Robust Model Predictive Control and Scheduling Co-design for Networked Cyber-physical Systems

by

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B. Eng., Hubei University of Science and Technology, 2013

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ABSTRACT

In modern cyber-physical systems (CPSs) where the control signals are generally transmitted via shared communication networks, there is a desire to balance the closed-loop control performance with the communication cost necessary to achieve it. In this context, aperiodic real-time scheduling of control tasks comes into being and has received increasing attention recently. It is well known that model predictive control (MPC) is currently widely utilized in industrial control systems and has greatly increased profits in comparison with the proportional-integral-derivative (PID) control. As communication and networks play more and more important roles in modern society, there is a great trend to upgrade and transform traditional industrial systems into CPSs, which naturally requires extending conventional MPC to communication-efficient MPC to save network resources.

Motivated by this fact, we in this thesis propose robust MPC and scheduling co-design algorithms to networked CPSs possibly affected by both parameter uncertainties and additive disturbances.

In Chapter 2, a dynamic event-triggered robust tube-based MPC for constrained linear systems with additive disturbances is developed, where a time-varying pre-stabilizing gain is obtained by interpolating multiple static state feedbacks and the interpolating coefficient is determined via optimization at the time instants when the MPC-based control is triggered. The original constraints are properly tightened to achieve robust constraint satisfaction and a sequence of dynamic sets used to test events are derived according to the optimized coefficient. We theoretically show that the proposed algorithm is recursively feasible and the closed-loop system is input-to-state stable (ISS) in the attraction region. Numerical results are presented to verify the design.

In Chapter 3, a self-triggered min-max MPC strategy is developed for constrained nonlinear systems subject to both parametric uncertainties and additive disturbances, where the robust constraint satisfaction is achieved by considering the worst case of all possible uncertainty realizations. First, we propose a new cost function that relaxes the penalty on the system state in a time period where the controller will not be invoked. With this cost function, the next triggering time instant can be obtained at current time instant by solving a min-max optimization problem where the maximum triggering period becomes a decision variable. The proposed strategy is proved to be input-to-state practical stable (ISpS) in the attraction region at triggering time instants under some standard assumptions. Extensions are made to linear systems with additive disturbances, for which the conditions reduce to a linear matrix inequality (LMI). Comprehensive numerical experiments are performed to verify the correctness of the theoretical results.

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Acronyms

CPS Cyber-physical system

MPC Model predictive control

PID Proportional-integral-derivative

ISS Input-to-state stability

ISpS Input-to-state practical stability

LMI Linear matrix inequality

RPI Robust positively invariant

MRPI Maximal robust positively invariant

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To my grandfather.

Chapter 1

Introduction

1.1 Networked Cyber-physical Systems and Aperiodic Control

The term networked cyber-physical systems (CPS) represents a new generation of systems with tightly integrated cyber and physical components that can interact with each other via wireless communication networks to achieve increased computational capability, flexibility and autonomy over conventional systems. An illustration of operation principles of modern CPSs can be found in Figure 1.1. The development of CPSs serves as a technical foundation to a lot of important engineering applications spanning automotive systems, industrial systems, smart grid and robotics. It is worth noting that the communication and control that help form the interplay between cyber and physical spaces in CPSs play a key role in advancing future developments of CPSs, which is also the main topic of this thesis.

In typical networked CPSs, the interacting system components are generally spatially distributed and connected via shared communication networks. In controller design of such systems, the communication cost used to realize feedback control should

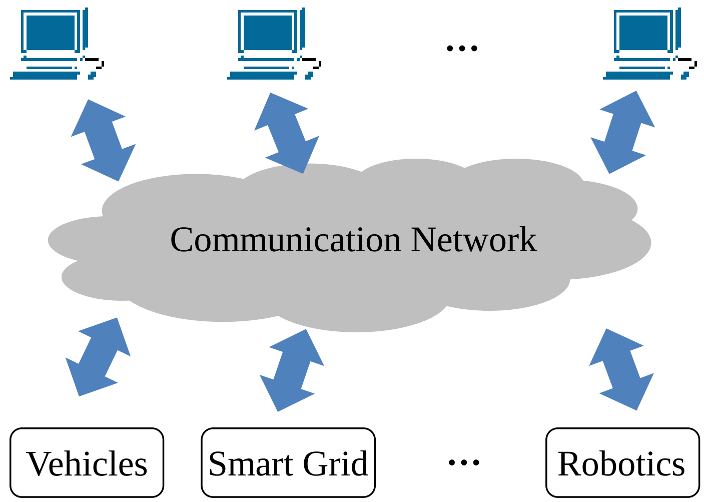

Figure 1.1: Operation principles of CPSs.

