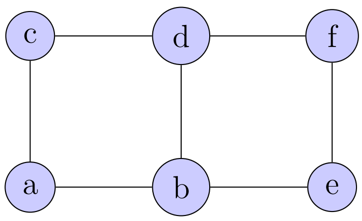
   (f) $K_{3,2}$ is isomorphic to C_5.

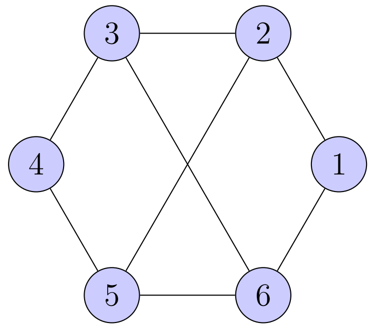
   (g) $K_{4,2}$ is isomorphic to $K_{2,4}$.

   (h) If G contains no cycles, all graphs isomorphic to G also have no cycles.

9. Determine whether the following pairs of graphs are isomorphic. If they are redraw one to look like the other, if not determine why.

(a)





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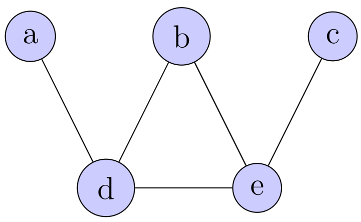

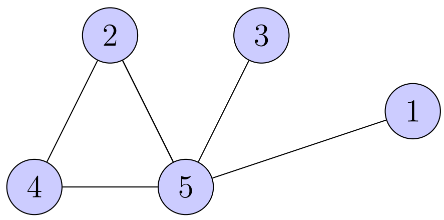

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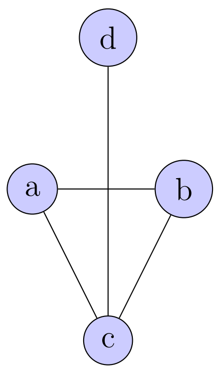

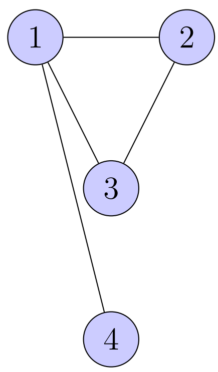

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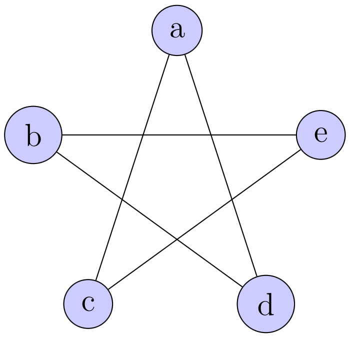

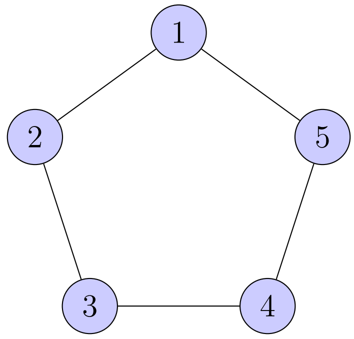

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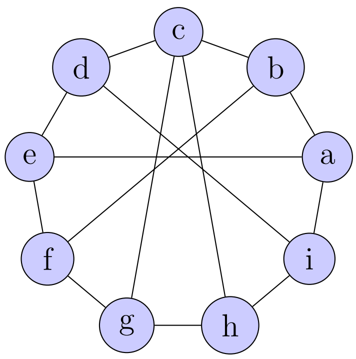
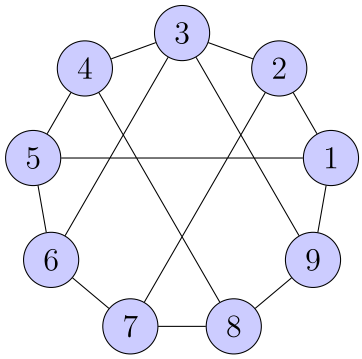




10. A *self-complimentary* graph is a graph where $G \cong \overline{G}$. Construct a self-complementary graph of order 8. Show the two graphs are isomorphic by drawing the complement to look the same as the original graph.

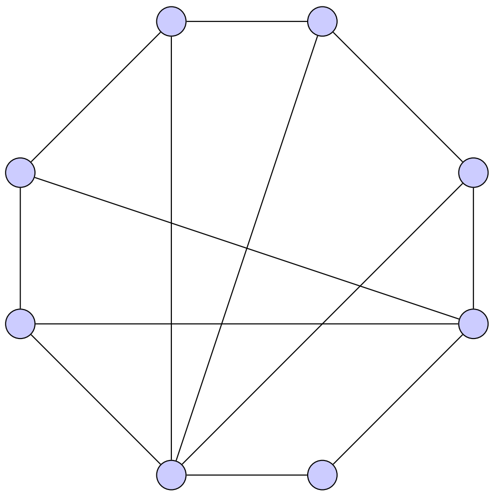
2.4 Eulerian Circuits

Questions:

1. In your own words, define an Eulerian circuit and an Eulerian trail.

2. Are the following graphs Eulerian? Does there exist an Eulerian trail?
   Hint: Use the parity of the vertex degrees.

   (a)



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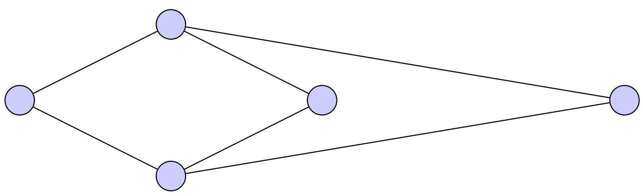

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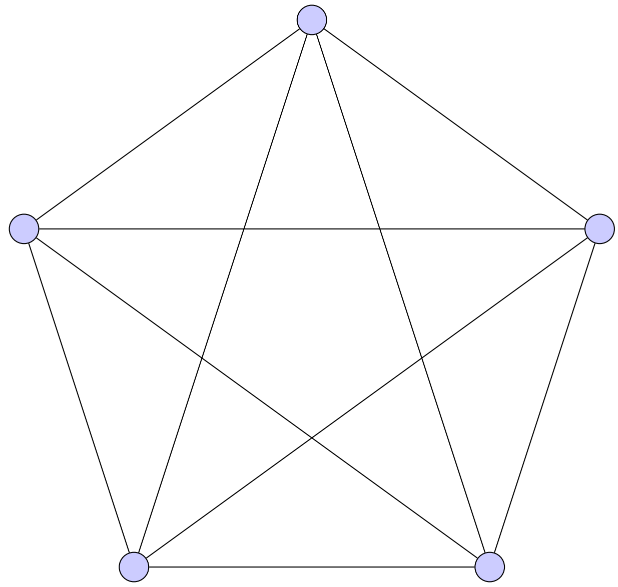

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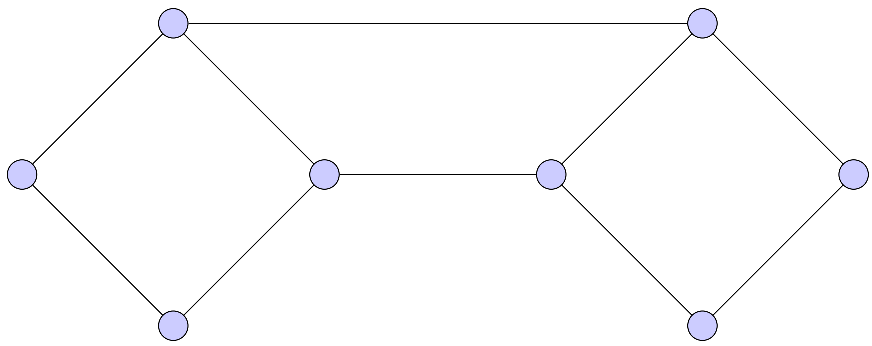

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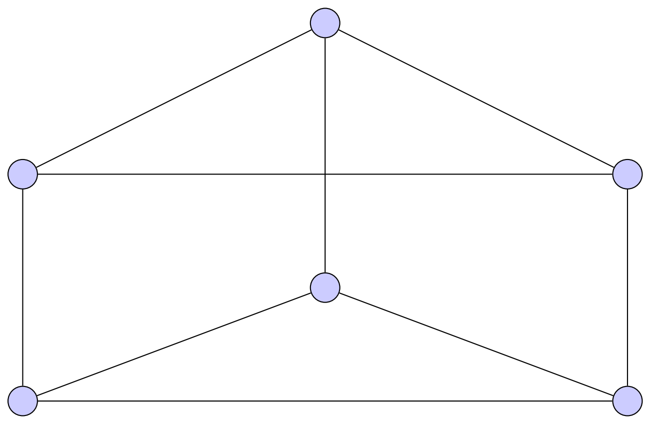

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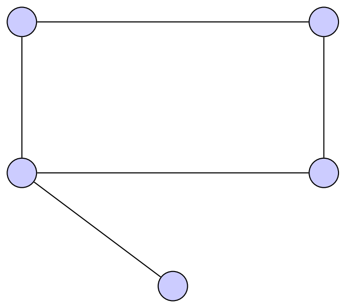


3. Suppose two graphs, G and H, are Eulerian. If an arbitrary vertex of G is made adjacent to an arbitrary vertex in H, is the new graph Eulerian?

4. Explain how the Königsberg Bridge problem is directly related to the study of Eulerian graphs.

5. Prove that there is a walk from vertex u to vertex v if and only if there is a uv path.
Hint: Induction in one direction.

6. Find a graph of order 7 such that both G and $\overline{G}$ contain Eulerian circuits.
Hint: For every $v \in V(G)$, $deg_G(v) + deg_{\overline{G}}(v) = 6$.

7. Determine if each statement is true or false. If true, provide a brief proof. If false, find a counterexample:

   (a) Any graph in which all vertices have even degree contains an Eulerian circuit.

