4.4 Logarithms

From our previous work, we know that if $b > 0$ then the range of the exponential function $f(x) = b(x)$ is the positive real numbers. If $b > 0$ then the graph of f increases as we move right on the x axis, and if $0 < b < 1$ then it increases as we move left on the x-axis. (The function $f(x) = b^x$ is not very exciting if $b = 1$.) In either case, that means for any positive real number y there exists a unique real number x such that $b^x = y$. This number is **the base-b logarithm of** y, and denoted by $\log_b(y)$. That is, if $b > 0$ and $b \neq 1$, then $\log_b(y)$ is the power to which b must be raised in order to get the positive number y.

It is important to notice that $\log_b(y)$ is only defined for positive numbers y. Logarithms are exponents. By definition $b^{\log_b(y)} = y$. Since b is positive, so is any power of b. Thus $\log_b(y)$ is undefined when y is negative because no power of a positive number can give us a negative number.

Logarithms have properties that follow immediately from the fact that they are exponents.

Properties of Logarithms. Suppose $b > 0$ and $b \neq 1$.

- $\log_b 1 = 0$ because $b^0 = 1$.
- $\log_b b = 1$ because $b^1 = b$.
- $b^x = y$ if and only if $\log_b(y) = x$.
- $\log_b(xy) = \log_b(x) + \log_b(y)$ because $b^{\log_b(x)+\log_b(y)} = b^{log_b(x)}b^{\log_b(y)} = xy$.
- $\log_b(a^x) = x\log_b(a)$ because $b^{x\log_b(a)} = \left(b^{\log_b(a)}\right)^x = a^x$.
- $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ because $b^{\log_b(x)-\log_b(y)} = b^{log_b(x)}b^{-\log_b(y)} = b^{log_b(x)} \cdot \frac{1}{b^{\log_b(y)}} = \frac{x}{y}$.

When the base b is omitted, as in $\log(100)$, it is assumed to be 10. Logarithms to base 10 are called **common logarithms**. Logarithms to base e are called **natural logarithms**, and denoted by $ln(x)$ rather than $log_e(x)$.

Example 49 *Evaluate each expression.*

- $\log_2(2) = 1$
- $\log(100) = \log(10^2) = \log_{10}(10^2) = 2$
- $\log_6(12) + \log_6(18) = \log_6(12 \cdot 18) = \log_6(216) = \log_6(6^3) = 3$
- $\log_5(2500) - \log_5(4) = \log_5(\frac{2500}{4}) = \log_5(625) = \log_5(5^4) = 4$
- $\log(5b) + \log(2c^2) = \log(10bc^2) = \log(10) + \log(bc^2) = 1 + \log(bc^2)$

Example 50 . *Solve* $\log_8(x) + \log_8(x - 12) = 2$.

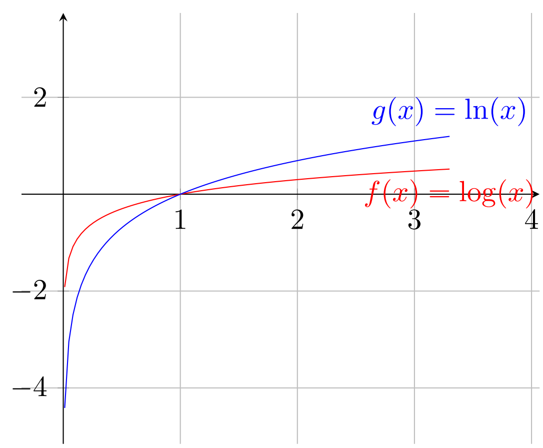
We have $2 = \log_8(x) + \log_8(x - 12) = \log_8(x(x - 12))$, so that $8^2 = x(x - 12)$, by the definition of logarithms. Thus $x^2 - 12 - 64 = 0$. By factoring (or using the quadratic formula first for help), this is the same as $(x - 16)(x + 4) = 0$. Therefore $x = 16$ or $x = -4$. But $x = -4$ is not a solution, as $\log_8(-4)$ is undefined. Hence the solution is $x = 16$.

For a positive real number $b \neq 1$, the **logarithm function with base** b is the function $f(x) = \log_b(x)$. Its domain is the set of positive real numbers. Its range is the set of all real numbers.

Notice that the functions $f(x) = b^x$ and $g(y) = \log_b(y)$ are inverses by definition: $b^x = y$ if and only if $\log_b(y) = x$. Thus, if $b \neq 1$, the logarithm function with base b is the inverse of the exponential function with base b. (The function $f(x) = 1^x$ does not have an inverse because, for example $1^2 = 1^3 = 1$.)

Finally, we observe that logarithm functions with different bases are just multiples of each other. We know $b^{\log_b(x)} = x$. Therefore, $\log_a\left(b^{\log_b(x)}\right) = \log_a(x)$. From one of the properties of logarithms, we know that $\log_a\left(b^{\log_b(x)}\right) = \log_b(x)\log_a(b)$, and so this is the same as $\log_b(x)\log_a(b) = \log_a(x)$. Since $\log_a(b)$ is a number, this says that $\log_a(x)$ is a multiple of $\log_b(x)$.

The graphs of $f(x) = log(x)$ and $g(x) = \ln(x)$ are shown below.

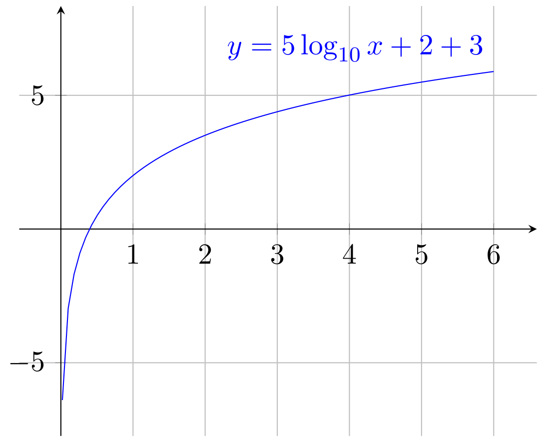


The same principles as before of shifting graphs or stretching them vertically or horizontally apply.

Example 51 *Sketch the graph of* $f(x) = 5\log(x + 2) + 3$.

This graph has the same basic shape as the graph of $y = \log(x)$. It is shifted upwards by 3 and left by 2. Also, it is stretched in the vertical direction by a factor of 5.

After plotting a few well chosen points and sketching a curve of the correct shape through them, one arrives at the graph below.



4.4.1 Practice Problems

In questions 1 to 5, use the properties of logarithms to to find an equivalent, arguably simpler, expression.

1. $\log_6 (12) + \log_6 (18)$
2. $\log_8(x) + \log_8(x - 12)$
3. $\log_5 (2500) - \log_5 (4)$
4. $\log(5b) + \log(2c^2)$
5. $\log_{25} (7) + \log_5 (3)$

6. Sketch the graph of $h(x) = 1 - 5\log(1 - \frac{x}{2})$.

4.4.2 Solutions

1. 3 2. $\log_8(x^2 - 12x)$ 3. 4 4. $\log(10bc^2)$ 5. $\frac{2\log(3)+\log(7)}{2\log(5)}$

6.