be taken into account. In this respect, the conventional periodic control may be not suitable, as it samples system state, calculates and delivers control input signals in a periodic way, possibly leading to unnecessary over-provisioning and therefore higher communication and computation costs. This problem, also faced by embedded control systems, becomes more serious for large-scale CPSs. To elaborate this, we consider the following discrete-time nonlinear system:

$$x_{t+1} = f(x_t, u_t) \tag{1.1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ represent the system state and control input, respectively, at time $t \in \mathbb{N}$. $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function satisfying $f(0, 0) = 0$. Let the sequence $\{t_k | k \in \mathbb{N}\} \in \mathbb{N}$ where $t_{k+1} > t_k$ be the time instants when the control input u_t needs to be updated. If the system is controlled by a periodic controller, one should derive in advance the maximum open-loop time period that the system equipped with such a controller can endure while preserving the closed-loop stability. This process, obviously, does not take the dynamical behavior of the closed-loop system into account and may give a conservative sampling strategy that leads to unnecessary use of computation and communication resources that are quite scarce in networked CPSs.

To tackle this problem, significant research has been devoted to the co-design of scheduling and control of CPSs, that is, generating and broadcasting control signals only when necessary. In particular, event-triggered control has been proposed and received considerable attention recently. In sharp contrast to periodic control, event-triggered control only generates network transmissions and closes the feedback loop when the system being controlled exhibits some undesired behaviors. In other words, the dynamical behavior of the real-time closed-loop system is taken into account to reduce the conservativeness of periodic schedulers. To be more specific, event-triggered control involves comparing the deviation between the actual state trajectory and the assumed trajectory at last triggering instant with a pre-defined, possibly time-varying threshold, thereby adapting the nonuniform sampling period in response to the system performance. A typical event-triggered control paradigm can be found in Figure 1.2. It is worth mentioning that continuous state measurement is necessary in event-triggered control. Intuitively speaking, the state deviation serves as a measure of how valuable the system state at current time instant is to the performance of the closed-loop system. If the deviation exceeds a pre-specified threshold, the current state is deemed as essential and will be used to generate control signals. Theoretical properties about how this threshold magnitude impacts the lower bound of the sampling period and the closed-loop system behavior are then analyzed by using different stability concepts in the literature. The hope of event-triggered control is to provide a larger average sampling period than periodic control while largely preserving the control performance. For a recent overview on event-triggered and self-triggered control, please refer to [28, 31, 37].

Early works on event-triggered control can be found in [1, 2, 29] for scalar systems.

Recently, there are some works addressing high-order systems using event-triggered control. For example, an event-triggered control strategy for a class of nonlinear systems based on the input-to-state stable (ISS) concept was developed in [61]. The event-triggered state-feedback control problem for linear systems was investigated in [46], where the performance was evaluated by using an emulation-based approach, i.e., comparing the event-triggered control with the corresponding continuous state-feedback. In [27], Heemels *et al.* proposed an event-triggered control strategy for linear systems where the event-triggered condition is only required to be verified periodically. In [14], Donkers *et al.* designed an output-based event-triggered control strategy for linear systems and studied the stability and $\mathcal{L}_\infty$-performance of the closed-loop system. Results on distributed event-triggered consensus were reported in [13] for first-order multi-agent systems and [64] for general linear models.