   (b) A closed walk contains a cycle.

   (c) A graph with multiple components can contain a Eulerian cycle.

(d) If a connected graph has $n = 2k$ vertices, for some positive integer k, all with odd degree, then there are k disjoint trails containing every edge.

8. For which integers m and n is $K_{m,n}$ Eulerian?

9. Is a Eulerian circuit necessarily a cycle? Prove or find a counterexample.

10. For which positive integers n does K_n have an

    (a) Eulerian circuit?

    (b) Eulerian trail?

11. If the following statement is true, prove it. If false, provide a counterexample. All circuits of order n contain a cycle and any circuit that is not isomorphic to C_n contains at least two cycles.

12. Define the relation, R, on the set of vertices of a graph, to be $u \ R \ v$ if and only if there exists a uv walk, where $u, v \in V(G)$. Prove that R is an equivalence relation on $V(G)$.

13. Prove that at least one of G and $\overline{G}$ is connected.

14. Prove that if for a graph, G, of order 9 every pair of distinct vertices $u, v \in V(G)$ $deg(v) + deg(u) \geq 8$ then G is connected.

15. If G is a connected graph on n vertices, what is the lower bound for the number of edges G?

16. Show if vertices u and v belong to a circuit of G that after the removal of any arbitrary edge of this circuit a uv trail will remain in the graph.

2.5 Hamiltonian Cycles

Questions:

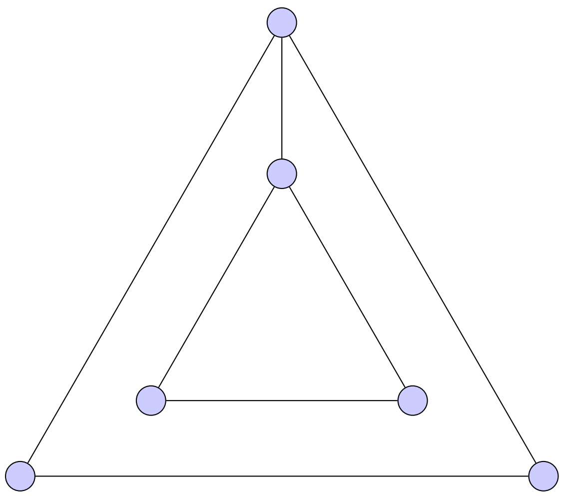
1. Explain the difference between an

   (a) Eulerian circuit and a Hamiltonian cycle.

   (b) Eulerian trail and a Hamiltonian path.

2. Which of the following graphs are Hamiltonian? If they are Hamiltonian identify a Hamiltonian cycle. If they are not, explain briefly why.

   (a)



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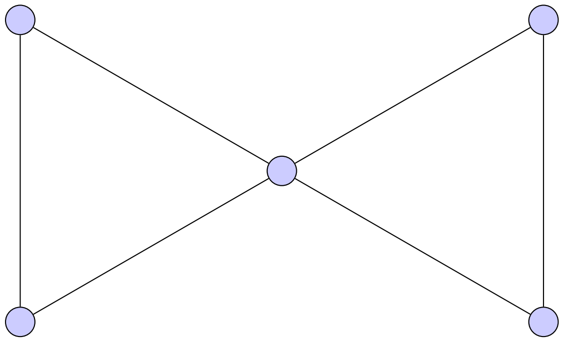

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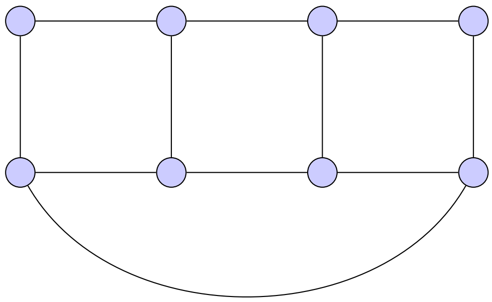

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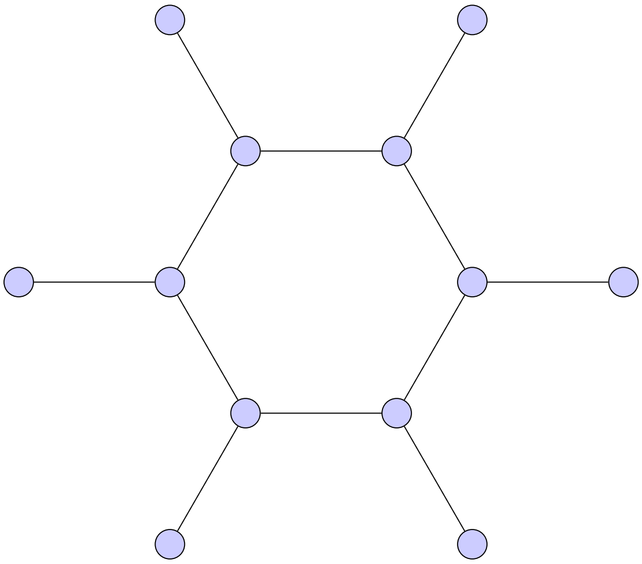

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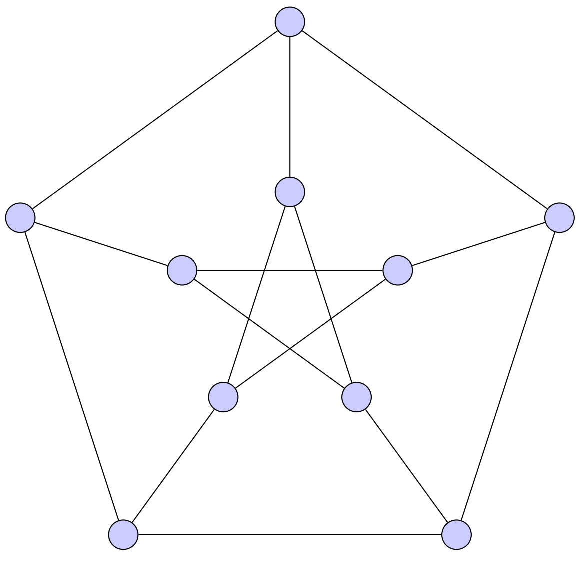

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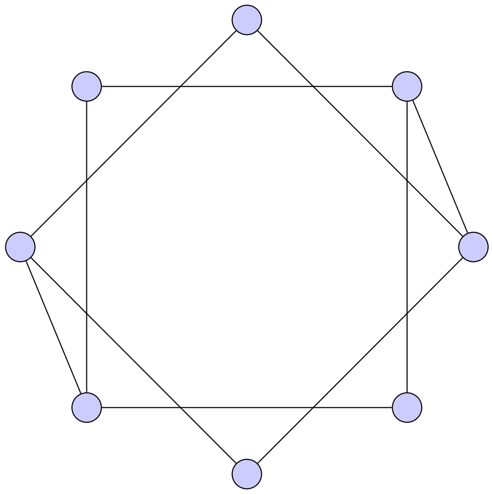

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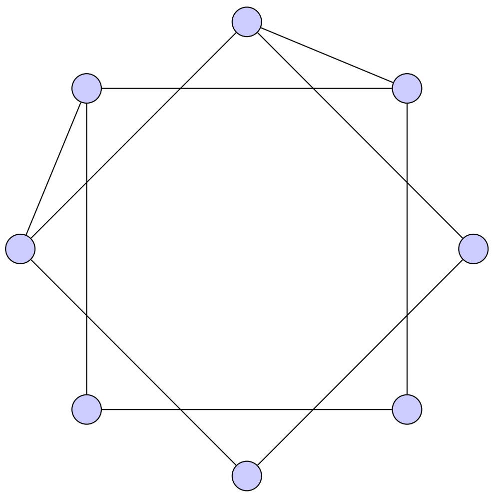


3. Recall that *Dirac's Theorem* states: if a graph G has at least 3 vertices such that every vertex has degree at least $\frac{n}{2}$, then G is Hamiltonian. Show that Dirac's Theorem does not hold if the minimum degree requirement is reduced to $\frac{n-1}{2}$.

4. Does there exist a graph that is both Eulerian and Hamiltonian? If so, find one. If not, explain why this is impossible.

5. A group of n people are going out to dinner, where $n \equiv 0 \pmod 2$ and $n \geq 3$. If every person going to dinner is friends with at least half the group, prove it is possible to seat the friends around a circular table so each person is seated next to two friends.

6. Let G be a graph with at least 3 vertices and $\binom{n-1}{2} + 2$ edges. Prove that G is Hamiltonian.

7. Determine if each statement is true or false. If true, provide a brief proof. If false, find an explicit counterexample.

   (a) A graph of order $n \geq 4$ that contains a triangle cannot be Hamiltonian.

   (b) Every Hamiltonian graph contains a Hamiltonian path.

   (c) If there exists a Hamiltonian path between any two vertices in a graph, then the graph is Hamiltonian.

8. Let G be a connected graph with 13 vertices and 76 edges. Show that G is Hamiltonian. Is G also Eulerian? Explain.

9. For what integers m and n is $K_{m,n}$ Hamiltonian? Explain.

10. Prove that if a cycle that begins and ends at vertex v goes through the vertex w, then there exists a cycle that begins and ends with the vertex w.

11. Prove that if G is a connected bipartite graph, with a Hamiltonian path, the orders of the partite sets differ by at most one.

12. Find a connected, cubic, non-Hamiltonian graph.

13. Show that any cubic graph of order 6 is Hamiltonian. Try to do this without using Ore's or Dirac's Theorem.
Hint: There are only two cubic graphs of order six.

14. Consider a cube. Identify a Hamiltonian cycle in the graph formed by its edges and vertices.

2.6 Trees and Their Properties

Questions:

1. Provide an example of a degree sequence of a tree with at least 3 vertices. Explain why this is a possible degree sequence.

