Chapter 2

Module 2

2.1 Functions

Let D be a set of real numbers. Any way of associating exactly one real number with each element of D is called a **function**. The set D is called the function's **domain**. For any x in D, we call $f(x)$ the **value of** f **at** x. The set $\{f(x) : x \text{ is in } D\} = \{y : y = f(x) \text{ for some x in } D\}$ is called the function's **range** – that is, the range of f is the set of all numbers that occur as values of f.

When the function f is described by giving a formula, then for any number in the domain the value of f at that number is obtained by substituting that number for the variable in the formula.

For example, if the function f is described by $f(x) = 5x^2 - x + 1$ and the domain is the set of all real numbers, $\mathbb{R}$, then to find $f(2)$ we substitute 2 into the formula for x, as in:

$$f(2) = 5(2)^2 - (2) + 1 = 5 \cdot 4 - 2 + 1 = 20 - 2 + 1 = 19.$$

Therefore, $f(2) = 19$.

Assumed domains. When the domain D is not explicitly given in the description of the function, we assume it is the largest collection of numbers for which the function makes sense (is defined).

Finding the range. When finding the range of a function that's described by an expression, there are usually two steps involved. The first step is to determine a set of possible values the function could take. The second step is to verify that each one of these is actually achieved for at least one x in the domain.

Example 7 For each function, find the assumed domain D, that is, the largest collection of numbers for which the function is defined. In addition, find the range of the function.

- $f(x) = x^2 - 5$.
  The expression $x^2 - 5$ is defined for all real numbers, so assumed domain is the set of all real numbers, $\mathbb{R}$.

  Since, for every real number x, the quantity x^2 is at least zero, $x^2 - 5$ is at least $0 - 5 = -5$. Therefore the possible numbers that can be in the range are the real numbers which are at least -5. Take any $y \geq -5$. Then $y = x^2 - 5$ if and only if $y + 5 = x^2$, and $y + 5 \geq 0$ so $x = \pm\sqrt{y+5}$. Hence there is always at least one number x the domain such that $f(x) = y$, so the range is $\{y : y \geq -5\}$.

- $f(x) = \sqrt{x-5} + 3$.
  The expression $\sqrt{x-5} + 3$ involves a square root, and the square root function only makes sense for non-negative numbers (it is only defined for non-negative numbers). That means we need to have $x - 5 \geq 0$, or equivalently $x \geq 5$. There are no other restrictions on the values of x that can be used. Therefore, the domain is $\{x : x \geq 5\}$.

  Since for every $x \geq 5$ the quantity $\sqrt{x-5} \geq 0$, we know that $\sqrt{x-5} + 3 \geq 0 + 3 = 3$. Therefore the possible numbers that can be in the range are the real numbers which are at least 3. Take any $y \geq 3$. Then $y = \sqrt{x-5} + 3$ if and only if $y - 3 = \sqrt{x-5}$, which implies $(y-3)^2 = x - 5$. Thus, if $f(x) = y$ then $x = (y-3)^2 + 5$. Since $y \geq 3$, the quantity $(y-3)^2 + 5$ is in the domain. Furthermore, if $x = (y-3)^2 + 5$, then

  $$f(x) = f((y-3)^2+5) = \sqrt{(y-3)^2+5-5}+3 = \sqrt{(y-3)^2}+3 = |y-3|+3 = (y-3)+3 = y.$$

  We've used $|y-3| = y-3$ because $y - 3 \geq 0$.

  Therefore, the range is $\{y : y \geq 3\}$.

A function can be defined by an equation. For example, for the equation $y = 10x - 7$, once we choose a value for x, then y is uniquely determined: as in the definition of a function, the equation $y = 10x - 7$ associates exactly one number y with each number x.

Example 8 *Does the equation $x = y^2 + 5$ define y as a function of x?*

No. For example, when $x = 6$, the statements $6 = (-1)^2 + 5$ and $6 = 1^2 + 5$ are both true. Thus this equation associates two values of y, namely ± 1, with $x = 6$, and so it does not define y as a function of x.

2.1.1 Practice Problems

1. Using $f(x) = 2x^2 - x + 1$, find:
   (a) $f(3)$
   (b) $f(x)$ when $x = 4$
   (c) $f(-3)$
2. Using $f(x) = x^3 - 3x^2 + 5$, find:
   (a) $f(0)$
   (b) $f(3)$
   (c) $f(-3)$
3. Let $f(x) = 10 - x^2$. Find:

(a) The (assumed) domain of f.

(b) The range of f.

4. Let $f(x) = 2 - \sqrt{x^2 - 25}$. Find:

   (a) The (assumed) domain of f.

   (b) The range of f.

5. Which of the following equations define y as a function of x?

   (a) $y = x^2 + 8$

   (b) $x^2 - y^2 = 16$

   (c) $y = 5$

2.1.2 Solutions

1. (a) 16
   (b) 29
   (c) 22

2. (a) 5
   (b) 5
   (c) -49

3. (a) $\mathbb{R}$
   (b) $\{y : y \leq 10\}$

4. (a) $(-\infty, -5] \cup [5, \infty)$
   (b) $\{y : y \leq 2\}$

5. (a) This equation does define y as a function of x.
   (b) This equation does not define y as a function of x.
   (c) This equation does define y as a function of x.