4.5 Systems of Inequalities

The solution to a system (or collection) of inequalities in one variable is the set of all numbers for which each inequality in the system is true. Thus, one way to find the solution to a system of inequalities is to solve each inequality, and then identify the sets of points where the solutions overlap. Another way to do it is

- First, rewrite each expression in a form where the inequality is with respect to 0.
- Find the collection of all numbers with the property that at least one expression is zero or an expression is undefined. These partition the real line into a number of intervals.
- Pick a test point in each interval and check if every inequality in the system is true at that point. If so, then all points on the interior of the interval belong to the solution set, and if not then no points in the interior of the interval belong to the solution set.
- Be careful about what happens at the endpoints of the intervals.

Example 52 *Find the solution set to the system of inequalities*

$$\begin{array}{rcl} x^2 - 4x & \geq & -3 \\ \dfrac{x-5}{x-7} & < & 0 \end{array}$$

The first inequality is equivalent to $x^2 - 4x + 3 \geq 0$. The second one is already in a form where it is with respect to 0.

Since $x^2 - 4x + 3 = (x-3)(x-1)$, the first expression equals 0 when $x = 1$ or when $x = 3$. The expression $\frac{x-5}{x-7}$ equals zero when $x = 5$, and is undefined when $x = 7$. Thus the set of points that partition the remainder of the real line into open intervals is $\{1, 3, 5, 7\}$.

We need to test a point in each of the five intervals in the partition, and then be careful about the endpoints of the intervals.

For the interval $(-\infty, 1)$ we pick $x = 0$ as the test point. The first inequality does not hold when $x = 0$, so no point in $(-\infty, 1)$ belongs to the solution set. The second inequality does not hold when $x = 1$, so this point is not in the solution set.

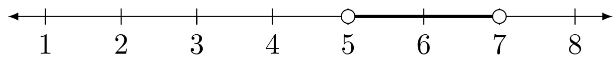
For $(1, 3)$, we pick $x = 2$ as the test point. The first inequality does not hold when $x = 2$, so no point in $(1, 3)$ belongs to the solution set.

For the interval $(3, 5)$, we choose $x = 4$ as the test point. The first inequality holds when $x = 4$, but the second one does not, so no point in $(3, 5)$ belongs to the solution set. The second inequality does not hold when $x = 5$, so this point is not in the solution set.

For the interval $(5, 7)$, we choose $x = 6$ as the test point. Both inequalities hold when $x = 6$, so all points in $(5, 7)$ belong to the solution set. The second inequality is undefined at $x = 7$, so this point is not in the solution set.

For the interval $(7, \infty)$, we choose $x = 8$ as the test point. The first equality holds when $x = 8$, but the second one does not, so no point in $(7, \infty)$ belongs to the solution set.

The solution set to the system of inequalities is therefore $(5, 7)$. Its graph on the real line is shown below.



The same ideas are used when graphing the solution to a system of inequalities in the plane.

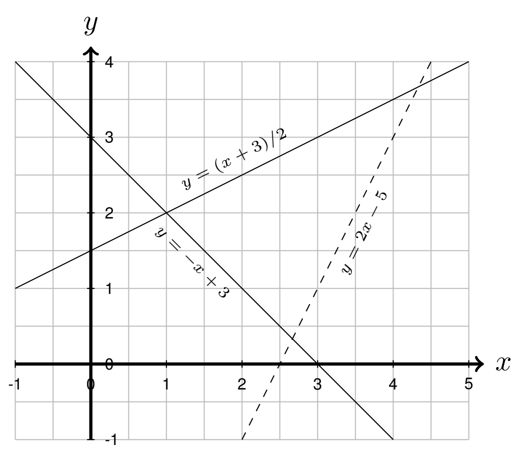
Example 53 *Graph the solution set to the system of inequalities*

$$\begin{aligned} x + y &\geq 3 \\ 2x - y &\leq 5 \\ 2y &\leq x + 3 \end{aligned}$$

The system of inequalities van be rewritten as

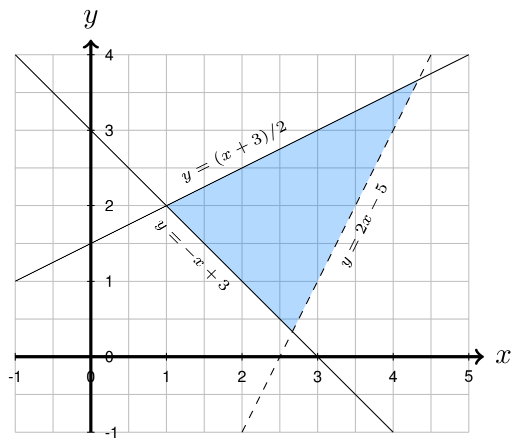
$$\begin{aligned} y &\geq -x + 3 \\ y &> 2x - 5 \\ y &\leq (x + 3)/2 \end{aligned}$$

The next step is to graph the three lines $y = -x + 3$, $y = 2x - 5$, and $y = (x + 3)/2$. Since the second inequality is $y > 2x - 5$, a dashed line is used for $y = 2x - 5$, to indicate that points on that line are not included in the solution set.



These three lines partition the remainder of the plane into 7 regions. We need to pick a test point in each region and see for which of these points all inequalities in the system hold.

We will spare you the details. The only region in what all inequalities in the system hold for the test point is the interior triangle. The graph of the solution set is shown below.



4.5.1 Practice Problems

Sketch the region

1. Find the solution set to the system of inequalities $3 - \frac{2x-4}{x} \geq 0, \quad x^2 < 9$, and graph it on the real line.

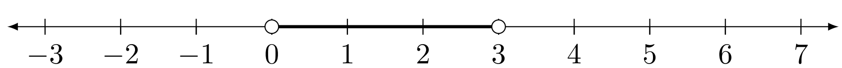
2. Find the solution set to the system of inequalities $(x-1)^3 \leq 0, \quad 3x \geq 5$, and graph it on the real line.

3. Graph the solution set to the system of inequalities $\frac{2}{3}x - 4 \leq y, \quad y \leq -\frac{1}{5}x + 4$.

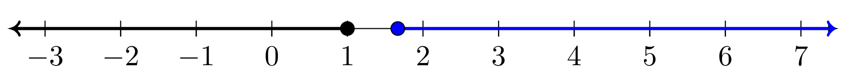
4. Graph the solution set to the system of inequalities $(x^2 - 7x + 10) \leq y, \quad y \leq -x + 3$

4.5.2 Solutions

1. The solution set is $(0, 3)$ and it is graphed on the real line below.



2. There is no solution set, but the graph below shows the two inequalities on the real line. The black line is $(x-1)^3 \leq 0$ and the blue line is $3x \geq 5$; because they do not intersect, there are no points in the solution set.



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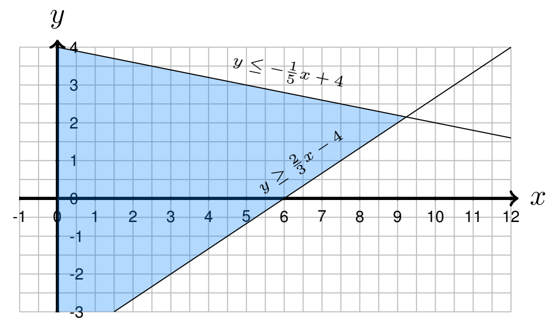

4.