2. Give three equivalent definitions of a tree.

3. Prove that the addition of any edge to a tree creates a cycle.

4. Draw all non-isomorphic trees of order n, where

   (a) $n = 4$
   Hint: There are exactly two.

   (b) $n = 5$
   Hint: There are exactly three.

   (c) $n = 6$.
   Hint: There are exactly six.

5. Does there exist a tree with a Hamiltonian trail? If yes, provide an example.

6. True or false: The subgraph of a tree is always a tree. Justify your answer.

7. A tree has 100 leaves, 20 vertices of degree 6, and half of the remaining vertices have degree 4. The left over vertices are degree 2, how many vertices are of degree 2?

8. A tree, T, with 35 vertices has 25 leaves, two vertices of degree 2, three vertices of degree 4, two vertices of degree 6 and three vertices of degree x. Solve for x.

9. Prove, by induction, that if a connected graph has n vertices and $n-1$ edges, then it is a tree.

10. Let T be a tree. Suppose $\deg(v) \in \{1, 5\}$ for all vertices of T. If T has 25 vertices of degree 5, how many vertices does T have?

11. Let T be a tree with 21 vertices such that $\deg(v) \in \{1, 3, 5, 6\}$ for every vertex of T. If T has 15 leaves and one vertex of degree 6, how many vertices with degree 5 are in T?

12. Prove that the deletion of any edge of a tree results in a disconnected graph. What can we say about the components of this new graph?

13. Prove that any tree with more than two vertices is bipartite.

14. Explain why any tree with two vertices of degree 3 has at least four leaves.

15. What is a necessary and sufficient condition for a tree to be a complete bipartite graph? Explain.

16. Determine a formula for the number of edges in a forest with order n and c components.

17. Find a graph with five vertices and four edges that is *not* a tree. What specific property of a tree fails?

18. Construct a tree with the following properties or explain why such a tree cannot exist.

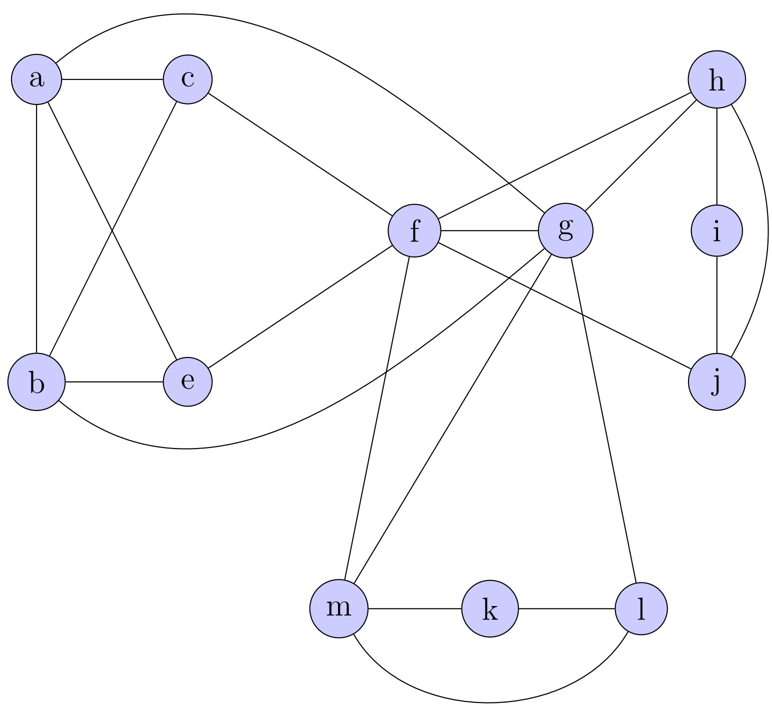
    (a) 10 vertices and the sum of degrees of vertices is 24.

    (b) 12 vertices and 15 edges.

    (c) 8 vertices and 7 edges.

    (d) 4 vertices and the sum of degrees of vertices is 3.

19. Consider the graph G below:



(a) Draw a spanning tree, T, of G that has two vertices of degree 6, or explain why such a spanning subgraph does not exist.

(b) Find an induced 4-cycle of G, or explain why such a subgraph does not exist.

2.7 Planar Graphs

Questions:

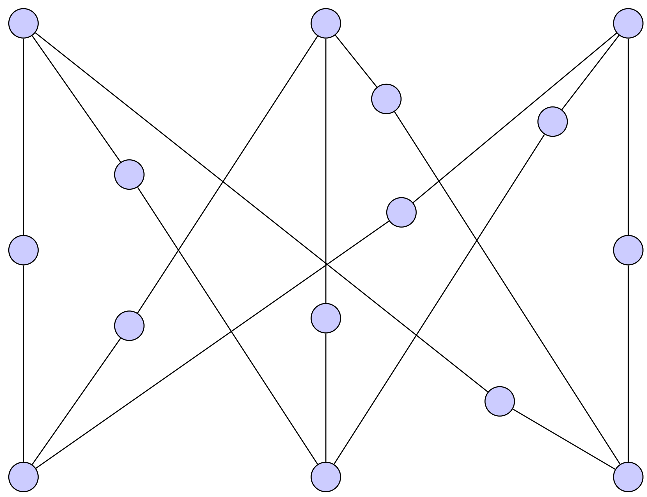
Throughout this section we will use V to denote the number of vertices of the graph, E the number of edges of the graph, and R the number of regions.

1. What is a planar graph?

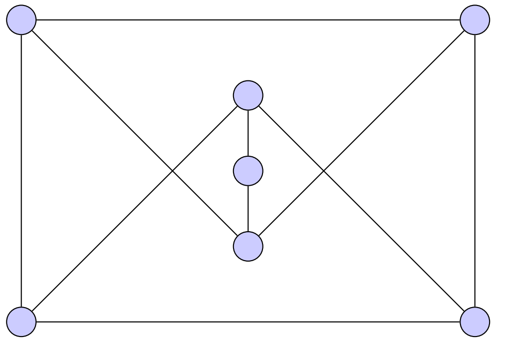
2. In your own words, define what it means for two graphs to be 'homeomorphic'?

3. Determine whether each of the following graphs is planar. If so, redraw it in the plane. If not, explain why using Kuratowski's Theorem.

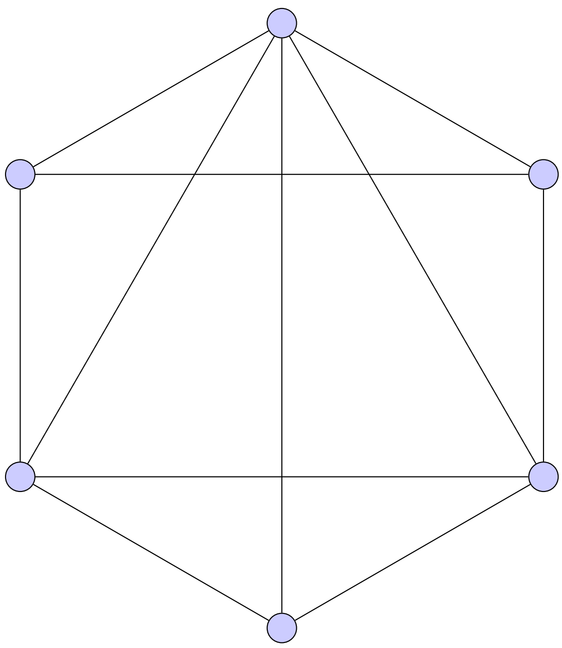
   (a)



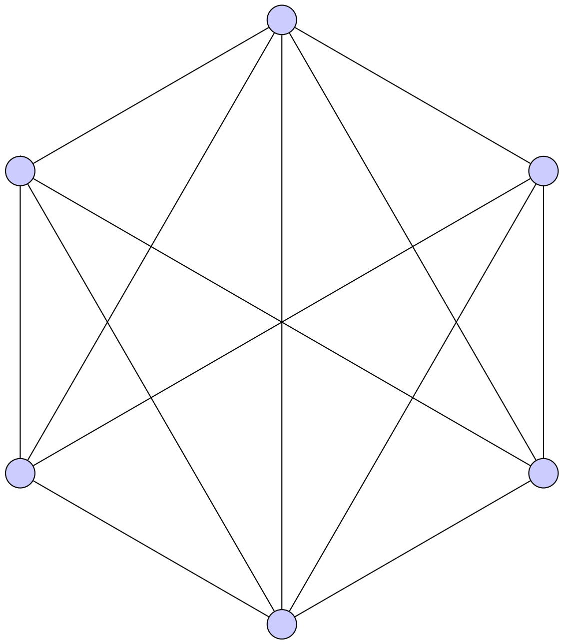
(b)



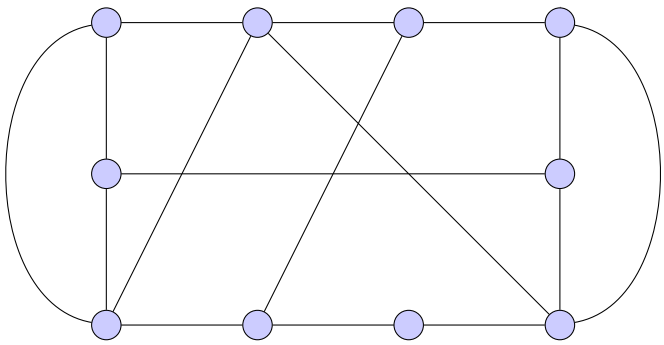
(c)



(d)



(e)



4. Prove that all trees are planar.

5. Let G be a connected, planar graph with at least 4 vertices. Prove that the number of regions is bounded above by $2V - 4$.

6. Prove that if G is a connected planar graph where $E = 3V - 6$ then every region of G is a triangle.

7. For which integers n is K_n planar?

8. For which integers m and n is $K_{m,n}$ planar?

9. Show that P_n is homemorphic to P_2 for all $n \geq 2$.

10. Let G be a planar graph where $\delta(G) \geq 5$. Show that G has at least 12 vertices.

11. Show that a connected, planar graph with order 22 has no more than 60 edges.

12. Determine if each statement is true or false. If true, provide a brief proof. If false, find an explicit counterexample.

