Algebra and Trigonometry
Review Material

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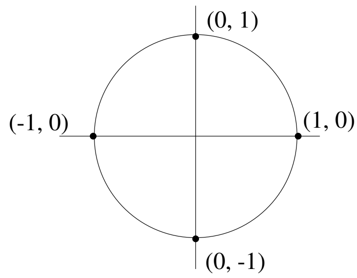
June 2, 2004

http://atlas.math.vanderbilt.edu/~pscrooke/calculus/calculus.html

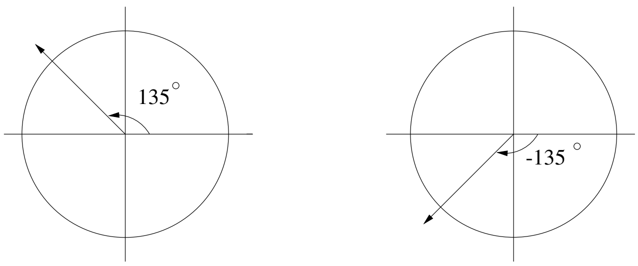
Trigonometry

The Unit Circle

The first key to understanding trigonometry is to know the unit circle. The **unit circle** is the circle centered at $(0,0)$ with radius 1.

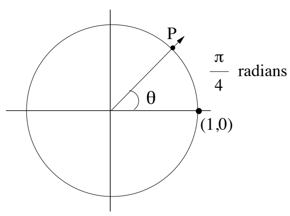


Consider an angle θ in the unit circle. The angle is positive if it is measured counterclockwise from the positive x-axis and negative if it is measured clockwise.

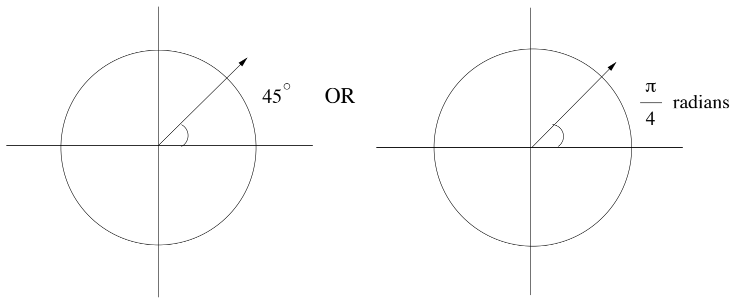


The above angles are measured using **degrees**. An angle θ may also be measured using **radians**. The radian measurement corresponds to a distance around the circumference, C, of the unit circle ($C = 2\pi$).

Let us measure an arc on the unit circle starting at $(1,0)$ of length $\frac{\pi}{4}$ and ending at a point P. If we draw a ray from the origin through point P, we have formed an angle θ, where $\theta = \frac{\pi}{4}$ radians.



The following two angles are the same.



To convert radians to degrees and vice versa, use the following equation.

$$\boxed{\pi \text{ radians} = 180°}$$

Example 1: Convert 30° to radians.

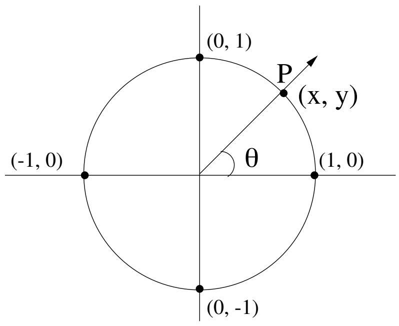
$$\begin{aligned} \pi \text{ radians} &= 180° \\ \frac{\pi}{180} \text{ radians} &= 1° \\ 30 \cdot \frac{\pi}{180} \text{ radians} &= 30 \cdot 1° \\ \frac{\pi}{6} \text{ radians} &= 30° \end{aligned}$$

Example 2: Convert $\frac{8\pi}{5}$ radians to degrees.

$$\begin{aligned} \pi \text{ radians} &= 180° \\ \frac{8}{5} \cdot \pi \text{ radians} &= \frac{8}{5} \cdot 180° \\ \frac{8\pi}{5} \text{ radians} &= 288° \end{aligned}$$

Trigonometric Functions

Consider the point $P = (x, y)$ where the angle of measure θ intersects the unit circle. We use the coordinates x and y of this point to define six **trigonometric functions** of θ.



