3.5 Exponential Functions

Let $b > 0$ be a positive real number. In previous sections we talked about the numbers b^r, where r is an integer or a rational number (a rational number is a fraction; a ratio of integers). In this section we consider numbers b^r, where r is any real number.

The precise definition of a number like $3^{\sqrt{2}}$ is a topic in mathematical analysis, and is beyond the scope of these notes. Here's a reasonable way to think about it. The number $\sqrt{2} = 1.4142\ldots$. Each of the numbers $1, 1.4, 1.41, 1.414, 1,4142, \ldots$ is rational, so the numbers $3^1, 3^{1.4}, 3^{1.41}, 3^{1.414}$, $3^{1,4142}, \ldots$ are all defined. These numbers are approximately $3, 4.6555, 4.70697, 4.727695, 4.7287, \ldots$. It turns out that as the sequence of rational approximations gets "close" to $\sqrt{2}$, the sequence of exponentials gets "close" to a particular number, and that number is defined to be $3^{\sqrt{2}}$.

The **properties of exponents** are the same no matter whether the exponent is an integer, a rational number or a real number:

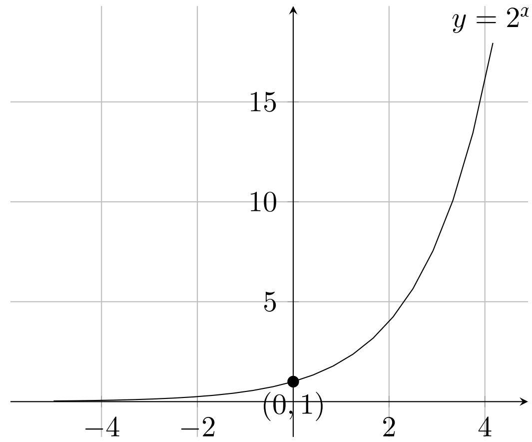
- When exponential expressions with the same base are multiplied, simplify by adding the powers, i.e.,
$$b^x b^y = b^{x+y}.$$
- When an exponential expression is itself raised to some power, simplify by multiplying the exponents, i.e.,
$$(b^x))^y = b^{xy}.$$

When $b > 0$ we can talk about b^x for any real number x. (This is not true if $b < 0$. For example, remember that $(-2)^{\frac{1}{2}}$ is undefined.) Thus we can talk about the **exponential function with base** b, $f(x) = b^x$.

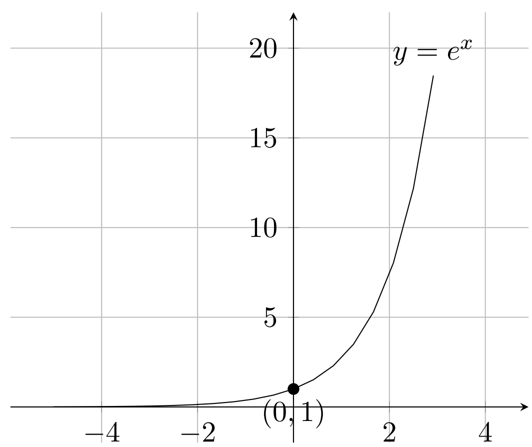
Properties of $f(x) = b^x$, where $b > 0$.

- The domain of $f(x)$ is the set of all real numbers.
- Since $b^0 = 1$, the graph of $f(x) = b^x$ always contains the point $(0, 1)$.
- There is no x such that $b^x = 0$, and in fact $b^x > 0$ for every real number x.
- If $b > 1$, then b^x gets larger and larger as x moves to the right on the x-axis, and gets close to zero as x moves to the left on the x-axis, and . (The first part is easy to see. To see the second part, think about the sequence of powers $b^{-1}, b^{-2}, \ldots = \frac{1}{b}, \frac{1}{b^2}, \ldots$; the denominators of the fractions get larger as the exponents do.)
- If $0 < b < 1$, then b^x decreases towards 0 as x gets large, and gets larger and larger as x gets small. The reasoning is the similar to the reasoning above because $\frac{1}{b} > 1$.
- The range of $f(x) = b^x$ is the $(0, \infty)$.

