19 Logarithms

From our previous work, we know that if \( b > 0 \) then the range of the exponential function \( f(x) = b^x \) is the positive real numbers. If \( b > 0 \) then the graph of \( f \) increases as we move right on the \( x \)-axis, and if \( 0 < b < 1 \) then it increases as we move left on the \( x \)-axis. (The function \( f(x) = b^x \) is not very exciting if \( b = 1 \).) In either case, that means for any positive real number \( y \) there exists a unique real number \( x \) such that \( b^x = y \). This number is the base-\( b \) logarithm of \( y \), and denoted by \( \log_b(y) \). That is, if \( b > 0 \) and \( b \neq 1 \), then \( \log_b(y) \) is the power to which \( b \) must be raised in order to get the positive number \( y \).

It is important to notice that \( \log_b(y) \) is only defined for positive numbers \( y \). Logarithms are exponents. By definition \( b^{\log_b(y)} = y \). Since \( b \) is positive, so is any power of \( b \). Thus \( \log_b(y) \) is undefined when \( y \) is negative because no power of a positive number can give us a negative number.

Logarithms have properties that follow immediately from the fact that they are exponents.

**Properties of Logarithms.** Suppose \( b > 0 \) and \( b \neq 1 \).

- \( \log_b 1 = 0 \) because \( b^0 = 1 \).
- \( \log_b b = 1 \) because \( b^1 = b \).
- \( b^x = y \) if and only if \( \log_b(y) = x \).
- \( \log_b(xy) = \log_b(x) + \log_b(y) \) because \( b^{\log_b(x)+\log_b(y)} = b^{\log_b(x)} b^{\log_b(y)} = xy \).
- \( \log_b(a^x) = x \log_b(a) \) because \( b^{x \log_b(a)} = (b^{\log_b(a)})^x = a^x \).
- \( \log_b \left( \frac{x}{y} \right) = \log_b(x) - \log_b(y) \) because \( b^{\log_b(x)-\log_b(y)} = b^{\log_b(x)} b^{-\log_b(y)} = b^{\log_b(x)} \cdot \frac{1}{b^{\log_b(y)}} = \frac{x}{y} \).

When the base \( b \) is omitted, as in \( \log(100) \), it is assumed to be 10. Logarithms to base 10 are called common logarithms. Logarithms to base \( e \) are called natural logarithms, and denoted by \( \ln(x) \) rather than \( \log_e(x) \).

**Example 1** Evaluate each expression.

- \( \log_2(2) = 1 \)
- \( \log(100) = \log(10^2) = \log_{10}(10^2) = 2 \)
- \( \log_6(12) + \log_6(18) = \log_6(12 \cdot 18) = \log_6(216) = \log_6(6^3) = 3 \)
- \( \log_5(2500) - \log_5(4) = \log_5 \left( \frac{2500}{4} \right) = \log_5(625) = \log_5(5^4) = 4 \)
- \( \log(5b) + \log(2c^2) = \log(10bc^2) = \log(10) + \log(bc^2) = 1 + \log(bc^2) \)
Example 2. Solve \( \log_8(x) + \log_8(x - 12) = 2 \).

We have \( 2 = \log_8(x) + \log_8(x - 12) = \log_8(x(x - 12)) \), so that \( 8^2 = x(x - 12) \), by the definition of logarithms. Thus \( x^2 - 12x - 64 = 0 \). By factoring (or using the quadratic formula first for help), this is the same as \( (x - 16)(x + 4) = 0 \). Therefore \( x = 16 \) or \( x = -4 \). But \( x = -4 \) is not a solution, as \( \log_8(-4) \) is undefined. Hence the solution is \( x = 16 \).

For a positive real number \( b \neq 1 \), the logarithm function with base \( b \) is the function \( f(x) = \log_b(x) \). Its domain is the set of positive real numbers. Its range is the set of all real numbers.

Notice that the functions \( f(x) = b^x \) and \( g(y) = \log_b(y) \) are inverses by definition: \( b^x = y \) if and only if \( \log_b(y) = x \). Thus, if \( b \neq 1 \), the logarithm function with base \( b \) is the inverse of the exponential function with base \( b \). (The function \( f(x) = 1^x \) does not have an inverse because, for example \( 1^2 = 1^3 = 1 \).)

Finally, we observe that logarithm functions with different bases are just multiples of each other. We know \( b^{\log_b(x)} = x \). Therefore, \( \log_a(b^{\log_b(x)}) = \log_a(x) \). From one of the properties of logarithms, we know that \( \log_a(b^{\log_b(x)}) = \log_a(x) \log_a(b) \), and so this is the same as \( \log_a(x) \log_a(b) = \log_a(x) \). Since \( \log_a(b) \) is a number, this says that \( \log_a(x) \) is a multiple of \( \log_b(x) \).

The graphs of \( f(x) = \log(x) \) and \( g(x) = \ln(x) \) are shown below.

The same principles as before of shifting graphs or stretching them vertically or horizontally apply.

Example 3 Sketch the graph of \( f(x) = 5 \log(x + 2) + 3 \).

This graph has the same basic shape as the graph of \( y = \log(x) \). It is shifted upwards by 3 and left by 2. Also, it is stretched in the vertical direction by a factor of 5.

After plotting a few well chosen points and sketching a curve of the correct shape through them, one arrives at the graph below.
19.1 Practice Problems

In questions 1 to 5, use the properties of logarithms to to find an equivalent, arguably simpler, expression.

1. \( \log_6 (12) + \log_6 (18) \)
2. \( \log_8 (x) + \log_8 (x - 12) \)
3. \( \log_5 (2500) - \log_5 (4) \)
4. \( \log(5b) + \log(2c^2) \)
5. \( \log_{25} (7) + \log_{5} (3) \)

6. Sketch the graph of \( h(x) = 1 - 5 \log \left( 1 - \frac{x}{2} \right) \).
19.2 Solutions

1. 3  2. $\log_8(x^2 - 12x)$  3. 4  4. $\log(10be^2)$  5. $\frac{2\log(3)+\log(7)}{2\log(5)}$

6.