We define the **cosine** of θ to be the x-coordinate of this point and the **sine** of θ to be the y-coordinate of this point. We use the abbreviations "cos" for cosine and "sin" for sine. Thus,

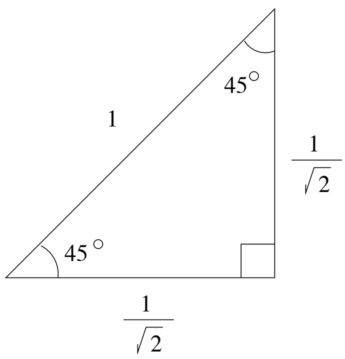
$$\cos\theta = x \qquad \text{and} \qquad \sin\theta = y.$$

Notice that the x and y coordinates of all points on the unit circle lie somewhere between -1 and 1. Thus

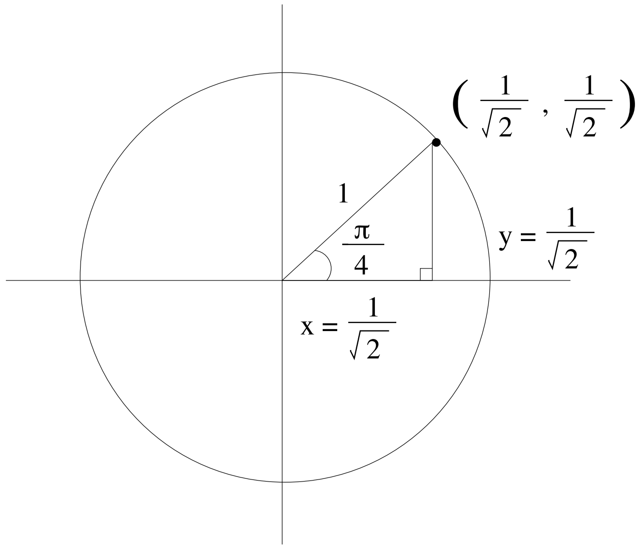
$$-1 \leq \cos\theta \leq 1 \quad \text{and} \quad -1 \leq \sin\theta \leq 1$$

no matter what θ is.

We can use geometry to determine $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$. Since $\frac{\pi}{4}$ radians equals 45°, consider a 45°-45°-90° triangle with a hypotenuse of length 1. Such a triangle must have legs each of length $\frac{1}{\sqrt{2}}$.



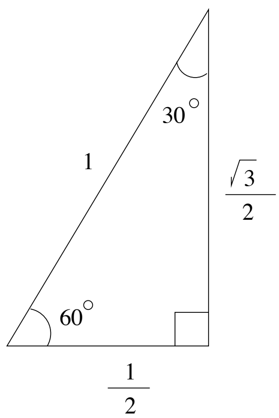
By moving this triangle into the unit circle and remembering that 45° equals $\frac{\pi}{4}$ radians, we find that the corresponding point on the unit circle has (x, y)-coordinates $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

