2.5 Polynomials

A **polynomial** is a function described by an expression of the form $f(x) = a_n x^n + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and $n \geq 0$ is an integer.

The expressions $a_n x^n, a_{n-1}x^{n-1}, \ldots, a_1 x, a_0$ are called the **terms** of the polynomial f. For each integer $k = 0, 1, \ldots, n$, the number a_k is called the **coefficient** of x^k in the polynomial f. The number a_0 is called the **constant term** of f. Notice that $a_0 = a_0 x^0$, so that a_0 is the coefficient of x^0.

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial in which at least one of the coefficients is not zero. Then, the **degree** of f is the largest integer k such that the coefficient of x^k is not zero. (Notice that such an integer k is guaranteed to exist.) In this case, $a_k x^k$ is the **leading term** of f, and a_k is the **leading coefficient** of f. Since $a_0 = a_0 x^0$, it is possible for a polynomial to have degree zero.

Example 20 *Let* $f(x) = 2x^3 - 10x + 9$. *Then,*

- the terms of f are $2x^3$, $0x^2$, $-10x$, and 9;
- the coefficients of f are the numbers $2, 0, -10$, and 9;
- the degree of f is 3, the leading term of f is $2x^3$, and the leading coefficient of f is 2;
- the constant term of f is 9.

Example 21 *Let* $f(x) = 14$. *Then,*

- the only term of f is $14 = 14x^0$;
- the only coefficient of f is 14;
- the degree of f is 0, the leading term of f is $14 = 14x^0$, and the leading coefficient of f is 14;
- the constant term of f is 14.

The polynomial in which all coefficients are zero, that is $f(x) = 0 + 0x + 0x^2 \cdots$, is called **the zero polynomial**. It is the only polynomial for which the degree is undefined.

An **important fact** is that if $f(x)$ is a polynomial, then $f(a) = 0$ if and only if $(x - a)$ divides $f(x)$. In other words, a is a root of the polynomial $f(x)$ if and only if $(x - a)$ is a factor of $f(x)$. Hence to check whether $x - a$ is a factor of $f(x)$, check whether $f(a) = 0$. Conversely, if $f(a) = 0$ then $(x - a)$ is a factor of $f(x)$.

Example 22 *Let* $f(x) = 2x^3 + x^2 - 1$. *Then,*

- $x - 1$ is not a factor of $f(x)$ because 1 is not a root of $f(x)$. We can calculate $f(1) = 2 \neq 0$;

- $x+1 = x-(-1)$ is a factor of $f(x)$ because $f(-1) = 0$. In fact, $f(x) = (x+1)(2x^2-x+1)$. We can check this by multiplication $(x+1)(2x^2-x+1) = 2x^3-x^2+x+2x^2-x-1 = 2x^3+x^2-1$. (The factors in this expression can be found by long division, as discussed below.)

A **useful fact** is that if a polynomial $f(x)$ has leading coefficient equal to 1, then an integer a can be a root of $f(x)$ only if a or $-a$ is a divisor of the constant term a_0. It may be that all, some, or none of these are actually roots of f. Also, f may have roots that are not integers.

Example 23 *Let $f(x) = x^3 - x^2 - 2x + 2$. Since the leading coefficient of f is 1, the only possible integers which could possibly be roots of f are the divisors of 2 and their negatives, that is $1, -1, 2,$ and -2. Of these possibilities, only 1 is a root (because $f(1) = 0$ whereas $f(-1), f(2)$ and $f(-2)$ are all non-zero). In fact, $f(x) = (x-1)(x^2-2)$*

Once we know a root of a polynomial, we can factor the polynomial using long division. Polynomials are divided just like numbers. We saw in Example 23 that 1 is a root of $f(x) = x^3-x^2-2x+2$, so we can divide $f(x)$ by $x-1$:

$$
\begin{array}{r|l}
 & \quad x^2 \qquad\quad - 2 \\ \hline
x-1 & \quad x^3 - x^2 - 2x + 2 \\
 & -x^3 + x^2 \\ \hline
 & \qquad\qquad -2x + 2 \\
 & \qquad\qquad\;\; 2x - 2 \\ \hline
 & \qquad\qquad\qquad\;\; 0
\end{array}
$$

Because the remainder is 0, this means that $x^3 - x^2 - 2x + 2 = (x-1)(x^2-2)$. We can factor further, because $(x^2-2) = (x-\sqrt{2})(x+\sqrt{2})$, so:

$$f(x) = (x-1)\left(x-\sqrt{2}\right)\left(x+\sqrt{2}\right).$$

Since a product of numbers equals zero if and only if one of them is zero, $f(x) = 0$ if and only if $x-1=0$ or $x-\sqrt{2}=0$ or $x+\sqrt{2}=0$, and so the roots of $f(x)$ are -1, $\sqrt{2}$ and $-\sqrt{2}$.

For any number x in its domain, and expression represents a number. If we remember that x just represents a number in the domain, then expressions involving a variable can be factored (using the commutative, associative and distributive laws).

Example 24

$$
\begin{aligned}
2x(x+1) + (x+1)^2 &= 2x(x+1) + (x+1)(x+1) \\
&= (x+1)[2x + (x+1)] \\
&= (x+1)(3x+1).
\end{aligned}
$$

Example 25

$$
\begin{aligned}
4x(x+1)^3 - 6x^2(x+1)^2 &= 2x(x+1)^2[2(x+1) - 3x] \\
&= 2x(x+1)^2[2-x].
\end{aligned}
$$

Notice that, in Examples 24 and 25, factoring would have been much more difficult if the expression had been multiplied out. Sometimes factoring requires creatively grouping the terms.

Example 26

$$\begin{aligned} 8x + 8x^3 + x^4 + x^6 &= 8x(1+x^2) + x^4(1+x^2) \\ &= (1+x^2)[8x + x^4] \\ &= (1+x^2)(x)(8+x^3). \end{aligned}$$

2.5.1 Practice Problems

State the terms, the coefficients, the degree, the leading term, the leading coefficient, and the constant term of the following functions.

1. $f(x) = 3x^2 - 10x$
2. $g(x) = 10x^5 + x^3 + 15$
3. Factor the following expressions:
   (a) $2x(x^2+1)^3 - 16(x^2+1)^5$
   (b) $8x + 8x^3 + x^4 + x^6$
4. Find all possible integer roots of the function and of those possibilities, determine which are roots of the function. $f(x) = x^2 + x - 6$

2.5.2 Solutions

1. The terms of f are $3x^2$, $-10x$, and 0; the coefficients of f are the numbers $3, -10$, and 0; the degree of f is 2, the leading term of f is $3x^2$, and the leading coefficient of f is 3; the constant term of f is 0.

2. The terms of g are $10x^5$,$0x^4$,x^3,$0x^2$, $-10x$, and 15; the coefficients of f are the numbers $10, 0, 1, 0, 0$, and 15; the degree of f is 5, the leading term of f is $10x^5$, and the leading coefficient of f is 10; the constant term of f is 15.

3. (a) $2(x^2+1)^3(x - 8(x^2+1)^2)$ (b) $x(1+x^2)(8+x^3) = x(1+x^2)(x+2)(x^2-2x+4)$

4. Possible roots: $1, -1, 2, -2, 3, -3, 6, -6$. Since $f(2) = 0$ and $f(-3) = 0$, 2 and -3 are roots of f.