    (a) If G is a graph with $E \leq 3V - 6$ then G is planar.

    (b) The subgraph of any planar graph is planar.

    (c) Every planar graph of order 4 or more contains at least one vertex of degree 5 or less.

    (d) If G has order 11, then at least one of G or $\overline{G}$ is non-planar.

13. For each of the following values, determine whether there exists a corresponding planar graph. If it exists, draw it. If not, explain why briefly.

(a) 7 vertices and 13 edges.

(b) 6 regions and 5 vertices.

(c) 8 vertices and 20 edges.

(d) 10 regions and 5 edges.

14. Does there exist a plane graph with 5 regions such that every region is bounded by exactly four edges. Explain.

15. Prove if there exists a circuit in a planar graph that contains two regions, both with an even number of boundary edges, then the circuit is of even length.

2.8 Colouring Graphs

Questions:

1. Describe what is meant by "colouring a graph".

2. What can be said about a graph with chromatic number 1?

3. Determine the chromatic number for the following graphs. Provide a brief explanation.

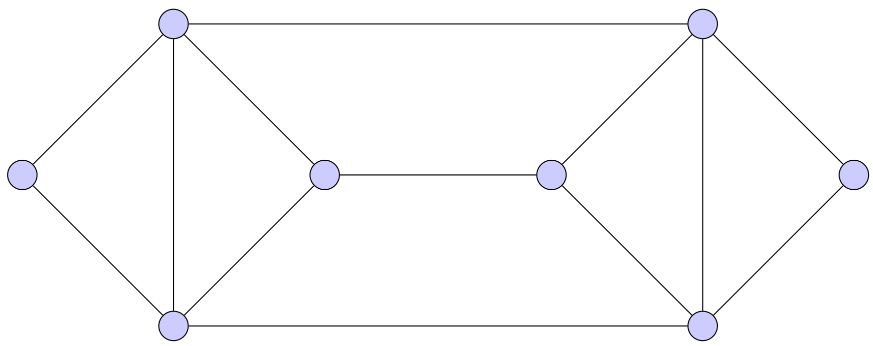
   (a) K_n.

   (b) $K_{m,n}$.

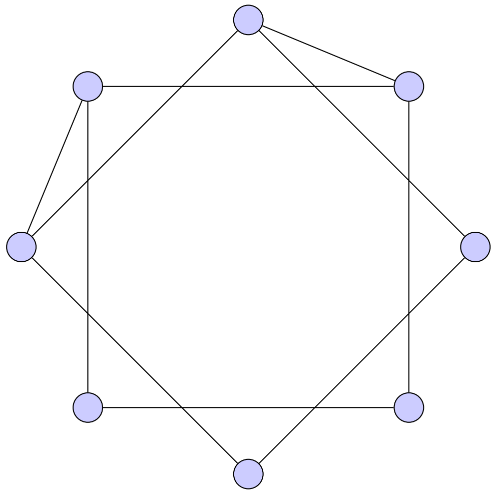
   (c) Any bipartite graph.

   (d) Any tree.

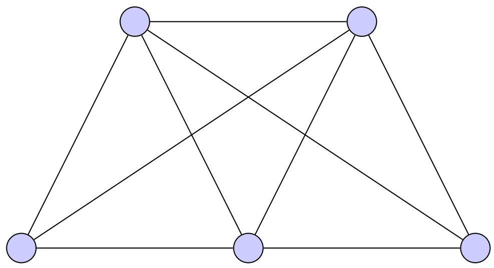
   (e)



(f)



(g)



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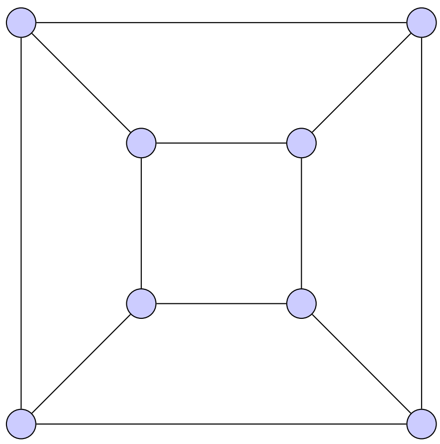
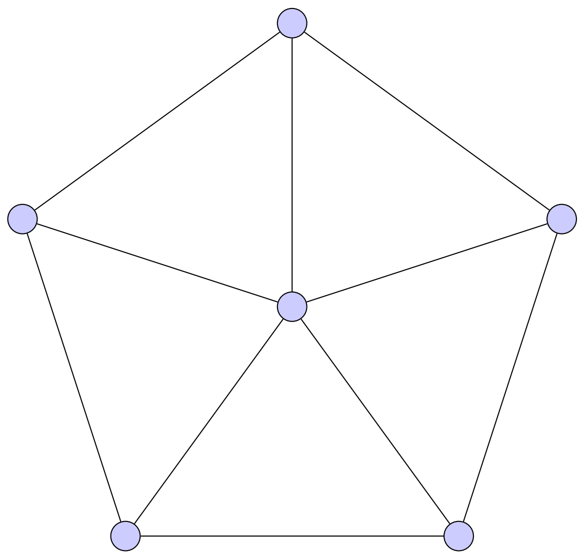


(i)



4. If the following is true, prove it. If false, provide a counterexample. Any graph with n or $n+1$ vertices and exactly n edges has chromatic number at most three.

5. True or False? Provide an explanation or find a counterexample.

(a) If $\chi(G) = 3$ then G contains a triangle.

(b) If a planar graph contains a triangle, then $\chi(G) = 3$.

(c) Isomorphic graphs have the same chromatic number.

(d) Homeomorphic graphs have the same chromatic number.

(e) Any Hamiltonian graph with $\chi(G) = 2$ is planar.

(f) A graph is bipartite if and only if it has chromatic number 2.

(g) If $\chi(G) \leq 4$ then G is planar.

(h) If $\chi(G) = n$, then G contains a subgraph isomorphic to K_n.

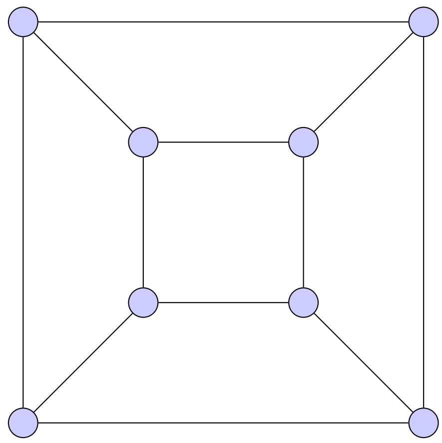
(i) If there exists a 4-colouring of G then $\chi(G) = 4$.

(j) If G contains a subgraph isomorphic to K_n then $\chi(G) \geq n$.

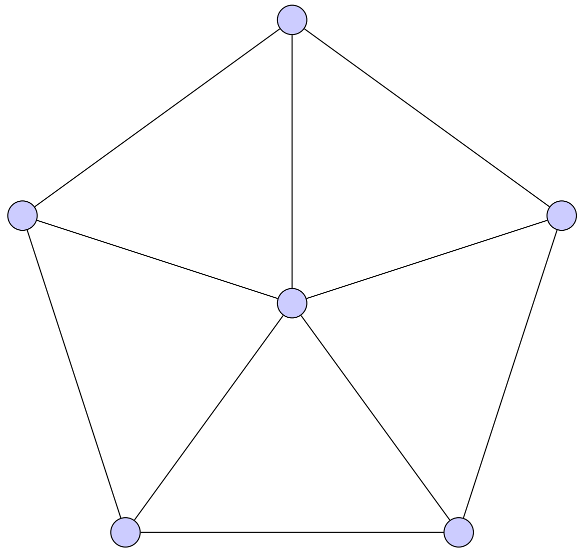
(k) If we can prove G has no 3-colouring then $\chi(G) = 4$.

6. An *edge colouring* of a graph is an assignment of colours to the edges of the graph such that no adjacent edges have the same colour. Find a minimum edge colouring for the following graphs. Does each graph have the same number of colours in a minimum edge colouring as a minimum vertex colouring?
Hint: You've already identified the chromatic number of these graphs in question 3.

(a)



(b)



7. Model and restate the following scenarios as a graph-colouring problem. Clearly indicate what represents the vertices, edges and colours.

   (a) A zoo plans to remodel by removing all cages and instead placing animals in large, open, enclosed areas. Any animals that cannot live together in harmony (i.e. a lion and a elk) must be put in different enclosures. The zoo would like to determine the minimum number of enclosures needed so that each animal can live peacefully.

   (b) The English department is scheduling courses for the upcoming year. Each student has made a list of the courses they would like to enrol in. The department would like to make a schedule so that every student can enrol in all of their desired courses without conflict.

8. Explain why a graph with 8 vertices and 17 edges has chromatic number more than two.

9. Use induction to prove that $\chi(G) + \chi(\overline{G}) \leq n + 1$, where n is the number of vertices of G.

10. If the following is true, prove it. If false, provide an explicit counterexample. If every region of a planar graph is bounded by an even number of edges, then there exists a 2-colouring of the graph.

11. Prove that if a graph has at most two cycles of odd length then it can be coloured with 3 colours.

12. Consider a colouring of a graph. What can you say about all vertices assigned the same colour?

3 Counting: Fundamental Topics

3.1 Basic Counting Principles

3.2 The Rules of Sum and Product

Questions:

For each of your solutions, when appropriate, explicitly identify if you are using the rule of sum, the rule of product, or both.

1. Explain the following in your own words.
   (a) The rule of sum for multiple events.
   (b) The rule of product for multiple events.