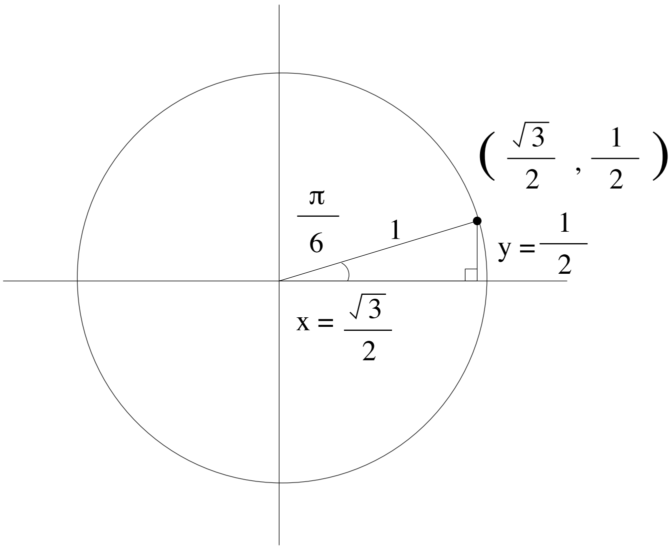


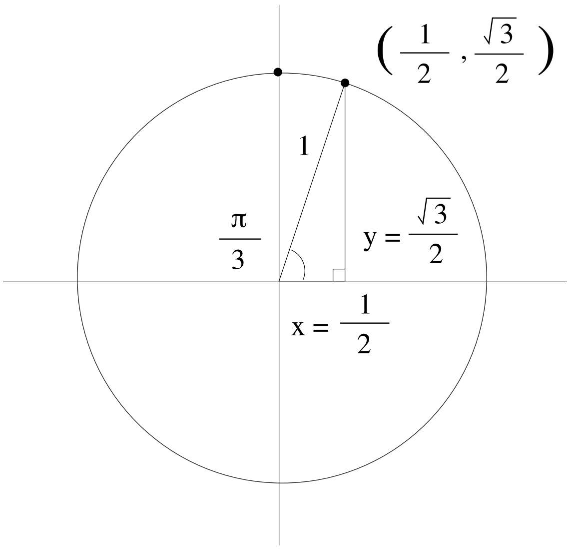
Thus

$$\sin\frac{\pi}{4} = y = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \cos\frac{\pi}{4} = x = \frac{1}{\sqrt{2}}.$$

We can find $\sin\frac{\pi}{3}$, $\cos\frac{\pi}{3}$, $\sin\frac{\pi}{6}$, and $\cos\frac{\pi}{6}$ in a similar fashion by noting that $\frac{\pi}{3}$ radians equals 60° and $\frac{\pi}{6}$ radians equals 30°. A 30°-60°-90° triangle with a hypotenuse of length 1 gives us the information we need.







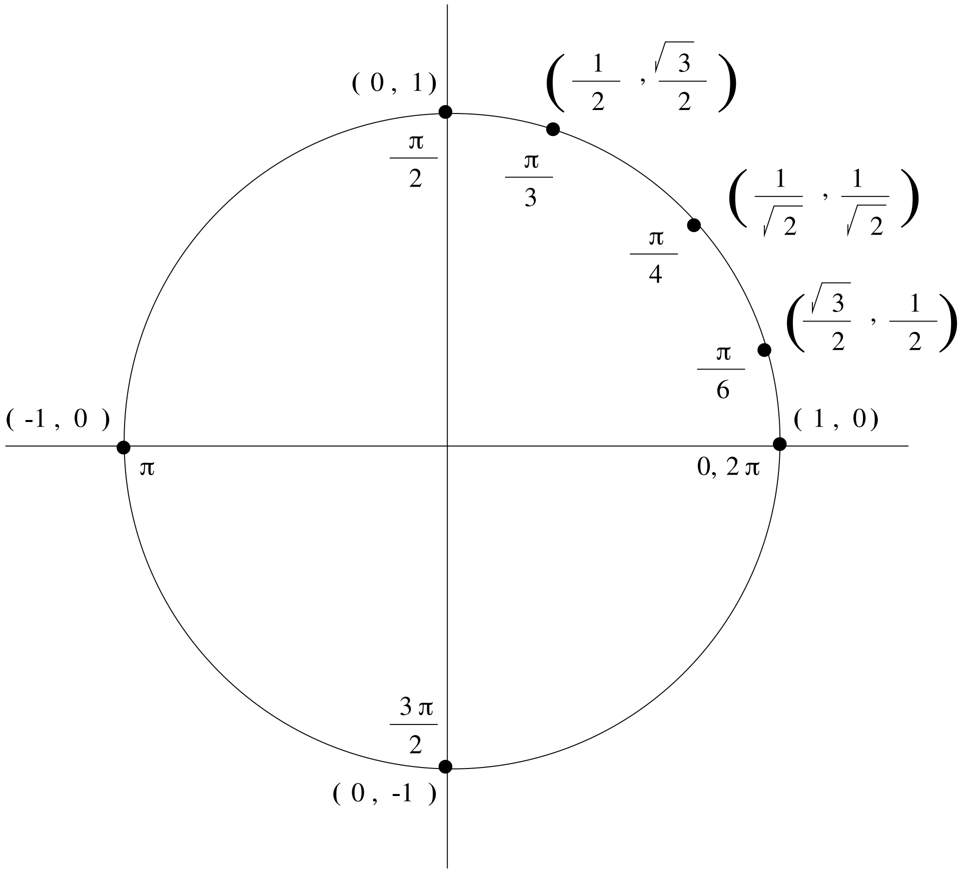
Thus

$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \text{and} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

and

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos\frac{\pi}{3} = \frac{1}{2}.$$

In calculus, we measure angles in *radians*, and we often use the trig values we just found, so it will be helpful to memorize the “enhanced” unit circle on the next page.



