2.2 Finding roots

A **root** of a function f is a number x such that $f(x) = 0$. Roots are also called **zeroes**. Some functions, such as $f(x) = x^2 + 1$, don't have roots that are real numbers. If $x = a$ is a root of the function $f(x)$, then the point $(a, 0)$ is an x-intercept of the graph $y = f(x)$.

Example 9 *Find the roots of* $f(x) = 3x - 2$.

Notice that $f(x) = 0$ exactly if $3x - 2 = 0$, and it is straightforward to solve this equation for x:

$$\begin{aligned} 3x - 2 &= 0 \\ 3x &= 2 \\ x &= \tfrac{2}{3}. \end{aligned}$$

Therefore $x = \frac{2}{3}$ is the only root of $f(x)$.

If $p(x)$ is a polynomial and $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$, so we can write $p(x) = (x - a)q(x)$ for some polynomial $q(x)$. Conversely, if $p(x) = (x - a)q(x)$ for some polynomial $q(x)$ then $p(x)$ is a polynomial and $p(a) = 0$. In order to find roots of a polynomial, it is therefore useful to be able to factor it – and to factor a polynomial it is useful to know one or more of its roots.

It is not always easy to factor a higher-degree polynomial, but using the quadratic formula you can factor any quadratic polynomial. If $p(x) = ax^2 + bx + c$, and $a \neq 0$, then the real roots of p are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$, provided these are real numbers. If $b^2 - 4ac = 0$ then there is only one root, because $\frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b}{2a}$. If $b^2 - 4ac < 0$ then there are no real roots, because $\sqrt{b^2 - 4ac}$ is not a real number.

Example 10 *Find the roots of* $f(x) = 3x^2 - 7x + 4$.

Again, $f(x) = 0$ exactly when $3x^2 - 7x + 4 = 0$. This time we cannot immediately solve the equation for x, but we can factor the quadratic: $3x^2-7x+4 = (3x-4)(x-1)$. Now, $(3x-4)(x-1) = 0$ if and only if $(3x - 4) = 0$ or $(x - 1) = 0$, i.e. when $x = \frac{4}{3}$ or $x = 1$.

These two roots correspond to factors of the original polynomial:

$$3x^2 - 7x + 4 = (3x - 4)(x - 1) = 3\left(x - \frac{4}{3}\right)(x - 1).$$

Example 11 *Find the roots of* $f(x) = x^2 - 6x + 7$.

This quadratic does not factor in an obvious way (because, at it will turn out, its roots are not rational). This time we should use the quadratic formula to find the roots:

$$\begin{aligned} x = \tfrac{-b\pm\sqrt{b^2-4ac}}{2a} &= \tfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(7)}}{2(1)} \\ &= \tfrac{6\pm\sqrt{36-28}}{2} \\ &= \tfrac{6\pm\sqrt{8}}{2} \\ &= \tfrac{6\pm2\sqrt{2}}{2} \\ &= 3 \pm \sqrt{2}, \end{aligned}$$

so the roots are $3+\sqrt{2}$ and $3-\sqrt{2}$.

You can check this answer by verifying that $f\left(3+\sqrt{2}\right)=0$ and $f\left(3-\sqrt{2}\right)=0$.

These two roots correspond to factors of the original polynomial:

$$x^2-6x+7=\left(x-\left(3-\sqrt{2}\right)\right)\left(x-\left(3+\sqrt{2}\right)\right).$$

Multiplying out the factored form is another way to verify the answer.

The next two examples involve functions which are not polynomials.

Example 12 *Find the roots of $f(x)=1-2^x$.*

Although we can not easily factor $1-2^x$, we can still think about the values of x at which $f(x)=0$. Notice that $1-2^x=0$ if and only if $1=2^x$. The only time 2^x is equal to 1 is when $x=0$, and so the only root of $f(x)$ is $x=0$. Because $f(x)$ is not a polynomial, it is not possible to write it in a factored form as we did in the previous examples.

Example 13 *Find the roots of $f(x)=\sqrt{x-1}$.*

If $0=\sqrt{x-1}$ then $0^2=x-1$, so $0=x-1$. Therefore, $x=1$ is the only possible root of $f(x)$, and it is a root of $f(x)$ because $f(1)=0$.

We will find roots of other types of functions later on.

2.2.1 Practice Problems

Find the roots of the following functions:

1. $f(x)=3x^2+5x+2$
2. $f(x)=x^2+5x+2$
3. $f(x)=\sqrt{x+2}-x+4$
4. $f(x)=(x+5)^2+(2x-7)^2-82$
5. $f(x)=x^3+x^2-2x$

2.2.2 Solutions

1. $-\frac{2}{3},-1$ 2. $x=\frac{-5+\sqrt{17}}{2},\frac{-5-\sqrt{17}}{2}$ 3. $x=2,7$ 4. $x=-\frac{2}{5},4$ 5. $x=0,1,-2$