3.2 Rational Exponents

For a real number x and positive integers m and n , we define $x^{\frac{m}{n}}$ to be the real number y such that $y^n = x^m$, if such a number y exists. Notice how this coincides with the definition of exponents $\frac{1}{n}$ when $m = 1$.

Example 31

- $9^{\frac{1}{2}} = 3$ because $3^2 = 9^1$
- $8^{\frac{2}{3}} = 4$ because $4^3 = 64 = 8^2$
- $(-3)^{\frac{5}{2}}$ is undefined since $(-3)^5 = -243$ and a negative number is not the square of any number (because $y^2 \geq 0$ for every real number y).

It follows from the definition that

$$x^{\frac{m}{n}} = (x^m)^{\frac{1}{n}}$$

and, if $x^{\frac{1}{n}}$ is defined, then

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m.$$

Example 32

- $5^{\frac{3}{2}} = (5^3)^{\frac{1}{2}} = \sqrt{125}$. Also $5^{\frac{3}{2}} = \left(5^{\frac{1}{2}}\right)^3 = \left(\sqrt{5}\right)^3 = \sqrt{125}$. Note that $\left(\sqrt{5}\right)^3 = \sqrt{125}$ because squaring both sides gives 125.
- $(-3)^{\frac{6}{2}} = \left((-3)^6\right)^{\frac{1}{2}} = \sqrt{729} = 27$, but $\left((-3)^{\frac{1}{2}}\right)^6$ is undefined because $(-3)^{\frac{1}{2}}$ is undefined.

When a rational exponent is not in lowest terms, for example as in the last bullet point of Example 32, it is tempting to reduce it to lowest terms. Care is needed, however, because that can be a terrible idea. Using that same example, $(-3)^{\frac{6}{2}} = 27$, but $(-3)^3 = -27$. Hence $(-3)^{\frac{6}{2}} \neq (-3)^3$ even though $\frac{6}{2} = 3$. Why? It is a question of applying the definition properly. The difficulty is that $(-3)^6$ is positive, so that $(-3)^{\frac{6}{2}} = \left((-3)^6\right)^{\frac{1}{2}}$ is positive, but $\frac{6}{2} = 3$ is an odd integer and a negative number raised to an odd power is negative. When reducing a rational exponent to lowest terms, one should always check whether the expression obtained has the same sign as the original expression. This issue only arises when the base is negative.

The familiar rules of exponents apply to expressions involving rational exponents. When exponential expressions with the same base are multiplied, the exponents are added, as in

$$2^{\frac{2}{3}}2^{\frac{4}{5}} = 2^{\frac{2}{3}+\frac{4}{5}} = 2^{\frac{22}{15}}.$$

Note that all numbers involved in positive, so each equality holds. When an exponential expression is raised to a higher power, the exponents are multiplied as in

$$((-3)^3)^{\frac{2}{5}} = (-3)^{\frac{6}{5}}.$$

Both the left hand side and right hand side of this statement are positive, so the equality holds.

Example 33

- Look at $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = 4$. Since $8^{\frac{1}{3}}$ is defined, we can also compute $(8^{\frac{1}{3}})^2 = 2^2 = 4$.
- Look at $625^{\frac{3}{4}} = (625^3)^{\frac{1}{4}} = (244140625)^{\frac{1}{3}} = 25$. Since $625^{\frac{1}{4}}$ is defined, we can also compute $(625^{\frac{1}{4}})^3 = 5^2 = 25$, which is arguably an easier calculation.
- Look at $(-49)^{\frac{4}{2}} = \left((-49)^4\right)^{\frac{1}{2}} = \sqrt{(-49)^4} = \sqrt{5764801} = 2401$. We can not compute as $\left((-49)^{\frac{1}{2}}\right)^4$ because $(-49)^{\frac{1}{2}}$ is not defined. We can compute as $(-49)^{\frac{4}{2}} = (-49)^2 = 2401$ because $(-49)^{\frac{4}{2}}$ is positive (because $(-49)^4$ is positive) and so is $(-49)^2$.
- Look at $(-5)^{\frac{10}{2}} = \sqrt{(-5)^{10}} = \sqrt{25^5} = \left(\sqrt{25}\right)^5 = 5^5$. We can not compute as $\left((-5)^{\frac{1}{2}}\right)^{10}$ because $(-5)^{\frac{1}{2}}$ is undefined. And in this case, since $(-5)^{10}$ is positive we have that $(-5)^{\frac{10}{2}}$ is positive, whereas $(-5)^5$ is negative, so $(-5)^{\frac{10}{2}} \neq (-5)^5$.

Finally, as before,

$$x^{-\frac{m}{n}} = \left(x^{-1}\right)^{\frac{m}{n}} = \left(\frac{1}{x}\right)^{\frac{m}{n}} = \left(\left(\frac{1}{x}\right)^m\right)^{\frac{1}{n}} = \left(\frac{1}{x^m}\right)^{\frac{1}{n}} = \frac{1}{x^{\frac{m}{n}}}$$

when $x^{-\frac{m}{n}}$ is defined.

Example 34 *Find x if $x^{\frac{3}{5}} = -7$.*

Since $x^{\frac{3}{5}} = -7$, we have that $x^3 = \left(x^{\frac{3}{5}}\right)^5 = (-7)^5$, so that

$$x = \left(x^3\right)^{\frac{1}{3}} = \left((-7)^5\right)^{\frac{1}{3}} = (-7)^{\frac{5}{3}}.$$

3.2.1 Exercises

In questions 1 to 5, find a simpler form of each expression.

1. $-16^{\frac{3}{2}}$
2. $(\frac{8}{343})^{-\frac{2}{3}}$
3. $(27)^{\frac{5}{3}}$
4. $(-6)^{\frac{6}{2}}$
5. $(-\frac{32}{243})^{\frac{2}{5}}$

6. Find x if $x^{\frac{7}{4}} = 9$.

3.2.2 Solutions

1. -64 2. $\frac{49}{4}$ 3. 243 4. 216 5. $\frac{4}{9}$ 6. $9^{\frac{4}{7}} = 3^{\frac{8}{7}}$.