Four other trigonometric functions are defined using sine and cosine. They are the **secant** ("sec"), **cosecant** ("csc"), **tangent** ("tan"), and **cotangent** ("cot") trigonometric functions, defined as follows.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

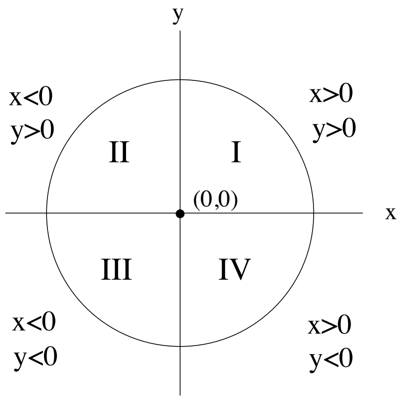
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

Since $\cos\theta$ and $\sin\theta$ are 0 for some values of θ, the trig functions $\sec\theta$, $\csc\theta$, $\tan\theta$, and $\cot\theta$ are undefined for some values of θ. For more information, see the graphs of the trig functions at the end of this section.

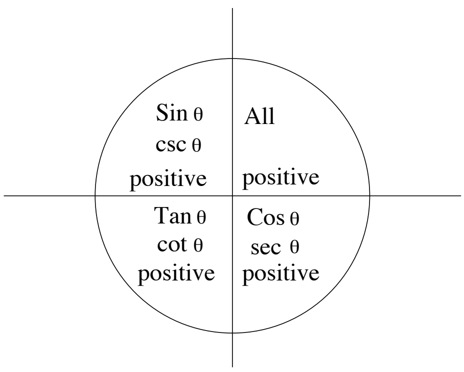
Example 3: Find $\cot \frac{\pi}{6}$.

$$\cot \frac{\pi}{6} = \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

The unit circle has four quadrants.

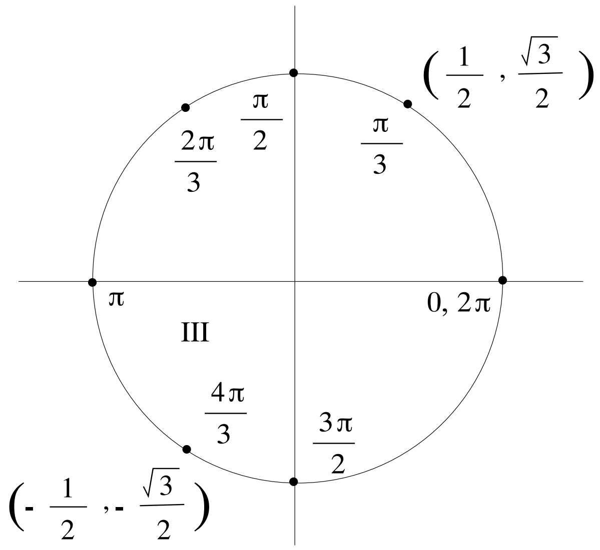


Note that a point in Quad II will have a negative x-value and a positive y-value. Thus, cosine is negative and sine is positive in Quad II. To remember which trig functions are positive in which quadrants, use the saying, "All Students Take Calculus."



Example 4: Find $\cos \frac{4\pi}{3}$.

We will first find in which quadrant $\frac{4\pi}{3}$ lies.



Due to symmetry, $\frac{\pi}{3}$ and $\frac{4\pi}{3}$ have the same coordinates except for the negative signs. Thus

$$\cos\frac{4\pi}{3} = -\frac{1}{2}.$$

Trigonometric Identities

The following trigonometric identities will be useful in calculus.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$
$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$
$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta\end{aligned}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$

It is *not* necessary to memorize all of the above identities! By knowing how an identity is derived, one can reduce the amount of memorization necessary.

