University of Victoria | Graduate Studies

Notice of the Final Oral Examination for the Degree of Master of Science

of

TAO GAEDE

BSc (University of British Columbia, Okanagan, 2018)

"Erdős-Deep Families of Arithmetic Progressions"

Department of Mathematics and Statistics

Tuesday, June 21, 2022
10:00 A.M.
David Strong Building
Room C128

Supervisory Committee:
Dr. Peter Dukes, Department of Mathematics and Statistics, University of Victoria (Supervisor)
Dr. Natasha Morrison, Department of Mathematics and Statistics, UVic (Member)

External Examiner:
Dr. Harald Krebs, School of Music, UVic

Chair of Oral Examination:
Dr. Rustom Bhiladvala, Department of Mechanical Engineering, UVic

Dr. Robin G. Hicks, Dean, Faculty of Graduate Studies

Abstract

The set $A \subseteq \mathbb{Z}_n$ with $|A| = k$ is called Erdős-deep if and only if the $\frac{k(k-1)}{2}$ pair-wise distances between distinct elements of A have multiplicities $1, 2, \dots, k-1$ Here, distance is measured in the usual way 'around the circle' in $\mathbb{Z}_n$, defined formally as $\mathrm{dist}(x, y) = |x - y|_n$, where $|z|_n = \min(z, n-z)$ for $z \in \{0, 1, \dots, n-1\}$. In other words, A is Erdős-deep if, for every $i \in \{1, 2, \dots, k-1\}$, there is a positive number d_i satisfying $\left|\{\{x, y\} \subseteq A : \mathrm{dist}(x, y) = d_i\}\right| = i$.

This property has also been considered in the geometric setting, where $\mathbb{Z}_n$ is replaced by the plane $\mathbb{R}^2$ (and distance replaced by the standard Euclidean metric).

Erdős-deep sets in $\mathbb{Z}_n$ have been previously classified as, with one exception, modular arithmetic $\{0, g, 2g, \cdots, (k-1)g\} \subseteq \mathbb{Z}_n$ for some generator g (or translates of this).

We introduce the notion of an Erdős-deep family $\mathcal{F} = \{A_1, A_2, \dots, A_s\}$, where each $A_j \subseteq \mathbb{Z}_n$ and the defining property is that the total multiplicities (over j), of distances occurring within the same set A_j, achieve precisely the values $1, 2, \dots, k-1$ for some k. That is, $\mathcal{F}$ is Erdős-deep if and only if, for every $i \in \{1, 2, \dots, k-1\}$, there is exactly one positive number d_i satisfying

$$\sum_{j=1}^{s} \left|\left\{\{x, y\} \subseteq A_j : \mathrm{dist}(x, y) = d_i\right\}\right| = i,$$

and no such d_i for any $i \geq k$.

We provide a complete existence theorem for Erdős-deep pairs of arithmetic progressions $A_1, A_2 \subseteq \mathbb{Z}_n$ and also give a conjectured classification for families of three arithmetic progressions. Using an interesting identity on triangular numbers, we show a general construction for larger families whose size s is the square of an integer. This suggests the existence of Erdős-deep families often relies on such number-theoretic identities.

We define an extremal case of the Erdős-deep family structure in which both the distances and multiplicities are in $\{1, \dots, k-1\}$; such families are called Winograd families. We conjecture that Winograd families do not exist over the integers under $\mathrm{dist}(x, y) = |x - y|$; that is, Winograd families only exist with a modulus.

Erdős-deep sets correspond to a class of interesting musical rhythms. We conclude this work with a variety of musical demonstrations and original compositions using Erdős-deep rhythm families as a device for composing multi-voiced rhythms.