Event-triggered control generally requires continuously sampling system state and then checking triggering conditions, which may be not feasible for practical implementation. An example of triggering times in event-triggered control is plotted in Figure 1.3. To overcome this drawback, the self-triggered control has been developed. In contrast to event-triggered control, it no longer monitors the closed-loop system behavior to detect the event but estimates the next triggering time instant based on the knowledge of system dynamics and state information at current triggering time instant. Please see Figure 1.4 for an example of self-triggering time instants. This, although leads to a relatively conservative sampling strategy, makes the practical implementation much easier. In [62], Wang *et al.* developed a self-triggered control strategy for linear time-invariant systems with additive disturbances where the control performance is evaluated by the finite-gain l_2 stability.

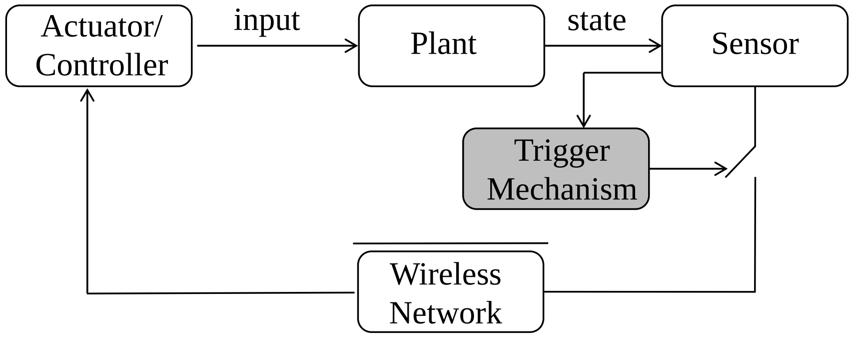

Figure 1.2: An event-triggered control paradigm.

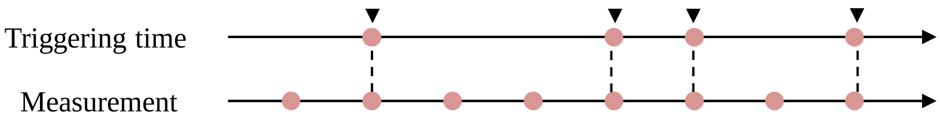

Figure 1.3: An example of triggering times in event-triggered control.

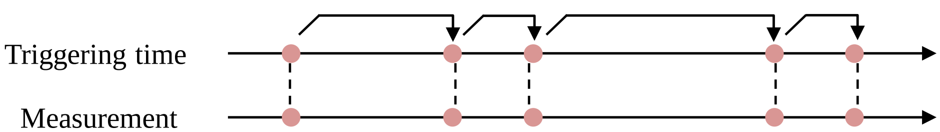

Figure 1.4: An example of triggering times in self-triggered control.

1.2 MPC and Aperiodic MPC

1.2.1 MPC

Model predictive control (MPC), also known as receding horizon control, is an advanced control strategy that combines the feedback mechanism with optimization. The control signal is derived by solving constrained optimization problems where the objective function is essentially a function of the system state at current time instant and a sequence of control inputs over a certain time horizon in the future, and the constraints are determined according to the limitations inherent in practical systems. MPC has now been widely used in various engineering areas such as process control systems [53] and motion control of autonomous vehicles [19]. Interestingly, the idea of iteratively optimizing a performance index has been also used in path planning for robotics [59].

Take the nonlinear system in (1.1) for example. Suppose that the system is subject to state constraints $x_t \in \mathcal{X} \subset \mathbb{R}^n$ and input constraints $u_t \in \mathcal{U} \subset \mathbb{R}^m$. The cost function to be minimized at each time instant can be set as

$$J(x_t, \mathbf{u}_{t,N}) = \sum_{i=0}^{N-1} L(x_{i,t}, u_{i,t}) + F(x_{N,t})$$

where N denotes the prediction horizon, $x_{i,t}$ and $u_{i,t}$ represent the predicted state and input trajectory emanating from time t and obey

$$\begin{aligned} x_{0,t} &= x_t \\ x_{i+1,t} &= f(x_{i,t}, u_{i,t}),\ i \in \mathbb{N}_{[0,N-1]}, \end{aligned} \tag{1.2}$$

and $\mathbf{u}_{t,N} = \left[u_{0,t}^{\mathrm{T}}, \cdots, u_{N-1,t}^{\mathrm{T}} \right]^{\mathrm{T}}$. $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $F : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are the stage cost function and terminal cost function, respectively. It is assumed that they are both continuous and satisfy $L(0,0) = 0$ and $F(0) = 0$. Then the control input at time t is derived by solving the following.