The equation of the unit circle is $x^2 + y^2 = 1$. The Pythagorean theorem also gives us the equation $x^2 + y^2 = 1$. This equation gives us

$$\cos^2\theta + \sin^2\theta = 1.$$

Dividing this equation by $\cos^2\theta$, we obtain

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \text{or} \qquad 1 + \tan^2\theta = \sec^2\theta.$$

Dividing $\cos^2\theta + \sin^2\theta = 1$ by $\sin^2\theta$ we obtain

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad \text{or} \qquad \cot^2\theta + 1 = \csc^2\theta.$$

Using

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

and

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

with $a = b = \theta$, we obtain the formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \text{and} \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta.$$

Also

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

and

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= (1 - \sin^2\theta) - \sin^2\theta \\ &= 1 - 2\sin^2\theta\end{aligned}$$

using the identity $\cos^2\theta + \sin^2\theta = 1$. We can then solve for $\cos^2\theta$ in $\cos 2\theta = 2\cos^2\theta - 1$ to get

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}.$$

Solving for $\sin^2\theta$ in $\cos 2\theta = 1 - 2\sin^2\theta$ gives us

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}.$$

Finding the Values of Trigonometric Functions

Given the values for one trig function, we can find the values for the other five trig functions. There are two ways to do this.

Method 1: Use identities.

Example 5: (Using Method 1) Given θ is in Quad III and $\cot\theta = 2$, find the values of the remaining trig functions.

First, cotangent and tangent are reciprocals, so

$$\tan\theta = \frac{1}{\cot\theta} = \frac{1}{2}.$$

Next use the identity $\cot^2\theta + 1 = \csc^2\theta$ to get $\csc\theta$:

$$\cot^2\theta + 1 = \csc^2\theta$$
$$2^2 + 1 = \csc^2\theta$$
$$\csc\theta = \pm\sqrt{5}$$
$$\csc\theta = -\sqrt{5}$$

since θ is in Quad III.

Cosecant and sine are reciprocals, so

$$\sin\theta = \frac{1}{\csc\theta} = -\frac{1}{\sqrt{5}}.$$

Using $\sin^2\theta + \cos^2\theta = 1$, we get

$$\cos^2\theta + \left(-\frac{1}{\sqrt{5}}\right)^2 = 1$$

$$\cos^2\theta = \frac{4}{5}$$
$$\cos\theta = \pm\frac{2}{\sqrt{5}}$$
$$\cos\theta = -\frac{2}{\sqrt{5}}$$

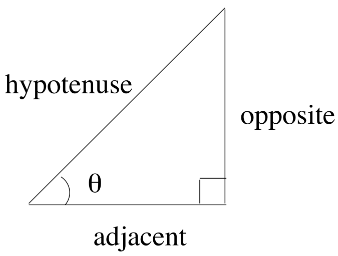
since θ is in Quad III.

Cosine and secant are reciprocals, so

$$\sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2}.$$

This gives us all six trig function values.

Method 2: Use a right triangle with 0 radians $< \theta < \frac{\pi}{2}$ radians.



This gives

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

One can remember these equations using the acronym "SOH CAH TOA" where S = sine, O = opposite, H = hypotenuse, C = cosine, A = adjacent, and T = tangent.

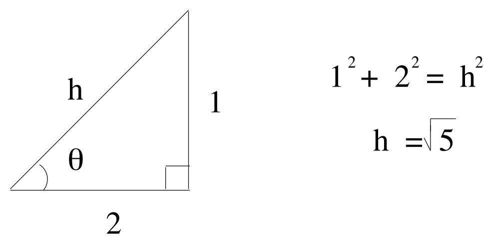
(In the previous triangle, we assumed θ was in Quad I. However, by using symmetry, we can assume θ is in any quadrant. But be careful of the SIGNS of the trig functions when θ is in Quad II, III, or IV!)

We will now work Example 5 again using Method 2.

Example 5: (Using Method 2) Given θ is in Quad III and $\cot\theta = 2$, find the values of the remaining trig functions.

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}} = \frac{2}{1}$$

We then have the following triangle.



Next note that θ is in Quad III. Thus

$$\cos\theta = -\frac{\text{adj}}{\text{hyp}} = -\frac{2}{\sqrt{5}} \qquad \sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{5}}{2}$$

$$\sin\theta = -\frac{\text{opp}}{\text{hyp}} = -\frac{1}{\sqrt{5}} \qquad \csc\theta = \frac{1}{\sin\theta} = -\sqrt{5}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{1}{2}$$

This gives us the values of all six trig functions.