2. In which counting scenarios will you need to apply both the rule of sum and product?

3. How many different licence plates with exactly 6 characters (numbers and lowercase letters) can be made given the following specifications?
   (a) No restrictions. This plate can have any arrangement of digits and letters and repetition is allowed.
   (b) The first two characters are digits and the last four are letters. Repetition is not allowed.
   (c) The characters alternate between letters and digits and no digit may be repeated.
   (d) The license plate includes no more than one digit.
   Hint: Consider all possible cases.
   (e) The first character must be either "T" or 0 and the last character must be either "J" or "Q".

4. A university student is looking to take out a book on either frogs or fireflies from their campus library. There are 45 books available covering frogs, and 13 discussing fireflies. How many books does this student have to choose from?

5. Joselyn stops by a sandwich shop on her way home from class. The shop sells 4 types of potato chips, 3 types of cookies, 7 different drinks and 10 different sandwiches. She is interested in determining how many different ways there are to order if she'd either like a drink and a cookie, or a meal which includes a sandwich, a drink, and chips.

6. Two 6-sided dice, each of a different colour, are rolled. Determine how many outcomes are possible.

7. How many nonempty sets of letters can be formed from 3 X's and 5 Y's?
   Hint: As these are sets, the order of the letters is irrelevant.

8. How many integers x are there where $10\ 000 \leq x \leq 99\ 999$ and:
   (a) x is even?
   (b) x contains exactly one digit 0?
   (c) x has at least one repeated digit?

9. How many ternary sequences (sequences using only the digits 0,1, and 2) of length 10 exist such that no consecutive digits are the same?

10. How many integers, x, between 100 and 999 are divisible by 5?

11. A palindrome is a word that has the same spelling when read forwards or backwards. Find the number of 7-letter palindromes.

12. Let $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ...b_n\}$. How many functions $f : A \to B$ are there such that:

   (a) $f(a_1) = f(a_2)$

   (b) $f(a_1) = b_1$ and $f(a_2) \neq b_1$

   (c) $f(a_1) \in \{b_1, b_2, b_3\}$

   (d) $f(a_1) = b_k$, for some $k \in \{1, 2, ...n\}$ and for all other $a_i, i \in \{2, 3, ...m\}$ $f(a_i) \neq b_k$

   (e) $f(a_1) \neq f(a_2)$

13. How many functions are there from a set with 5 elements to a set with 3 elements?

14. How many different ways are there to answer a true or false test with 25 questions, assuming every question is answered?

15. The math department is hosting an event. They are randomly inviting one professor and one student to give a speech together. If there are 1500 students and 50 professors, how many different pairs could give a speech? What about if only one person gives a speech and it could be a student or a professor?

16. If $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, $C = \{\alpha, \beta, \gamma, \chi, \lambda\}$, how many distinct 3-tuples are there in the set $A \times B \times C$?

17. Jamie is buying a combination lock to lock up her work-out gear at the gym. Jamie would like to pick the most secure lock to protect her valuables. Lock

1 advertises that its combination is an ordered sequence of numbers between 1 and 35 such that the first number cannot be the third number. Lock 2 advertises that its combination is an ordered sequence of 4 numbers between 1 and 25 where the first three numbers are all distinct and the fourth number must be the same as one of the previous three numbers. Which lock should Jamie purchase?

18. How many words (strings of letters) exist that are length 1, 2 or 3?

3.3 Permutations

Questions:

1. In your own words, describe a permutation.

2. Explain why the permute function requires distinct objects when counting.

3. How many different permutations of the word MATHS are there of length,

   (a) 5?

   (b) 3?

   (c) If repetitions are allowed, how many different 10 letter 'words' can we make using the letters from MATHS?

4. How many different 6 letter permutations of the word COFFEE are there?
   Hint: Be aware of repeated letters.

5. (a) You and seven friends dine at a circular table at a fancy restaurant. How many different ways can the eight of you seat yourselves around this table?

   (b) What if two people insist on sitting together?

   Hint: The arrangement is considered the same if everyone sits next to the same two people.

6. There are 25 people competing in the school swim race including Nia, Andre, and Katie.

   (a) At the race, the first, second, third, fourth, and fifth fastest swimmers receive medals. How many possible ways can these medals be distributed?

(b) How many possible ways can these medals be distributed if Nia, Andre, and Katie always place in the top three positions?

7. A group of eight would sit in a row at the movie theater, how many ways can arrange themselves if Andrew and Asiya refuse to sit beside each other?

8. In how many ways can the numbers $3, 4, 4, 5, 6, 7, 8$ be arranged to create numbers less than 6000000?

9. Leora has 20 books in her room. Her three friends each want to borrow two books from her. Tomorrow they're all coming over to pick them up, in how many different ways can Leora loan out the books such that the order she gives each friend their books is the order in which they read them?

10. José lost the last two digits of his friend's phone number. How many different phone numbers will José potentially have to call before calling his friend?

11. How many 7-letter words, with no repeated letters, are there such that:

    (a) There are no additional restrictions?

    (b) T must occur in the word?

    (c) A must be the first letter?

    (d) Exactly one of X, Y must be in the word?

12. In a group of teenagers m of them are naturally brunette and n of them were not born with brown hair. How many different ways can these teenagers be arranged in a line such that the m brunette's are all together?

13. Using the definition of a permutation, show that $P(n,n) = n!$.

14. Explain when you would use a permutation instead of the Rule of Product.

15. Create a counting problem that has the solution:

    (a) $P(7,2)$.

    (b) $P(10,9)$.

16. Prove that for an integer $n \geq 2$ that

$$P(n+1,2) - P(n,2) = 2 \cdot P(n,1)$$

.

17. A K-pop fan has 10 different posters to arrange (in a line) on their wall. Three posters are from one band, four from a different group, and three from a third group. How many ways can the posters be lined up such that posters from the same group are together?

18. How many ways can the letters of MISSISSIPPI be permuted?

19. What if the functions from question 12, section 3.2, had to be one-to-one? What must first be said about the cardinality of A and B?

20. (a) How many ways can the letters in BOOKKEEPER be rearranged?

    (b) What if the E's cannot be consecutive?

    (c) What if the E's had to be consecutive?

    (d) What if the vowels had to occur consecutively?

3.4 Combinations and the Binomial Theorem

Questions:

1. In your own words, explain the differences and similarities between a permutation and a combination and describe when each one is used.

2. A lottery ticket consists of five unordered, distinct numbers between 1 and 69 and one letter. A winning ticket must contain all the numbers and the letter drawn by the lottery company. If the prize is \$10,000,000 and the tickets cost \$0.50 is it worth buying all the tickets to ensure a win?

3. The local college's intramural basketball team accepts 21 players. This year 80 students tried out. They want to arbitrarily decide who to let on the team. In each scenario, determine how many different possible teams there are.

   (a) No further restrictions.

   (b) The school boasts about the opportunities available for first year students so, the team wants to make 10 out of the 21 team-members first year students. Out of the 80 players who tried out, 40 of them are first year students.

   (c) While the intramural team is non-competitive, they enjoy beating the neighbouring college's team, so they guarantee the two highest scoring players from last year's team a spot.

   (d) The school wants to have a mix of students who played last year and students who didn't. 65 of the students who registered did not play last year, while 15 students did. The school wants 10 students who did not play last year and 11 who did.

   (e) The coach wants to make sure there is a good mix of types of players on the team,. Each student tells the coach which position they play: 20 students play centre, 15 play shooting guard, 10 play point guard, 20 play small forward, and 15 play power forward. The coach wants to ensure the team has 5 people who play shooting guard and 4 people of every other position.

(f) There are 5 students who are graduating this year. The coach wants to ensure at least 3 of them get to play.

4. A teacher randomly selects 4 numbers from 1 to n. There are exactly 2672670 possible sets of 4 numbers that can be chosen. Determine n.

5. How many distinct, three-element subsets of A are there, where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$?

6. (a) If there are 12 students in a class and the teacher would like to create groups of 6, how many ways can the groups be arranged?

   (b) What if two students refuse to work together?

7. Eli wakes up every morning and makes himself a smoothie with frozen fruit. He picks 3 fruits everyday to make his smoothie with out of the 10 options types of fruit in his freezer. He likes any combination of fruit in his smoothie except banana with apple. How many ways are there for Eli to make his smoothie?

8. UVic is picking what first year math courses they should offer next year. They can only offer both 'Logic and Foundations' and 'Linear Algebra' if they are also offering 'Calculus 1'. If there are 15 possible first year math courses and they will offer 7 courses, how many different ways can they offer courses?

9. Robert is picking the group from his dance class to perform the opening act at the upcoming show. The opening act will have 8 students out of a class of 20, how many possible groups of dancers are there given each of the following scenarios:

   (a) No further restrictions.

   (b) The opening act must be half advanced dancers and half beginner dancers. There are ten students of each level in the class.

   (c) Charlotte and Mohammad do not want to dance together.

   (d) The opening act has a solo at the end that one of the 8 dancers will perform.

10. Give an algebraic and a combinatorial proof of:

$$m\binom{n}{m} = n\binom{n-1}{m-1}$$

Recall: A combinatorial proof is an arbitrary scenario where the same thing can be counted two different ways.

