4.3 Inverse functions

A pair of functions f and g with the property that $f(x) = y$ if and only if $g(y) = x$ are called **inverses**.

Informally, if f and g are inverses, then g "undoes" f by sending y back to x. Similarly, f "undoes" g. A more formal way to state this involves function composition: if f and g are inverses, then

$$(g \circ f)(x) = g(f(x)) = g(y) = x$$

by definition, and similarly

$$(f \circ g)(y) = f(g(y)) = f(x) = y.$$

If f and g are inverses, then g is called the **inverse function** of f, and denoted by f^{-1}. We could also call f the inverse function of g, and denote it by g^{-1}.

If f and g are inverses, then the range of f must be the same as the domain of g, and the range of g must be the same as the domain of f. Furthermore, there can not be different real numbers x_1 and x_2 such that $f(x_1) = f(x_2) = y$ for some real number y because we would need to have $g(y)$ equal to both x_1 and x_2, which is impossible because g is a function. Similarly, there can not be different real numbers y_1 and y_2 such that $g(y_1) = g(y_2) = x$ for some real number x.

Not every function has an inverse. For example, $f(x) = x^2$ does not have an inverse because $f(1) = f(-2)$.

Example 46 *Show that the functions* $f(x) = 3x - 6$ *and* $g(x) = \frac{x+6}{3}$ *are inverses.*

We need to show that $f(x) = y$ if and only if $g(y) = x$.

Suppose that $f(x) = y$. Then $y = 3x - 6$. Thus

$$g(y) = g(3x - 6) = \frac{(3x - 6) + 6}{3} = x,$$

as required.

Now suppose that $g(y) = x$. Then $x = \frac{y+6}{3}$. Thus

$$f(x) = f(\frac{y + 6}{3}) = 3\left(\frac{y + 6}{3}\right) - 6 = y,$$

as required.

Therefore f and g are inverses.

In Example 1, notice that the inverse of f is described by the expression obtained by solving the equation $y = f(x)$ for x. If $y = f(x)$ then $y = 3x - 6$. That means $y + 6 = 3x$, and so $x = \frac{y+6}{3} = g(y)$. This computation can be viewed as telling us what x needs to be in order for $f(x) = y$ to be true. Similarly, the inverse of g is described by the expression obtained by solving the equation $x = g(y)$ for y.

Example 47 *Let $f(x) = \frac{2x}{4x-1}$. Find the inverse of f, if it exists.*

The domain of f is the set of all real numbers except $\frac{1}{4}$. To find an expression for the inverse of f, if it exists, we need to solve the equation $y = f(x)$ for x.

If $y = f(x)$, then $y = \frac{2x}{4x-1}$ and $x \neq \frac{1}{4}$. Thus $y(4x-1) = 2x$, so $4xy - y = 2x$. This equation can be rearranged to obtain $y = 4xy - 2x = x(4y-2)$, so that $x = \frac{y}{4y-2}$.

Therefore, if f has an inverse, it is $g(y) = \frac{y}{4y-2}$. The domain of g is the set of all real numbers except $\frac{1}{2}$. We need to check that $f(x) = y$ if and only if $g(y) = x$.

Suppose $f(x) = y$. Then $y = \frac{2x}{4x-1}$ and $x \neq \frac{1}{4}$, so that

$$g(y) = g\left(\frac{2x}{4x-1}\right) = \frac{\frac{2x}{4x-1}}{4\frac{2x}{4x-1} - 2} = \frac{\frac{2x}{4x-1}}{4\frac{2x}{4x-1} - 2\frac{4x-1}{4x-1}} = \frac{\left(\frac{2x}{4x-1}\right)}{\left(\frac{2}{4x-1}\right)} = x.$$

Now suppose that $g(y) = x$. Then $x = \frac{y}{4y-2}$ and $y \neq \frac{1}{2}$, so that

$$f(x) = f\left(\frac{y}{4y-2}\right) = \frac{2\frac{y}{4y-2}}{4\frac{y}{4y-2} - 1} = \frac{\frac{2y}{4y-2}}{4\frac{y}{4y-2} - \frac{4y-2}{4y-2}} = \frac{\frac{2y}{4y-2}}{4\frac{y}{4y-2} - \frac{4y-2}{4y-2}} = \frac{\left(\frac{2y}{4y-2}\right)}{\left(\frac{2}{4y-2}\right)} = y.$$

Therefore, f has an inverse, and is the function g. That is, $f^{-1}(y) = \frac{y}{4y-2}$.

Example 48 *Let $f(x) = x^2 + 1$. Find the inverse of f, if it exists.*

We could possibly observe right away that $f(1) = f(-2) = 2$, which means that f can not have an inverse. If we were lucky enough to see that, then there is nothing more to do. If we weren't, then we could proceed as above and try to solve $y = f(x)$ for x. Doing that leads to $x = \pm\sqrt{y-1}$, which tells us there may be two different values of x that give the same value of y. For example, if we choose $y = 2$, then the equation says we must have $x = \pm 1$, that is, that $f(1) = f(-1) = 2$.

The lesson from Example 48 is that, if it is not possible to see right away whether f has an inverse, then we can proceed by trying to find one on the assumption that it exists. Either that will be successful, or the calculation will reach a point where it becomes clear that there are two different values of x that give the same value of y. In the first case, f has an inverse described by the expression obtained. In the second case, f does not have an inverse.

4.3.1 Practice Problems

Find the inverse, when it exists.

1. $y = x^4$
2. $y = 7x - 3$
3. $y = \frac{2x}{4x-1}, x \neq \frac{1}{4}$
4. $y = \sqrt{x-5}$
5. $y = 2 - x^3$

4.3.2 Solutions

1. An inverse does not exist. 2. $y = \frac{x+3}{7}$ 3. $y = \frac{x}{4x-2}, x \neq \frac{1}{2}$ 4. $y = x^2 + 5$ 5.$y = \sqrt[3]{2-x}$