Solving Equations Involving Trig Functions

We will begin by considering equations with one term involving a trigonometric function.

Example 6: Solve $2\sin x = 1$.

We want to find which values of x make this equation true. We will *not* rewrite this equation as $x = \cdots$. We will isolate the trig function instead:

$$\sin x = \frac{1}{2}$$

To find the solutions to this equation, we find the radian value(s) that will give us sine equal to $\frac{1}{2}$.

For x in $[0, 2\pi]$, we have 2 solutions: $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

However, if x can be any number, note that $\frac{\pi}{6} + 2\pi$, $\frac{\pi}{6} + 4\pi$, $\frac{\pi}{6} - 2\pi$, and $\frac{5\pi}{6} + 2\pi$ are also solutions. In fact, for x in $(-\infty, \infty)$, we have an infinite number of solutions that can be represented as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi$$

where n is any integer.

Example 7: Solve $2 \sin 2x = -1$ for x in $[0, 2\pi]$.

We will begin by solving for $\sin 2x$.

$$\sin 2x = -\frac{1}{2}$$

We want solutions x for which

$$0 \leq x \leq 2\pi \quad \text{or} \quad 0 \leq 2x \leq 4\pi.$$

For which radian values θ between 0 and 4π does $\sin \theta$ equal $-\frac{1}{2}$?

Between 0 and 2π, $\sin \theta = -\frac{1}{2}$ for

$$\theta = \frac{7\pi}{6} \text{ and } \theta = \frac{11\pi}{6}.$$

Between 2π and 4π, $\sin \theta = -\frac{1}{2}$ for

$$\theta = \frac{7\pi}{6} + 2\pi = \frac{19\pi}{6} \text{ and } \theta = \frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}.$$

To compensate for the $2x$ in $\sin 2x$, we will set $\theta = 2x$.

$$\sin \frac{7\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad 2x = \frac{7\pi}{6} \qquad \Rightarrow \qquad x = \frac{7\pi}{12}$$

Similarly,

$$\sin \frac{11\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad x = \frac{11\pi}{12},$$
$$\sin \frac{19\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad x = \frac{19\pi}{12},$$

and

$$\sin\frac{23\pi}{6} = -\frac{1}{2} = \sin 2x \qquad \Rightarrow \qquad x = \frac{23\pi}{12}.$$

Thus the solutions for x in $[0, 2\pi]$ are

$$x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}.$$

We will now consider equations with more than one term involving a trigonometric function. The key concept involved in these types of equations is factoring.

Example 8: Solve $2\cos^2 x \tan x - \tan x = 0$ for x in $[0, 2\pi]$.

We begin by factoring $\tan x$ out of each term.

$$\tan x(2\cos^2 x - 1) = 0.$$

Thus either

$\tan x = 0$ or $2\cos^2 x - 1 = 0.$

If $\tan x = 0$, then
$x = 0, \pi, 2\pi.$

If $2\cos^2 x - 1 = 0$, then
$\cos^2 x = \frac{1}{2}$
and $\cos x = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}.$
So for x in $[0, 2\pi]$,
$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$

The solution set is

$$\left\{0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}.$$

In some problems we will first use identities and then factor.

Example 9: Solve $\cos x = \cos 2x$ for x in $[0, 2\pi]$.

We will first use the identity $\cos 2x = 2\cos^2 x - 1$.

$$\cos x = 2\cos^2 x - 1$$

$$2\cos^2 x - 1 - \cos x = 0$$

Next, factor.

$$2\cos^2 x - \cos x - 1 = 0$$

$$(2\cos x + 1)(\cos x - 1) = 0$$

Thus either

$$2\cos x + 1 = 0 \qquad \text{or} \qquad \cos x - 1 = 0.$$

If $2\cos x + 1 = 0$, then
$\cos x = -\frac{1}{2}$
and $x = \frac{2\pi}{3}, \frac{4\pi}{3}$.