11. Give a combinatorial proof of the identity:

$$\sum_{k=0}^{n}\binom{n}{k} = 2^n$$

12. Give a combinatorial proof of Pascal's Identity:
For any integer $n \geq 2$ and each integer k such that $0 < k < n$:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

13. Use the binomial theorem to efficiently expand the binomial:

    (a) $(x+y)^n$

    (b) $(3-x)^6$

    (c) $(2x-3y)^7$

    (d) $(4x+7y)^{15}$

14. Determine the coefficient of x^9y^4 in the expansion of:

    (a) $(x+y)^{13}$

    (b) $(2x+y)^{13}$

    (c) $(4x-3y)^{13}$

15. Determine the coeffiecient of:

    (a) x^7y^4 in $(2x-3y)^{11}$

    (b) x^7y^2 in $(2x+5y)^9$

    (c) x^5 in $(3x-y)^5$

    (d) x^3y^9 in $(-2x+2y)^{12}$

    (e) xy^6 in $(2x-4y)^7$

16. Evaluate the following using the binomial theorem:

    (a) $\sum_{k=2}^{n} 2^k \binom{n}{k}, n \geq 2$

    (b) $\sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!}, n \geq 1$

17. Using the binomial theorem, prove $(1+i)^n + (1-i)^n$ is an integer for all $n \geq 0$, where $i^2 = -1$.

18. Evaluate the following for all positive integers n:

    (a) $\sum_{k=0}^{n} 6^k \binom{n}{k}$

    (b) $\sum_{k=0}^{n} 4^n \binom{n}{k}$

    (c) $\sum_{k=0}^{n} (-3)^k (2)^{n-k} \binom{n}{k}$

    (d) $\sum_{k=0}^{n} 6\binom{n}{k}$

    (e) $\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}$

Recall: **Multinomial Theorem**: For positive integers n, m, the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ is

$$\frac{n!}{n_1! n_2! \cdots n_m!},$$

where $n_i \in \mathbb{Z}$ for $0 \leq n_i \leq n$, for every $i = 1, 2, ..., m$ and $n_1 + n_2 + ... + n_m = n$.

19. When is the multinomial theorem used instead of the binomial theorem?

20. Use the multinomial theorem to determine the coefficient of

    (a) x^2yz in $(x + y + z)^4$

    (b) $w^2x^2y^2z^2$ in $(2w - x + 3y - 3z)^8$

    (c) xyz in $(3x - y - z)^{10}$

3.5 Combinations with Repetitions

Questions:

1. Find two statements equivalent to: "The number of combinations of n objects taken r at a time, where repetition is allowed."

2. When using the *combinations with repetitions* formula,

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1},$$

what does n represent? r?

3. Seven teammates go to a fast food joint between tournament games. The menu offers: cheeseburgers, hamburgers, hot dogs, fries, and onion rings. If each individual orders only one thing, how many different ways can the group order?

4. A tea shop offers twenty varieties of teas. Assuming they will not run out of tea, how many combinations of tea, with repetition allowed, can 12 teas be purchased?

5. A store has a sale on basic T-shirts, offering 50% off any purchase of exactly 12 shirts. They have 40 different colours to pick from. In how many ways can someone purchase 12 shirts such that,

   (a) they would like every shirt to be a different colour?

   (b) they can purchase the same colour shirt more than once?

6. How many ways are there to distribute 20 toy cars to m children if:

   (a) the toy cars are identical?

   (b) the toy cars are distinct?

7. Parents are distributing the last of the Halloween candies between their four children. There are seven packs of Skittles and six chocolate bars, in how many ways can these parents distribute the candy such that each child gets at least one pack of Skittles.

8. Create a situation that results in counting the number of integer solutions to:

$$x_1 + x_2 + x_3 = 10,$$

   where $x_i \geq 0$ for $i = 1, 2, 3$.

9. How many integer solutions are there to the inequality,

$$x_1 + x_2 + x_3 + x_4 + x_5 < 20,$$

   where $x_i \geq 0$ for $1 \leq i \leq 5$?

10. How many integer solutions are there to

$$x_1 + x_2 + x_3 + x_4 = 8,$$

   where $x_1 \geq 1, x_2 > 1$ and $x_3, x_4 \geq 0$.

11. How many integer solutions are there to:

$$x_1 + x_2 + x_3 + x_4 = 30,$$

where $x_i \geq 1$ for $i \in \{1, 2\}$, $x_3 \geq 4$, and $x_4 \geq 0$?

12. How many integer solutions are there to:

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 21,$$

where $x_i \geq 0$ for $i \in 1, 2, 3$, $x_4 \geq 1$ and $x_5 \geq 3$?

13. How many ways can we arrange ten identical towels in a house with five bathrooms where each bathroom needs one towel?

14. For which positive integer, n, will the two equations,

$$x_1 + x_2 + x_3 + ... + x_{22} = n, \quad \text{and}$$

$$y_1 + y_2 + y_3 + ... + y_{51} = n,$$

have the same number of positive integer solutions?

15. Hannah is buying two dozen loaves of bread at a bakery out of n options. The bakery has more than three dozen of each type and Hannah is okay with repetition. If she can select the bread in 593 775 different ways, how many different types of bread does the bakery have?
Note: one dozen = 12.

16. Prove, either algebraically or with a combinatorial proof, that

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}.$$

17. Which formula would you use to count:

    (a) An ordered arrangement of r distinct objects from a set of n distinct objects with no repetition?

    (b) The un-ordered arrangement of r objects from n objects where repetition is not allowed?

    (c) The distribution of r distinct objects from n distinct objects, where unlimited repetition is allowed?

    (d) The distribution of r identical objects into n distinct containers?

18. How many integer solutions exist to,

$$x_1 + x_2 + x_3 = 0,$$

with the restriction that $x_i \geq -5$ for all $i = 1, 2, 3$?

19. Write the following problems as an equivalent problem of counting the number of integer solutions of an equation. (You do not need to count the number of solutions).
Hint: Refer to question 1 from this section.

    (a) The selection of seven marbles from a group of 3 red, 4 blue and 2 green marbles.

    (b) Distributing 30 identical poker chips to 5 different players.

    (c) Picking 12 apples from 4 different varieties of apples, where at least two of each type is selected.

    (d) The distribution of 15 identical markers into 4 distinct boxes such that the number of markers in the first and second boxes are equal.

20. How many ways can someone distribute x identical marbles into n distinct boxes such that there are m boxes empty, $m \leq n$?

21. How many ways are there to distribute r identical shoes into n distinct shoe boxes with the first m boxes collectively holding at least s shoes, where $m \leq n$ and $s \leq r$?

3.6 The Pigeonhole Principle

Questions:

1. In your own words, describe the Pigeonhole Principle.

2. State the Pigeonhole Principle in terms of sets, functions and domains.

3. What can you say about the number of pigeons occupying each pigeonhole if there are n pigeons and m pigeonholes, where $m \geq n$?

4. Let k and m be positive integers. Explain why the existence of $km + 1$ pigeons and m pigeonholes results in at least one pigeonhole housing $k + 1$ pigeons.

5. Apply the pigeonhole principle to solve the following problems. Describe the 'pigeons' and the 'pigeonholes'.

   (a) There are 367 individuals attending a mathematics seminar, is it possible that everyone has a different birthday? Explain.

   (b) Consider a subset of the positive integers with 29 elements. Prove that at least two elements in this set will have the same remainder when divided by 28.

   (c) You are handed a bag with 9 pairs of shoes in it. If you take shoes one at a time, how many shoes must you take out to guarantee that you have found a pair?

   (d) You are given a list of 17 500 three letter "words" (strings of letters of length 3, repetition is allowed). Are all of these words distinct? Explain.

6. Prove that any subset of $A = \{1, 2, ..., 9\}$ with 6 or more elements contains two elements whose sum is 10.

7. An ice cream parlour sells 15 different ice cream flavours. A parent brings 8 children to the parlour and lets them each get a double-scoop of ice cream with the requirement that each scoop must be different flavor. Is it possible for no flavour to be ordered more than once?

8. Prove that if more than 1001 integers are selected from $\{1, ..., 2000\}$ then:

   (a) there are two integers with the property that one number divides the other.

   (b) there are two integers that are relatively prime (i.e. there exist two integers, say m and n, such that $gcd(m, n) = 1$).
   Hint: Every pair of consecutive integers are relatively prime.

9. How many people must attend a conference to ensure that at least two attendees share the same first and last initial?

10. While trying to apply for scholarships to pay for college, Brynn spends six weeks sending out applications. She sends out at least one application daily, but less than 60 were sent out over the course of these six weeks. Prove that there was a period of consecutive days where Brynn applied for 23 scholarships.

11. Show that any subset of the positive integers with more than three elements, will contain two distinct elements whose sum is even.

12. The local library has 12 computers available. There are 42 people who signed up to use them today. Each person may only use one computer, and to minimize the strain on the computers, the library does not allow more than six people to use a single computer in a day. Show that there are at least five computers used by three or more individuals.

13. At a party there are n people, where $n \geq 2$. Prove that it is guaranteed that two people will speak to the exact same number of people.

14. Prove that in any set of exactly 13 integers 12 divides the difference of two numbers from that set.

15. Farmer Mary has 32 cows in a rectangular paddock measuring 15 metres by 24 metres. Show that at any given moment, there are two cows that are no more than 5 metres apart.

16. How many integers must you pick in $A = \{1, 2, ..., 200\}$ to ensure that there is at least one number divisible by 5?

17. How many integers in $X = \{0, .., 60\}$ must be chosen to ensure that an odd integer is selected?

4 Inclusion and Exclusion

4.1 The Principle of Inclusion-Exclusion

Questions:

Where appropriate, indicate the set and the conditions you will be working with.

1. In your own words explain the Principle of Inclusion-Exclusion.

2. Determine whether the following mathematical statement is true or false. If it is true, prove it. If it is false, provide a counterexample.

$$N(\overline{c_1}\ \ \overline{c_2}) = N(\overline{c_1 c_2}).$$

3. A kindergarten class with 30 students was surveyed about which activities they enjoy. 20 students enjoy nap time, and 14 students enjoy colouring. Determine
   (a) How many students don't enjoy colouring?
   (b) If 7 students like both activities, how many students like either colouring, nap-time or both?
   (c) How many students enjoy exactly one activity?

4. There are 500 families that live in the neighbourhood of South Brambleton. 100 of these families have no children and no pets. 300 families have pets, and 400 have children. How many homes in South Brambleton have both of children and pets living in them?

