Strategy to Calculate Limits

To compute $\lim_{x\to a} f(x)$:

1. Try to plug the value of a directly into the function.
   - If we get a number or the limit 'blows up' then we are done!
   - You *should* be so lucky. Typically the value is undefined, having the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.
2. If we do not get a number then we need to simplify the expression. Simplification can involve any number of techniques including but certainly not limited to:
   - Use the definition of the limit
   - Use of the limit rules
   - Factoring
   - Multiplying by the conjugate
   - Finding a common denominator
   - Using the Squeeze Theorem
   - Applying some memorized limit such as $\lim_{x\to 0} \frac{\sin x}{x} = 1$
3. Try to plug the number into the function once again. If we get a number or $\pm\infty$ then we are done. Otherwise go back to step 2.

A quick word of warning .... l'Hôpital's theorem should not be used at this point since it involves taking derivatives. Most instructors don't give any credit for limits found using this method at this point.

Like anything else, the best way to get proficient at finding limits is with practice. We conclude with a few examples.

Examples

1. Find $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x, & x < 1 \\ 0, & x = 1 \\ -x+2, & x > 1. \end{cases}$

This one requires the definition of the limit. From the left one has

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1 = L^-$$

and from the right,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (-x+2) = 1 = L^+$$

since $L^+ = L^- = 1$, $\lim_{x\to 1} f(x) = 1$.

2. Find $\lim_{x\to -1} \frac{\sqrt{x^2+8}}{2x+4}$.

For this one we just plug in to find $\sqrt{9}/2 = 3/2$.

3. Find $\lim_{x\to-\infty}\dfrac{x^4+1}{x^3-1}$. If we try to plug in we get the indeterminate form ∞/∞. The trick here is to divide the top and bottom by the highest power in the denominator.

$$\lim_{x\to-\infty}\frac{x^4+1}{x^3-1}=\lim_{x\to-\infty}\frac{x^4+1}{x^3-1}\frac{(1/x^3)}{(1/x^3)}=\lim_{x\to-\infty}\frac{x+1/x^3}{1-1/x^3}\to-\infty$$

so the limit does not exist.

4. Find $\lim_{x\to4}\dfrac{x-4}{\sqrt{x}-2}$.
This limit requires that we multiply by the conjugate.

$$\lim_{x\to4}\frac{x-4}{\sqrt{x}-2}\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}=\lim_{x\to4}\frac{(x-4)(\sqrt{x}+2)}{x-4}=\lim_{x\to4}\frac{\sqrt{x}+2}{1}=2+2=4$$

allowing us to conclude that the limit is 4.

5. Find $\lim_{x\to0}\dfrac{\sec x-1}{x^2}$.
This one is a bit of a challenge. We start with multiplying by the conjugate, then use the trigonometric identity $\sec^2 x-1=\tan^2 x$ and finally we separate into three pieces.

$$\begin{aligned}\lim_{x\to0}\frac{\sec x-1}{x^2} &= \lim_{x\to0}\frac{\sec x-1}{x^2}\frac{(\sec x+1)}{(\sec x+1)}=\lim_{x\to0}\frac{\tan^2 x}{x^2(\sec x+1)}\\ &= \lim_{x\to0}\left(\frac{\sin x}{x}\right)^2\lim_{x\to0}\left(\frac{1}{\cos^2 x}\right)\lim_{x\to0}\left(\frac{1}{\sec x+1}\right)\\ &= 1^2\cdot\frac{1}{1}\cdot\frac{1}{2}=\frac{1}{2}.\end{aligned}$$