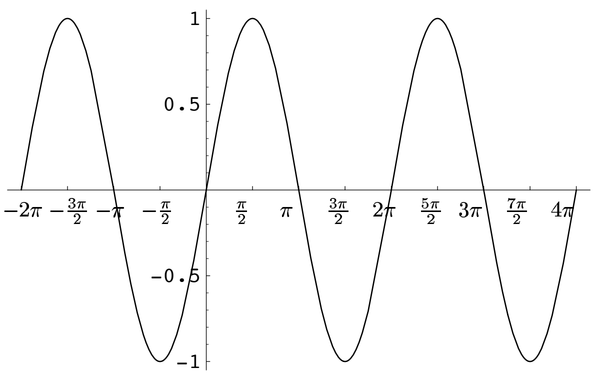
If $\cos x - 1 = 0$, then
$\cos x = 1$
and $x = 0, 2\pi$.

So the solution set is

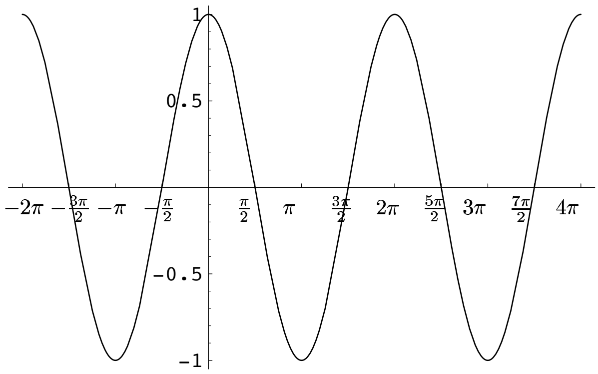
$$\left\{0, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}.$$

Graphs of Trig Functions

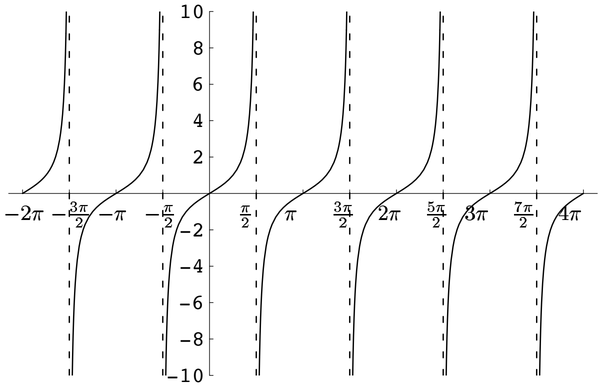
The graphs of the six trig functions are shown below. The trig functions are all periodic. (A function is **periodic** with period p if $f(x+p) = f(x)$ for all real numbers x. Such functions repeat every p units along the x-axis.) Sine and cosine have periods of 2π. Tangent, cotangent, secant, and cosecant all have periods of π.