5. Suppose there are 100 different cookies at the bake-sale. There are 40 cookies with chocolate chips and 25 with raisins.

   (a) If 11 cookies have both chocolate chips and raisins, how many cookies have neither chocolate chips nor raisins?

   (b) Suppose there are also 30 cookies with oatmeal, 10 of which also have chocolate, and 15 of which have raisins. If there are only 6 cookies with all three ingredients in them, how many cookies do not contain any oatmeal, raisins, and chocolate chips?

6. A school is having a year end barbeque. Each family is asked to contribute at least one of the three following foods: salads, sandwiches, and juices. If there are 150 families attending and you know that 25 brought salads and sandwiches, 30 brought sandwiches and juice, and 40 brought salad and juice. Further, each item is brought by 50 families. Lastly, 60 families bring at least two items and 20 families bring all three items.

   (a) How many families brought exactly one item?

   (b) How many different families brought only juice?

   (c) How many families did not only bring sandwiches?

7. At a local twin convention, the attendees are seated at circular tables. Each table can sit 12 people so 6 sets of twins. In how many ways can they be arranged so that no individual sits beside their twin?
   Note: two seating arrangements are the same if one is simply a rotation of the other.

8. Determine the number positive integers in $A = \{1, 2, 3, ...3000\}$ that are:

   (a) divisible by 7 or 2.

   (b) divisible by 7 and not 2.

(c) not divisible by 3 or 5.

(d) divisible by 2, 3 and 7, but not divisible by 11.

9. How many nine-digit sequences contain all of the digits 1,2, and 3 appearing at least once?

10. How many 10-digit permutations of the digits $0, 1, ..., 9$ exist such that the first digit is at least 2 and the last digit is less than or equal to 7?

11. How many sequences of 10 distinct letters do not contain any of the words: GAINS, BUG, SNAP?

12. How many sequences of 12 distinct letters do not contain any of the words: DOG, SPUN, or DREAM?

13. How many ways can the letters of MISSISSIPPI be arranged such that none of the following are true: all of the I's are consecutive, all the P's are consecutive, and all the S's are consecutive?

14. How many non-negative integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 25$ if

    (a) $0 \leq x_i$ for $i = 1, 2, 3, 4$?

    (b) $0 \leq x_i \leq 9$ for $i = 1, 2, 3, 4$?

    (c) $0 \leq x_1 \leq 5$, $0 \leq x_2 \leq 3$, $2 \leq x_3 \leq 7$, $3 \leq x_4 \leq 11$?

15. How many ways are there to distribute 25 identical marbles into 6 distinct boxes so that each of the first three boxes have no more than six balls?

16. Consider a finite set, S, with $|S| = k$. Let c_1, c_2, c_3, c_4 be four conditions, each of which may be satisfied by one or more elements of the set S. Use the Principle of Inclusion-Exclusion to prove that

$$N(\overline{c_2}\ \overline{c_3}\ \overline{c_4}) = N(c_1\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}) + N(\overline{c_1}\ \overline{c_2}\ \overline{c_3}\ \overline{c_4}).$$

Hint: Use a combinatorial proof and consider an arbitrary element of the set.

4.2 Derangements: Nothing in its Right Place

Questions:

1. Explain, in your own words, what a derangement is. Give a simple example.

2. Explain how a derangement is an application of the Principle of Inclusion and Exclusion.

3. How many ways can we permute the alphabet such that no letter is in its usual place?

4. A high school decides to host a gift exchange for their students. If 150 students participate in the exchange, how many different ways can gift-givers be assigned such that,

   (a) no student is assigned to themselves?

   (b) there are 50 grade 9's, 30 grade 10's, 30 grade 11's, and 40 grade 12's participating, and the students only draw names from students in their own grade?

5. Kalil is interviewing for jobs at 5 different companies where each job has a two-part interview. He has 5 interview-appropriate outfits. Kalil wants to wear each outfit once for each round of interviews, but does not want to wear the same outfit to the second-part of an interview as he wore to the first. How many ways can he do this?

6. How many different ways can you arrange the numbers $\{1, 2, 3, ..., 9, 10\}$ such that no *even* number is in its original position?

7. For the set of positive integers $\{1, 2, 3, 4, ..., n-1, n\}$, we know that the first 6 digits appear in the first 6 positions. If there are 2 385 derangement of this set, what is the value of n?

8. A waiter (who is not particularly good at their job) has 8 customers at lunch. Every person order a different meal. How many different ways can the waiter bring people their food such that:

   (a) no one get the meal they ordered?

   (b) at least one person gets the food they ordered?

   (c) exactly two people get the food they ordered?

   (d) exactly one person gets someone else's food?

9. Twelve friends host a potluck (a *party* where everyone brings a dish), six of them are vegetarians while the other six are not. Every individual bring both a drink and a main dish (suppose that the vegetarians only bring vegetarian main dishes). How many ways can these friends bring home leftovers such that each friend brings home one drink and one main dish and

   (a) no friend brings home either of the things they brought to the potluck?

   (b) the vegetarians all bring home the meal they brought but not their drink, while the non-vegetarians bring home the drink they brought but not a different main dish?

   (c) No one brings home *both* of the items they brought.
   Hint: Use PIE instead of trying to adapt the derangement formula.

10. Give a combinatorial proof of:

$$n! = \sum_{k=0}^{n} \binom{n}{k} \cdot d(k)$$

for every $1 \leq k \leq n$, with $n \in \mathbb{Z}^+$, where $d(k)$ represents the number of derangement's of k elements.

4.3 Onto Functions and Stirling Numbers of the Second Kind

Questions:

1. In your own words, explain what an onto function is.

2. Using the language of "objects" and "containers", what is counting the number of surjective functions equivalent to?

3. Give an example of an onto function.

4. Let $f : A \to B$ be a surjective function, what can you say about $|A|$ compared to $|B|$?

5. Count the number of surjective functions from C to D where $|C| = n$ and $|D| = n + 1$.

6. How many surjective functions $f : A \longrightarrow B$ exist where A is the first 13 letters of the alphabet and $B = \{1, 2, 3, ..., 9\}$?

7. Consider the function $g : X \to Y$ where $X = \{1, 2, 3, 4\}$ and $Y = \{\alpha, \beta\}$. How many functions g are *not* surjective?

8. A middle school social studies teacher wants the students to learn about the seven different continents of the world. Between the 27 students they must split into groups such that every continent has at least one student studying it. How many different ways can the students group themselves?
   Note: You may leave your solution in terms of a summation without evaluating it.

9. At an engineering firm there are 4 professional engineers. Currently they have 7 different clients, one of which is by far the most profitable. In how many ways can each engineer work on at least one account with the condition that Billi, the most senior engineer, is always given the most valuable client.

10. In your own words, define what a Stirling number of the Second Kind is.

11. Express the number of onto functions from a set of size k to a set of size j where $k \geq j$ using Stirling numbers.

12. You have been asked to distribute 10 different stuffed animals between 5 labelled bins.

    (a) How many ways can the stuffed animals be distributed so that every bin has at least one stuffed animal in it?

    (b) One of the stuffed animals is a collectable. In how many ways can we distribute the stuffed animals such that the collectable is in the first bin, and no bins are left empty?

    (c) Suppose the bins are identical. How many ways can the stuffed animals be distributed amongst the bins, with any number bins left empty?

13. Consider the integer $55\ 335 = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 31$.

    (a) In how many ways can 55 335 be written as the product of two factors, where each factor must be greater than 1?

    (b) In how many ways can 55 335 be written as the product of two *or more* factors, with every factor greater than 1?

    Note: The order of the factors is irrelevant since multiplication is commutative.

14. Using the definition of Stirling numbers, algebraically prove:

   (a) $S(n, 1) = 1$.

   (b) $S(n, 2) = 2^{n-1} - 1$.

   (c) $S(n, n-1) = \binom{n}{2}$.

5 Generating Functions

5.1 Introductory Examples

5.2 Definition and Examples: Calculating Techniques

Questions:

1. Given the following sequences, determine the corresponding generating function as a summation and in closed form (as a formula).

   (a) $1, 2, 3, 4, ...$

   (b) $5, 4, 3, 0, 0, ...$

   (c) $1, -1, 1, -1, 1, -1, ...$

   (d) $\binom{10}{10}, \binom{11}{10}, \binom{12}{10}, \binom{13}{10} ...$

   (e) $\binom{10}{10}, -\binom{11}{10}, \binom{12}{10}, -\binom{13}{10} ...$

   (f) $1, 0, 1, 0, 1, ...$

   (g) $1, -2, 4, -8, 16, -32, 0, 0, 0, 0, ...$

2. Given the following generating functions, determine the sequence that represents it.

   (a) $f(x) = 0$

   (b) $f(x) = x$

   (c) $f(x) = 4 + 3x - 10x^2 + 55x^3$

   (d) $f(x) = (3x - 4)^3$

   (e) $f(x) = \frac{3x}{1-x}$

   (f) $f(x) = \frac{1}{(1-3x)^2}$

3. Determine the coefficient of the specified term in the expansion of the given the function.

   (a) x^3 in $\frac{1}{1-x}$.

   (b) x^2 in $\frac{1}{(1-2x)^3}$.

   (c) x^5 in $\frac{(1-x^8)}{1-x}$.

   (d) x^4 in $\frac{1}{(1+3x)^{10}}$.

4. In how many ways can 1000 identical pamphlets be distributed to five different counselling centers, where pamphlets are put in stacks of 50, such that each center receives at least 50 but no more than 500 pamphlets?

5. In how many ways can 20 identical balls be distributed between 3 distinct boxes such that,

   (a) There are at least two balls assigned to box?

   (b) There are at least three, but no more than 10 balls assigned to each box?

   (c) Using the same condition as in part b, how many distributions are possible if there were 25 balls instead of 20?

