University of Victoria | Graduate Studies

Notice of the Final Oral Examination for the Degree of Master of Science

of

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"Tight Bounds on 3-Neighbor Bootstrap Percolation"

Department of Mathematics and Statistics

Monday, August 29, 2022
11:00 A.M.
David Strong Building
Room C114

Supervisory Committee:
Dr. Peter Dukes, Department of Mathematics and Statistics, University of Victoria (Co-Supervisor)
Dr. Jonathan Noel, Department of Mathematics and Statistics, UVic (Co-Supervisor)

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Chair of Oral Examination:
Dr. Nishant Mehta, Department of Computer Science, UVic

Dr. Robin G. Hicks, Dean, Faculty of Graduate Studies

Abstract

Consider infecting a subset $A_0 \subseteq V(G)$ of the vertices of a graph G. Let an uninfected vertex $v \in V(G)$ become infected if $|N_G(v) \cap A_0| \geq r$, for some integer r. Define $A_t = A_{t-1} \cup \{v \in V(G): |N_G(v) \cap A_{t-1}| \geq r\}$, and say that the set A_0 is *lethal* under r-neighbor percolation if there exists a t such that $A_t = V(G)$. For a graph G, let $m(G, r)$ be the size of the smallest lethal set in G under r-neighbor percolation.

The problem of determining $m(G, r)$ has been extensively studied for grids G of various dimensions. We define

$$m\left(\coprod_{t=1}^{d}[a_i], r\right) = m(a_1, \dots, a_d, r)$$

for ease of notation. Famously, a lower bound of $m(a_1, \dots, a_d, d) \geq \frac{\sum_{j=1}^{d} \prod_{i \neq j} a_i}{d}$ is given by a beautiful argument regarding the high-dimensional "surface area" of $G = [a_1] \times \cdots \times [a_d]$. While exact values of $m(G, r)$ are known in some specific cases, general results are difficult to come by.

In this thesis, we introduce a novel technique for viewing d-neighbor lethal sets on d-dimensional grids in terms of lethal sets in $(d-1)$ dimensions. We also provide a strategy for recursively building up large lethal sets from existing small constructions. Using these techniques, we determine the exact size of all lethal sets under 3-neighbor percolation in three-dimensional grids $[a_1] \times [a_2] \times [a_3]$, for $a_1, a_2, a_3 \geq 11$

The problem of determining $m([n] \times [n], 3)$ is discussed by Benevides, Bermond, Lesfari and Nisse in [7]. The authors determine the exact value of $m([n] \times [n], 3)$ for even n, and show that, for odd n,

$$\left\lceil \frac{n^2 + 2n}{3} \right\rceil \leq m([n] \times [n], 3) \leq \left\lceil \frac{n^2 + 2n}{3} \right\rceil + 1.$$

We prove that $m([n] \times [n], 3) = \left\lceil \frac{n^2 + 2n}{3} \right\rceil$ if and only if $n = 2^k - 1$, for some $k > 0$.

Finally, we prove that for $a_1, a_2, a_3 \geq 12$, bounds on the minimum lethal set on the torus $G = C_{a_1} \square C_{a_2} \square C_{a_3}$ are given by

$$m(G, 3) \geq \frac{a_1 a_2 + a_2 a_3 + a_3 a_1 - 2(a_1 + a_2 + a_3)}{3} + 2$$

and

$$m(G, 3) \leq \frac{a_1 a_2 + a_2 a_3 + a_3 a_1 - 2(a_1 + a_2 + a_3)}{3} + 3.$$