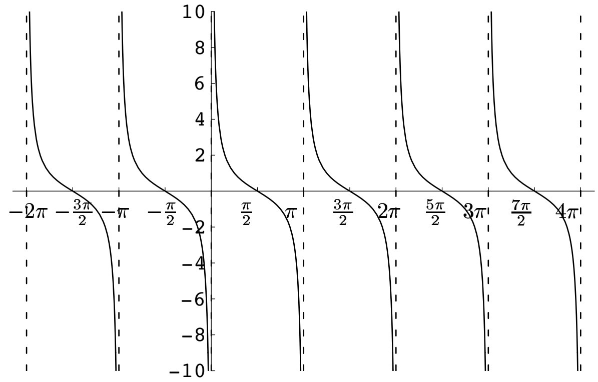
$y = \sin x$



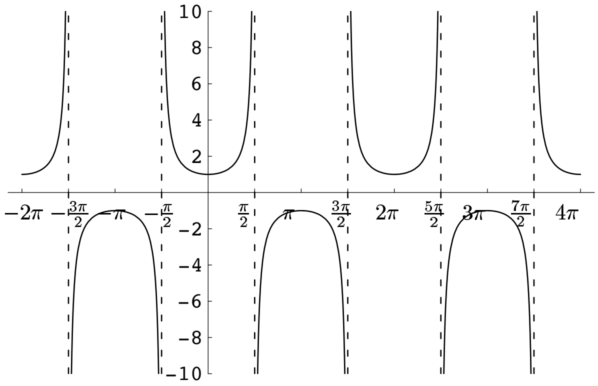
$y = \cos x$



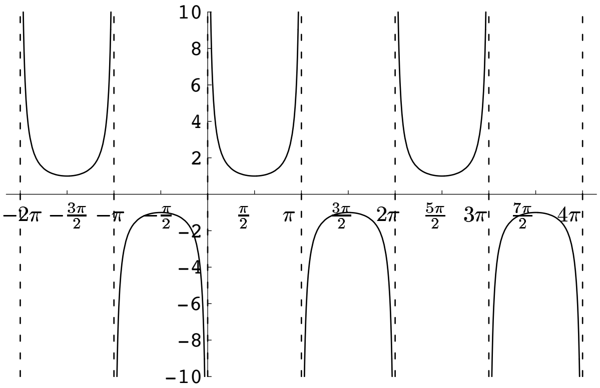
$y = \tan x$



$y = \cot x$



$y = \sec x$



$y = \csc x$

Exercises

Change to radian measure:

1. 50° 2. 120° 3. 375° 4. -12°

Change to degree measure:

5. $-\frac{5\pi}{6}$ 6. $\frac{35\pi}{12}$ 7. $\frac{7\pi}{8}$ 8. $-\frac{2\pi}{3}$

Find the function values:

9. $\sin \frac{5\pi}{3}$ 10. $\tan \frac{\pi}{6}$ 11. $\csc \frac{11\pi}{4}$ 12. $\cos\left(-\frac{2\pi}{3}\right)$

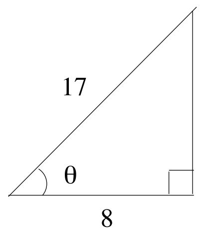
13. $\sec \frac{11\pi}{6}$ 14. $\sin\left(-\frac{3\pi}{2}\right)$ 15. $\cot \frac{5\pi}{4}$ 16. $\cos \frac{5\pi}{6}$

A function value and a quadrant are specified. Find the other five function values.

17. $\sin\theta = \frac{1}{3}$, II 18. $\sec\theta = \frac{5}{3}$, I

19. $\tan\theta = 5$, III 20. $\cot\theta = -4$, IV

21. Find the six trigonometric function values for the following θ:



Solve, finding all solutions:

22. $\tan x = \sqrt{3}$ 23. $2\cos^2 x = 1$ 24. $2\sin^2 x - 5\sin x + 2 = 0$

Solve, finding all solutions in $[0, 2\pi]$.

25. $\sec^2 x - 4 = 0$ 26. $2\sin^3 x = \sin x$ 27. $\cos 2x \sin x + \sin x = 0$

28. $\sec^2 x = 4\tan^2 x$ 29. $\cos 2x - \sin x = 1$

Answers to odd-numbered exercises

1. $\dfrac{5\pi}{18}$
3. $\dfrac{25\pi}{12}$
5. $-150°$
7. $157.5°$
9. $-\dfrac{\sqrt{3}}{2}$
11. $\sqrt{2}$
13. $\dfrac{2}{\sqrt{3}}$
15. 1
17. $\cos\theta = -\dfrac{2\sqrt{2}}{3}$, $\tan\theta = -\dfrac{1}{2\sqrt{2}}$, $\sec\theta = -\dfrac{3}{2\sqrt{2}}$, $\cot\theta = -2\sqrt{2}$, $\csc\theta = 3$
19. $\cos\theta = -\dfrac{1}{\sqrt{26}}$, $\sin\theta = -\dfrac{5}{\sqrt{26}}$, $\sec\theta = -\sqrt{26}$, $\csc\theta = -\dfrac{\sqrt{26}}{5}$, $\cot\theta = \dfrac{1}{5}$
21. $\cos\theta = \dfrac{8}{17}$, $\sin\theta = \dfrac{15}{17}$, $\tan\theta = \dfrac{15}{8}$, $\sec\theta = \dfrac{17}{8}$, $\csc\theta = \dfrac{17}{15}$, $\cot\theta = \dfrac{8}{15}$
23. $\left\{\dfrac{\pi}{4} + 2n\pi, \dfrac{3\pi}{4} + 2n\pi\dfrac{5\pi}{4} + 2n\pi, \dfrac{7\pi}{4} + 2n\pi\right\}$ or $\left\{\dfrac{\pi}{4} + \dfrac{n}{2}\pi\right\}$
25. $\left\{\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}\right\}$
27. $\left\{0, \pi, 2\pi, \dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}$
29. $\left\{0, \pi, 2\pi, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right\}$