6. Determine the number of ways that \$12 in loonies can be distributed between a father's three children so that the eldest gets at least four dollars, the middle and youngest child are both guaranteed at least two dollars, but the youngest cannot receive any more than \$5 since he will spend it all on candy and rot his teeth.

7. In how many ways can n balls be selected from a supply of pink, orange and black balls such that the number of black balls selected must be even?
   Hint: Partial fractions may come in handy.

8. A restaurant just closed for the night and they had an extra 12 orders of fries and 16 mini-desserts left over. The restaurant manger decides to split this left over food between the four employees closing that night. How can the manager do this so that the head chef receives at least one order of fries and exactly three mini-desserts, while the three other closing-staff are guaranteed at least two orders of fries but less than 5 desserts?

9. Use generating functions to determine the number of four-element subsets of $\{1, 2, 3, ..., 15\}$ that contain no consecutive integers.

10. A student is picking out a handful of gummy bears from a large container. There are red, yellow, and green gummy bears in the container. The student wishes to pick out an even number of red gummy bears, an odd number that is at least 3 of yellow gummy bears, and either 4 or 6 green gummy bears.

    (a) Determine the appropriate generating function that models this situation.

    (b) How many ways can the student pick out gummy bears if they pick out:

        i. 15?

        ii. 22?

11. Determine the generating function for the following equations, where no $x_i < 0$:

    (a) $x_1 + x_2 + x_3 + x_4 = k$, where $2 \leq x_3 \leq 5$, and $4 \leq x_4$

    (b) $x_1 + x_2 + x_3 + x_4 = k$, where x_1 and x_2 are even, $x_3 \leq 5$ and $x_4 \leq 2$

    (c) $x_1 + x_2 + x_3 + x_4 = k$, where $x_i \geq i$ for $i = 1, 2, 3, 4$

    (d) $2x_1 + 3x_2 + x_3 + 3x_4 = k$

12. Someone buys a chocolate bar and receives 50 cents in change. Create a generating function that could determine the number of ways they could receive their change in any combination of pennies, nickles, dimes, and quarters? The coefficient of which term will give the desired solution?
Note: You are *not* being asked to determine how many ways this is possible.

13. A deck of cards has 52 cards in total. Half of the deck is red and half is black. A quarter of the deck has the symbol hearts, a quarter has the symbol diamonds, a quarter has the symbol spades, and a quarter has the symbol clubs. How many ways are there to pick 15 cards if:
    (a) You wish to pick an even number of black cards and an odd number of red cards?
    (b) You wish to pick at least two of each symbol, but no more than 5 hearts and 6 spades?

14. Three students are running for student body president: Krishna, and Jamar, and Bonnie. Find the generating function used to determine the possible distribution of n students' votes
    (a) with no further restrictions?
    (b) if every student running votes for themselves?

15. How many ways are there to obtain a sum of 7 if 2 distinct 6-sided dice, numbered $1, 2, 3, 4, 5, 6$, are thrown?

5.3 Partitions of Integers

Questions:

1. In your own words, define a partition on a positive integer n.

2. Explain why generating functions are helpful in determining the number of possible partitions of integers.

3. What is a Ferrers diagram?

4. Give an example of a valid partition of 54.

5. Find *all* partitions of 5. Which of these partitions use only distinct summands?

6. Find the generating function that represents the number of ways of distributing an unlimited supply balls into 5 identical boxes. What method would be used to find the ways of distributing 10 balls?

7. Find the generating function for each of the following partitions of the integer r such that:

   (a) the largest summand is *equal* to k?

   (b) the largest summand is equal to $(2k + 1)$ and all summands are odd?

   (c) the summands are all odd and distinct?

   (d) there is at least one summand of size 2?

   (e) if a summand is even it is distinct?

   (f) every summand is distinct?

   (g) the summands cannot occur more than five times?

(h) no summand can exceed 12, and summands cannot occur more than five times?

8. Prove that the number of odd partitions of a natural number, n, is the same as the number of distinct partitions of n.

9. Prove that, given a positive integer n, the number of partitions of n in which no even summand is repeated (there may be odd summands that repeat) is equal to the number of partitions of n where no summand appears more than 3 times.

10. Use a Ferrers diagram to show that the number of partitions of n is equal to the number of ways to partition $2n$ into n parts.

11. Use a Ferrers diagram to prove that the number of partitions of n is equal to the number of ways to partition $(n + m)$ into m parts, where $n \leq m$.

6 Recurrence Relations

6.1 First-Order Linear Recurrence Relations

Questions:

1. In your own words, describe what a recurrence relation is.

2. What does it mean to *solve* a recurrence relation?

3. Suppose $a_0 = 2$, $a_1 = 7$ and $a_{n+1} = -a_n + 5a_{n-1}$ for $n \geq 1$. Find a_6 *without* solving the recurrence relation.

   Why is it better to *solve* a recurrence relation rather than just find the desired terms as necessary?

4. Solve the recurrence relation $a_n = -2a_{n-1}$, where $a_0 = 5$.

5. Solve the recurrence relation $a_n = \frac{1}{3}a_{n-1}$, where $n \geq 1$ and $a_2 = 101$.

6. Solve the recurrence relation $5a_n + 6a_{n-1} = 0$ where $n \geq 1$.

7. Solve the recurrence relation $2a_n - 7a_{n-1} = 0$ where $n \geq 1$ and $a_4 = 81$.

8. Leora puts money in a high interest savings account to help save for university. The interest is 8% annually and compounds monthly. If she deposits \$1500.00 on the day she opens the account, how much money will she have after 16 months? Use recurrence relations to solve this problem.

9. By making a substitution, transform the following non-linear recurrence relation into a linear recurrence relation and then solve it.

$$a_{n+1}^2 = 3a_n^2, \text{where } a_n > 0 \text{ and } a_0 = 5$$

10. Given the following geometric progressions, find a recurrence relation with an initial condition that satisfies the progression.

    (a) $0, 2, 6, 12, 20, 30, 42....$

    (b) $7, \frac{14}{5}, \frac{28}{25}, \frac{56}{125}, ...$

11. Given the following recurrence relation and initial conditions, solve for d:

$$a_{n+1} - d \cdot a_n = 0, \text{ where } a_3 = \frac{-8}{343} \text{ and } a_5 = \frac{-32}{16807}$$

12. Suppose the amount of bacteria in a container triples every hour. If initially there are only 5 bacteria, how many bacteria are in the container after a day and a half?

13. Solve the following recursive functions:

    (a) $a_0 = 1, a_n = -5a_{n-1}$ for $n > 0$

    (b) $a_1 = 1, a_n = 4a_{n-1}$ for $n > 1$

6.2 Second Order Linear Homogeneous Recurrence Relations with Constant Coefficients

Questions:

1. When solving a second order linear homogeneous recurrence relation with constant coefficients, how many initial terms must be known to find its unique solution? Explain.

2. What is the characteristic equation of

$$C_0 a_n = -C_1 a_{n-1} + C_2 a_{n-2}$$

for $n \geq 2$?

3. Solve the following recurrence relations.

   (a) $a_n = 9a_{n-1} - 20a_{n-2}$ for $n \geq 2$, with $a_0 = 5, a_1 = 6$.

   (b) $a_n + a_{n-2} = 0$ for $n \geq 2$, with $a_0 = 0, a_1 = -2$.

   (c) $5a_n = -30a_{n-1} - 45a_{n-2}$ for $n \geq 2$, with $a_0 = 7, a_2 = 20$.

   (d) $a_n = 2a_{n-1} - 2a_{n-2}$ for $n \geq 2$, with $a_0 = a_1 = 0$.

   (e) $a_n - 2\sqrt{3}a_{n-1} + 3a_{n-2} = 0$ for $n \geq 2$, with $a_0 = -1, a_1 = 5$.

   (f) $a_0 = 2, a_1 = 6, a_n = 7a_{n-1} - 12a_{n-2}$ for $n > 1$

   (g) $a_0 = 0, a_1 = 3, 2a_n = 12a_{n-1} - 20a_{n-2}$ for $n > 1$

   (h) $a_0 = -3, a_1 = 1, a_n = 4a_{n-1} - 4a_{n-2}$ for $n > 1$

   (i) $a_0 = 3, a_1 = 4, a_n = 4a_{n-2}$ for $n > 1$

4. Suppose $a_0 = -2, a_1 = 5, a_2 = 14$, and $a_3 = 39$ are initial conditions for a recurrence relation of the form $a_{n+2} + ba_{n+1} + ca_n = 0$ for $n \geq 0$. Solve for the constants b and c and then solve the recurrence relation.

Note: Do not be concerned if you are not getting nice results!

5. Determine a recurrence relation with characteristic equation:

$$3r^2 - 5r + 11 = 0$$

6. Here are the first 10 terms of the Fibonacci sequence,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$

This sequence can be represented by a second order linear recurrence relation with constant coefficients. Determine and then solve this recurrence relation.

7. Find a recurrence relation for the number of binary sequences of length n that have no consecutive 0's.
Note: A binary sequence is sequence made up of only the digits "0" and "1".

8. Suppose a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has a general solution $a_n = A_1 3^n + A_2 6^n$. Find c_1, c_2.

9. Determine and then solve a recurrence relation that determines the value of a stock market indicator where the change in value in any given year, from the previous, is twice the change noticed in the previous year.

10. Find the recurrence relation o_n for the number of n-letter words made from the letters O, W, N that contain at least one O using generating functions. Check your answer using a straightforward counting argument.

11. Solve the recurrence equation $a_n = 3a_{n-1} + n, n \geq 1, a_0 = 1$ using generating functions.

12. Set up recurrence relation, with initial conditions, for:

    (a) u_n, the number of n-letter words using the letters B,A,R that contain no consecutive A's, $n \geq 0$

    (b) v_n, the number of n-letter words using the letters B,A,R such that each B and each A can only be followed directly by an R.