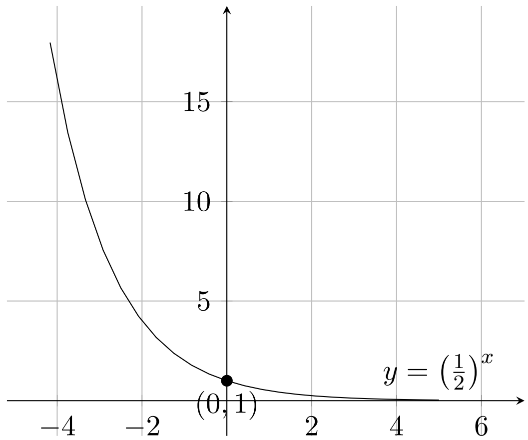
The graph of $f(x) = 2^x$ is shown below.



The **natural exponential function** is is $f(x) = e^x$, where $e \approx 2.718$. The reasons why this function is important are best explain in calculus. The graph of $f(x) = e^x$ is shown below.



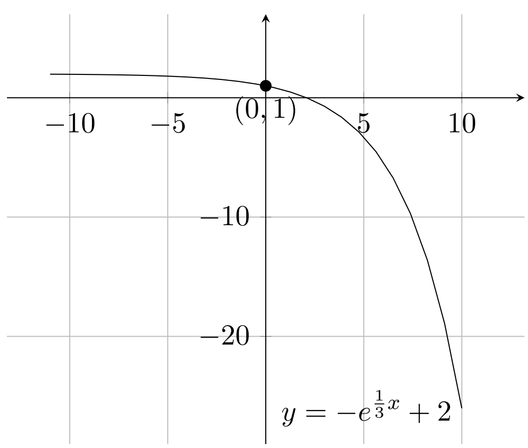
When $0 < b < 0$ the graph of $f(x) = b(x)$ has a different shape. As noted above, it decreases towards 0 as x gets large, and gets larger and larger as x gets small. The graph of $f(x) = \left(\frac{1}{2}\right)^x$ is shown below.



The principles of shifting or stretching graphs apply to the graph of any function, so in particular they apply to the graph of exponential functions.

Example 40 *Sketch the graph of $f(x) = -e^{\frac{1}{3}x} + 2$.*

The graph is similar to the graph of $y = e^x$, except that it is inverted (the height is scaled by -1), shifted 2 units up, stretched by a factor of 3. After plotting a few points and sketching a curve of the correct shape through them, one arrives at the graph below.



Example 41 *Find the positive number x such that $x^{\sqrt{2}} = 3^5$.*

We have $x^{\sqrt{2}} = 3^5$, so $\left(x^{\sqrt{2}}\right)^{\sqrt{2}} = \left(3^5\right)^{\sqrt{2}}$. From the properties of exponents, this is the same as $x^2 = 3^{5\sqrt{2}}$. Since x is positive,

$$x = \sqrt{3^{5\sqrt{2}}} = \left(3^{5\sqrt{2}}\right)^{\frac{1}{2}} = 3^{\frac{5}{2}\sqrt{2}}.$$

The number $3^{\frac{5}{2}\sqrt{2}} \approx 48.627$.

3.5.1 Practice Problems

1. Using $f(x) = 4^x$, find:

   (a) $f(-2)$

   (b) $f(x)$ when $x = 1$

   (c) $f(3/2)$

2. Using $f(x) = (1/3)^x$, find:

   (a) $f(0)$

   (b) $f(3)$

   (c) $f(-3)$

3. Sketch the following functions on the same graph

   (a) $f(x) = 3^x$

   (b) $g(x) = (\frac{1}{3})^x$

4. Find the positive number x such that $x^{\sqrt{3}} = 8^2$.

5. Sketch the graph of $h(x) = 1 - 3e^{x+2}$.

3.5.2 Solutions

1. (a) $1/16$

   (b) 4

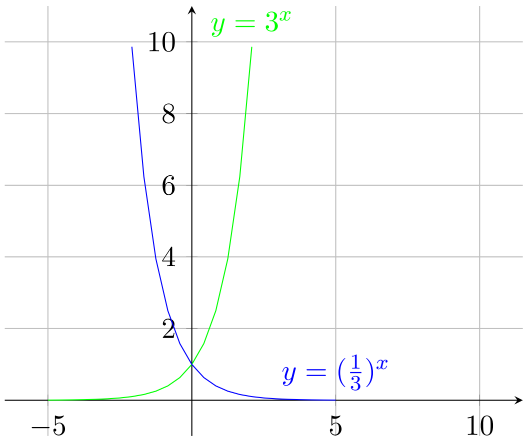
   (c) 8

2. (a) 1

   (b) $1/27$

   (c) 27

3. The graphs are shown below.



4. $x = 8^{\frac{2}{3}\sqrt{3}} = 4^{\sqrt{3}}$

5.