2.4 Exponents $\frac{1}{n}$

For a positive integer n, a real number y is an **n-th root of** x if $y^n = x$.

If n is odd, every real number x has exactly one n-th root.

If n is even, a real number x has no n-th root when $x < 0$, exactly one n-th root when $x = 0$, and exactly two n-th roots when $x > 0$. If x has two n-th roots, one is positive and the other is its negative.

The symbol $\sqrt[n]{x}$ is defined to be the n-th root of x when n is odd, or the non-negative n-th root of x, if one exists, when n is even.

Notice that if n is odd, $\sqrt[n]{x}$ exists and has the same sign as x.

Example 17

- $\sqrt[3]{64} = 4$
- $\sqrt[3]{-64} = -4$
- $\sqrt[2]{64} = 8$
- $-\sqrt[2]{64} = (-1)\sqrt[2]{64} = -8$
- $\sqrt[2]{-64}$ is not defined.

The expression $x^{\frac{1}{n}}$ is just another way of writing $\sqrt[n]{x}$. Notice that if $x^{\frac{1}{n}}$ is defined, then $\left(x^{\frac{1}{n}}\right)^n = x$.

If n is even and x is non-negative, then $(x^n)^{\frac{1}{n}} = x$, because x^n is non-negative and $(x^n)^{\frac{1}{n}}$ is the non-negative n-th root of x^n. If n is even and x is negative, then x^n is positive, so that $(x^n)^{\frac{1}{n}} =| x |$.

For example $((-3)^2)^{\frac{1}{2}} = 9^{\frac{1}{2}} = 3$. Notice that, in this case, the expression $\left((-3)^{\frac{1}{2}}\right)^2$ does not make sense because $(-3)^{\frac{1}{2}}$ is not defined.

When n is odd, x^n has the same sign as x, so that $(x^n)^{\frac{1}{n}}$ also has the same sign as x. Therefore $(x^n)^{\frac{1}{n}} = x$. For example $((-3)^3)^{\frac{1}{3}} = (-27)^{\frac{1}{3}} = -3$ and, since $(-3)^{\frac{1}{3}}$ is defined, $\left((-3)^{\frac{1}{3}}\right)^3 = -3$.

Example 18 *Find x if $x^{\frac{1}{4}} = \frac{3}{2}$.*

Since $x^{\frac{1}{4}} = \frac{3}{2}$, we know that

$$x = \left(x^{\frac{1}{4}}\right)^4 = \left(\frac{3}{2}\right)^4 = \frac{3^4}{2^4} = \frac{81}{16}.$$

Example 19 *Find x if $x^{\frac{1}{3}} = \left(\frac{5}{4}\right)^{\frac{1}{7}}$.*

Since $x^{\frac{1}{3}} = \left(\frac{5}{4}\right)^{\frac{1}{7}}$, we know that

$$x = \left(x^{\frac{1}{3}}\right)^3 = \left(\left(\frac{5}{4}\right)^{\frac{1}{7}}\right)^3 = \left(\sqrt[7]{\frac{5}{4}}\right)^3 = \sqrt[7]{\frac{125}{64}}.$$

2.4.1 Practice Problems

In questions 1 - 5, use the rules of exponents to find an expression equivalent to given expression.

1. $36^{\frac{1}{2}}$
2. $(-125)^{\frac{1}{3}}$
3. $(\frac{9}{4})^{\frac{1}{2}}$
4. $x^{\frac{1}{4}}x^{-\frac{1}{5}}$
5. $(m^{\frac{1}{2}}n^{-\frac{1}{3}})^{-\frac{1}{6}}$

6. Find x if $x^{\frac{1}{5}} = \left(\frac{2}{3}\right)^{\frac{1}{2}}$.

2.4.2 Solutions

1. 6 2. -5 3. $\frac{3}{2}$ 4. $x^{\frac{1}{20}}$ 5. $\dfrac{n^{1/18}}{m^{1/12}}$ 6. $\sqrt{\frac{32}{243}} = \frac{4}{9}\sqrt{\frac{2}{3}}$