9 Exponents \( \frac{1}{n} \)

For a positive integer \( n \), a real number \( y \) is an \( n \)-th root of \( x \) if \( y^n = x \).

If \( n \) is odd, every real number \( x \) has exactly one \( n \)-th root.

If \( n \) is even, a real number \( x \) has no \( n \)-th root when \( x < 0 \), exactly one \( n \)-th root when \( x = 0 \), and exactly two \( n \)-th roots when \( x > 0 \). If \( x \) has two \( n \)-th roots, one is positive and the other is its negative.

The symbol \( \sqrt[n]{x} \) is defined to be the \( n \)-th root of \( x \) when \( n \) is odd, or the non-negative \( n \)-th root of \( x \), if one exists, when \( n \) is even.

Notice that if \( n \) is odd, \( \sqrt[n]{x} \) exists and has the same sign as \( x \).

Example 1

- \( \sqrt[3]{64} = 4 \)
- \( \sqrt[3]{-64} = -4 \)
- \( \sqrt[2]{64} = 8 \)
- \( -\sqrt[3]{64} = (-1)\sqrt[3]{64} = -8 \)
- \( \sqrt[2]{-64} \) is not defined.

The expression \( x^{1/n} \) is just another way of writing \( \sqrt[n]{x} \). Notice that if \( x^{1/n} \) is defined, then \( (x^{1/n})^n = x \).

If \( n \) is even and \( x \) is non-negative, then \( (x^n)^{1/n} = x \), because \( x^n \) is non-negative and \( (x^n)^{1/n} \) is the non-negative \( n \)-th root of \( x^n \). If \( n \) is even and \( x \) is negative, then \( x^n \) is positive, so that \( (x^n)^{1/n} = |x| \).

For example \( ((-3)^2)^{1/2} = 9^{1/2} = 3 \). Notice that, in this case, the expression \( ((-3)^{1/2})^2 \) does not make sense because \( (-3)^{1/2} \) is not defined.

When \( n \) is odd, \( x^n \) has the same sign as \( x \), so that \( (x^n)^{1/n} \) also has the same sign as \( x \). Therefore \( (x^n)^{1/n} = x \). For example \( ((-3)^3)^{1/3} = (-27)^{1/3} = -3 \) and, since \( (-3)^{1/3} \) is defined, \( ((-3)^{1/3})^3 = -3 \).

Example 2 Find \( x \) if \( x^{1/2} = \frac{3}{2} \).

Since \( x^{1/2} = \frac{3}{2} \), we know that

\[
x = \left(\frac{3}{2}\right)^4 = \frac{81}{16}.
\]

Example 3 Find \( x \) if \( x^{1/3} = \left(\frac{5}{4}\right)^{1/2} \).
Since \( x^{\frac{1}{3}} = \left( \frac{5}{4} \right)^{\frac{1}{7}} \), we know that
\[
x = \left( x^{\frac{1}{3}} \right)^3 = \left( \left( \frac{5}{4} \right)^{\frac{1}{7}} \right)^3 = \left( \sqrt[7]{\frac{5}{4}} \right)^3 = \sqrt[7]{\frac{125}{64}}.
\]

9.1 Practice Problems

In questions 1 - 5, use the rules of exponents to find an expression equivalent to given expression.

1. \( 36^{\frac{1}{2}} \)
2. \( (-125)^{\frac{1}{3}} \)
3. \( \left( \frac{9}{4} \right)^{\frac{1}{2}} \)
4. \( x^{\frac{3}{2}}x^{-\frac{1}{2}} \)
5. \( (m^{\frac{1}{2}}n^{-\frac{1}{3}})^{-\frac{1}{6}} \)

6. Find \( x \) if \( x^{\frac{1}{5}} = \left( \frac{2}{3} \right)^{\frac{1}{7}}. \)

9.2 Solutions

1. 6  2. -5  3. \( \frac{3}{2} \)  4. \( x^{\frac{1}{20}} \)  5. \( \frac{n^{1/18}}{m^{1/12}} \)  6. \( \frac{32}{243} = \frac{4}{9} \sqrt[3]{\frac{2}{3}} \)