$$\begin{aligned} \mathbf{u}_{t,N}^* &= \arg \min_{u_{0,t} \in \mathcal{U}, \cdots, u_{N-1,t} \in \mathcal{U}} J(x_t, \mathbf{u}_{t,N}) \\ \text{s.t.} \quad & (1.2) \\ & u_{i,t} = \mathcal{U}, x_{i,t} \in \mathcal{X}, i \in \mathbb{N}_{[0,N-1]} \\ & x_{N,t} \in \mathcal{X}. \end{aligned}$$

Once a sequence of future control inputs, i.e., $\mathbf{u}_{t,N}^*$, is derived, the first element of it, i.e., $u_{0,t}^*$, is applied to the system. As time evolves, the MPC law can be obtained by re-sampling the system state and re-activating the optimization iteratively. Please see [49] for more details on MPC. Note that the objective and constraints in MPC-based controllers are usually set as functions of future system states and inputs to conveniently encode the desired control performance and system constraints in practice. However, given a system model, the future system states are functions of the current state and the future control actions. This implies that, at each time instant, the decision variable of the optimization problem becomes only the future inputs since the variable "current state" is fixed.

In the literature, there are some typical MPC schemes that are carefully designed in order to provide performance guarantees, e.g., recursive feasibility of optimization problems, closed-loop stability and robustness against additive disturbance and/or parametric uncertainties.

- First, to ensure recursive feasibility and stability, the authors in [10] proposed to add some tailored terminal ingredients including usually a terminal state penalty and terminal state constraints to the optimization problem in MPC-based controllers. The essential idea of this stabilizing MPC framework is that, by assuming the linearization of the original system is stabilizable, a static feedback law that stabilizes the linearization also works for the original nonlinear system *locally* and can be used to produce feasible control input solutions to optimization problems. The stability then follows from the use of this feasible control input and optimality. There are also some other stabilizing MPC strategies. For example, a Lyapunov-based constraint characterized by a stabilizing control law was used to ensure the feasibility and stability of MPC in [12].

- Second, there are three typical robust MPC schemes in the literature, that is, *robust MPC with nominal cost* [42, 48], *robust MPC with min-max cost* [43, 54], and *tube-based MPC* [11, 50].

1. In the first method, the Lipschitz continuity of the cost function [48] or the exponential stability of the local feedback [42] was explored to establish some degree of inherent robustness and the constraint satisfaction in presence of additive disturbances was achieved by properly tightening the original constraints. This approach generally yields conservative robustness margins as the prediction in this scheme is conducted in an open-loop fashion with which the disturbance effect exponentially increases according to the Gronwall-Bellman inequality [33].

2. In the second strategy, the controllers consider the worst case of all possible disturbance and/or uncertainty realizations to ensure robust constraint satisfaction and solve a min-max optimization problem to generate control inputs. This strategy provides larger robustness margins due to the so-called feedback prediction process [54] but also becomes computationally expensive. Trade-offs between computation and performance in min-max MPC were made in [43, 44]. Note that in the above two methods, the control input is purely optimization-based (Opt-based).

3. In robust tube-based MPC, the control law is composed by a pre-stabilizing linear feedback and the optimization-based input, in which the static linear feedback helps attenuate disturbance impacts and the latter contributes to the constraint satisfaction. It is worth mentioning that, with a pre-stabilizing feedback in the prediction model, the conservativeness caused by the constraint tightening procedure in [48] can be alleviated, especially for unstable linear systems and nonlinear systems where the model is Lipschitz continuous with a constant larger than 1.

An overview of typical MPC algorithms can be found in Table 1.1.

MPC schemes		Optimization	Constraints	Control input
Standard MPC in [10]		Minimization	Original	Opt-based
Robust MPC	[48]	Minimization	Tightened	Opt-based
	[54]	Min-max	Original	Opt-based
	[50]	Minimization	Tightened	Opt-based + pre-stabilizing

Table 1.1: An overview of typical MPC algorithms.

