1.3 Integer Exponents

For a non-negative integer n:

- The notation a^n represents a multiplied by itself n times. ex: $5^3 = 5 \times 5 \times 5$.
- If $a \neq 0$, then $a^0 = 1$ because a multiplied by itself zero times is a product that has no terms, and a product that has no terms equals 1. Note: 0^0 is not a number; it is an indeterminate form that will be studied in calculus.
- **Multiplying exponents:** When multiplying powers of the same base, add the exponents, because:

  $$a^n \times a^m = \underbrace{aa\cdots a}_{n \text{ times}} \times \underbrace{aa\cdots a}_{m \text{ times}} = \underbrace{aa\cdots a \times aa\cdots a}_{n+m \text{ times}} = a^{n+m}$$

  When multiplying exponential expressions of different bases but of the same power, multiply by the bases together and raise it to the exponent, because:

  $$a^n \times b^n = \underbrace{aa\cdots a}_{n \text{ times}} \times \underbrace{bb\cdots b}_{n \text{ times}} = \underbrace{abab\cdots ab}_{n \text{ times}} = (ab)^n$$

- **Powers of Powers:** When raising a power to another power, multiply the exponents, because:

  $$(a^n)^m = \underbrace{a^n a^n \cdots a^n}_{m \text{ times}} = \underbrace{aaaa\cdots aaa \times aaa\cdots aa}_{nm \text{ times}} = a^{nm}$$

- **Negative exponents** are a shorthand for a power of the reciprocal, and only make sense if the base is not zero. That is, for $n > 0$ and $a \neq 0$,

  $$a^{-n} = \left(a^{-1}\right)^n = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}.$$

  Note that $a^{-1} = \frac{1}{a}$.
- **Dividing exponents:** When dividing powers of the same base, subtract the exponents because of the meaning of negative exponents (above):

  $$a^n \div a^m = \frac{a^n}{a^m} = a^n\left(\frac{1}{a^m}\right) = a^n \times a^{-m} = a^{n-m}.$$

- If you have different bases to different powers, sometimes you can combine by factoring. See below.

Example 3

- $6^3 \cdot 6^7 = 6^{3+7} = 6^{10}$

- $2^4 \cdot 2^7 = 2^{4+7} = 2^{11}$. Notice that 2^{11} can arise other ways; for example $2^{11} = 2^{8+3} = 2^8 \cdot 2^3$
- $(4^{60})^2 = 4^{60\times 2} = 4^{120}$
- $\frac{1}{343} = \frac{1}{7^3} = 7^{-3}$
- $9^6 \div 9^2 = 9^{6-2} = 9^4$
- $\frac{3^{-4}}{3^{-5}} = 3^{-4-(-5)} = 3^1 = 3$
- $4^3 5^3 = (4 \times 5)^3 = 20^3$
- $2^5 \cdot 8^6 = 2^5 \cdot (2^3)^6 = 2^5 \cdot 2^{18} = 2^{23}$
- $3^2 \cdot 9^{-4} = 3^2 \cdot (3^2)^{-4} = 3^2 \cdot (3^{-8}) = 3^{-6}$
- $(35)^2 \cdot 7^9 = (5 \cdot 7)^2 \cdot 7^9 = 5^2 \cdot 7^{11}$
- $12^3 \cdot 18^4 = (2 \cdot 6)^3 \cdot (3 \cdot 6)^4 = 2^3 \cdot 6^3 \cdot 3^4 \cdot 6^4 = 2^3 \cdot 3^4 \cdot 6^7 = 3 \cdot 6^3 \cdot 6^7 = 3 \cdot 6^{10}$

1.3.1 Practice Problems

Use the rules of exponents to find an expression equivalent to the following:

1. $z^4 \cdot z^{-5}$
2. $(9^3)^4$
3. $(2^{-2})^{-4}$
4. $\frac{q^5}{q^2}$
5. $(-4)^3 \cdot 5^3$

1.3.2 Solutions

1. z^{-1} 2. $9^{12} = 3^{24}$ 3. 2^8 4. q^3 5. $(-20)^3 = -(20^3) = -20^3$
In each case there are other correct solutions.