1.2.2 Event-triggered MPC and self-triggered MPC

It is well known that MPC is currently widely utilized in the industrial control systems and has greatly increased profits in comparison with the proportional-integral-derivative (PID) control. As communication and networks play more and more important roles in modern society, there is a great trend to upgrade and transform traditional industrial systems into CPSs, which naturally requires extending conventional MPC to communication-efficient MPC to save network resources. In this context, aperiodic MPC comes into being and has received increasing attention recently.

One widely used methodology in existing works on event-triggered MPC is to make use of the cost function to derive event-triggering conditions. For example, the event-triggered mechanisms, recursive feasibility and closed-loop stability in [15, 16, 24–26] were developed by taking advantage of the Lipschitz continuity of the cost function; specifically the authors in [24–26] considered the sample-and-hold implementation of the control law with different hold mechanisms. The authors in [24] further proposed a computationally efficient method for adaptively selecting sampling intervals while ensuring some degree of sub-optimality. Moreover, the robust constraint satisfaction therein was achieved by properly tightening the original constraints according to the Gronwall-Bellman inequality [33]. The authors in [20, 39, 43, 63] introduced a new variable that provides a degree of freedom to balance the communication cost and the control performance to the standard MPC cost function, and by solving a more complex optimization problem, the next triggering time can be explicitly determined at current triggering time instant. The essential idea is to relax the state cost penalty in a certain time period by multiplying the cost by a constant smaller than 1 if the controller during this period will not be triggered. The decrease in the optimal cost caused by the relaxed penalty may be seen as a reward due to a larger sampling period. By performing optimization, a trade-off between communication and control performance is sought. Amongst them, references [5, 20, 39] considered nonlinear systems without disturbances; the authors in [3, 6, 63] considered disturbed linear systems and [43] dealt with uncertain nonlinear systems.

Another standard routine, known as the emulation-based event-triggered control, involves setting a threshold that limits the deviation between the actual state and the predicted state at last triggering time instant and investigating how this threshold will affect the recursive feasibility and closed-loop stability of MPC algorithms; see [7, 8, 22, 23, 38, 40, 42] for example. The MPC-based control in these schemes should have some degree of robustness. This is primarily because that these works either considered systems with zero-order hold control inputs or/and additive disturbances. In this respect, these strategies differ from each other by the different types of robust MPC strategies used. In particular, the works in [22,23,38,40,42] recruited the robust MPC with nominal cost mentioned in the last subsection and [7, 8] used the robust tube-based MPC. Note that the solution proposed in [7, 8] may be less conservative since the tube-based MPC can better cope with the disturbance thanks to the pre-stabilizing linear feedback. When dealing with continuous-time systems within this framework, the effect caused by bounded additive disturbances is usually explored in order to make the event trigger Zeno-free [23, 38, 40, 42].

Aperiodic MPC		Mechanism	Disturbance	Uncertainty
Cost-based	[24, 26]	Self-triggered	Yes	No
	[25]	Event-triggered	Yes	No
	[20, 39]	Self-triggered	No	No
Emulation-based	[8, 23, 38]	Event-triggered	Yes	No

Table 1.2: An overview of aperiodic MPC algorithms.

1.3 Motivations

Although the event-triggered and self-triggered MPC have received enormous attention recently and great progress has been made in the literature, the existing schemes mostly presented a separate design of MPC and triggering strategy, as surveyed in the previous section. Notable exceptions include [20] where undisturbed nonlinear systems were considered and [3, 6] addressing linear systems with additive disturbances. These methods cannot be easily extended to disturbed nonlinear systems with or without parametric uncertainties primarily because the tube-based MPC framework mainly applies to linear systems and cannot handle parameter uncertainties. This two reasons motivate the research in this thesis to present a robust MPC and scheduling co-design framework for general nonlinear systems subject to both additive disturbances and parametric uncertainties. Specifically, the main motivations are summarized in the following two aspects.

- **Dynamic event-triggered tube-based MPC.** The co-design frameworks in [3, 6, 20] are all self trigger-based. In other words, the event-triggered schedulers and MPC in the literature are all separately designed in the sense that the threshold that characterizes the event trigger does not relates to the constrained optimization problem in the MPC framework. Considering that the optimization problem lies at the core of MPC, it would make perfect sense that the event-triggered threshold and the optimization problem can be jointly designed, i.e., the dynamic threshold is determined by the optimization problem at each triggering time instant. A better trade-off between communication and control performance can be expected due to the new optimization-based dynamic event trigger. This idea will be pursued in the first part of the thesis.

- **Self-triggered min-max MPC**. None of the existing results can handle general nonlinear systems affected by parametric uncertainties, although model uncertainties are almost unavoidable in system modeling. This is mainly due to the robust MPC schemes on which the existing results build are the robust MPC with nominal cost and the tube-based MPC, and cannot handle parametric uncertainties. Besides, the prediction in these two schemes is performed in an open-loop sense, leading to conservative attraction regions in presence of uncertainties. Robust min-max MPC can well handle general nonlinear systems with both parametric uncertainties and additive disturbances and provides relatively large attraction regions mainly thanks to the feedback prediction. However, how to introduce self-triggered schedulers to min-max MPC is unexplored and will be investigated in the second part of the thesis.

1.4 Contributions

The co-design problem of robust MPC and scheduling for networked CPSs is investigated in the thesis. The main contributions are summarized as follows.

- **Dynamic event-triggered tube-based MPC for disturbed unconstrained linear systems.** The first part of the thesis is concerned with the robust event-triggered MPC of discrete-time constrained linear systems subject to bounded additive disturbances. We make use of the interpolation technique to construct a feedback policy and tighten the original system constraint accordingly to fulfill robust constraint satisfaction. A dynamic event trigger that allows the controller to solve the optimization problem only at triggering time instants is developed, where the triggering threshold is related to the interpolating coefficient of the feedback policy and determined via optimization. We show that the proposed algorithm is recursively feasible and the closed-loop system is ISS in the attraction region. Finally, a numerical example is provided to verify the theoretical results.

- **Self-triggered min-max MPC for uncertain constrained nonlinear systems.** In the second part, we propose a robust self-triggered MPC algorithm for constrained discrete-time nonlinear systems subject to parametric uncertainties and disturbances. To fulfill robust constraint satisfaction, we take advantage of the min-max MPC framework to consider the worst case of all possible uncertainty realizations. In this framework, a novel cost function is designed based on which a self-triggered strategy is introduced via optimization. The conditions on ensuring algorithm feasibility and closed-loop stability are developed. In particular, we show that the closed-loop system is input-to-state practical stable (ISpS) in the attraction region at triggering time instants. In addition, we show that the main feasibility and stability conditions reduce to a linear matrix inequality (LMI) for linear case. Finally, numerical simulations and comparison studies are performed to verify the proposed control strategy.

1.5 Thesis Organization

The remainder of the thesis is organized as follows. In **Chapter 2**, the co-design of event trigger and the tube-based MPC for constrained linear systems with additive disturbances is investigated. A self-triggered min-max MPC strategy for uncertain constrained nonlinear systems is proposed in **Chapter 3**. **Chapter 4** concludes the thesis and gives some future research directions.

Chapter 2

Dynamic Event-triggered Tube-based MPC for Disturbed Constrained Linear Systems

2.1 Introduction

In this chapter, the focus is on event-triggered MPC of discrete-time constrained linear systems subject to bounded additive disturbances. When additive disturbance is considered in the MPC framework, the original state constraint should be tightened to achieve robust constraint satisfaction as the actual state and the predicted state do not coincide necessarily. The authors in [25, 42, 48] quantified the effect caused by the worst case disturbance on the system state by taking advantage of the Lipschitz continuity of the nonlinear system model; by set subtraction, a sequence of time-varying tightened constraints can be obtained. However, the use of the open-loop prediction strategy and the Lipschitz continuity essentially results in conservative attractive regions. To better attenuate the disturbance effect, the feedback prediction strategy [43, 54] can be employed to limit the growth of the disturbance effect along the prediction horizon. With this strategy, the well-known min-max MPC framework was developed in [58], where the controllers consider the worst case of all possible disturbance realizations to achieve constraint satisfaction and performs min-max optimization to derive optimal control policies. However, such a min-max optimization problem is computationally intractable, and parameterization of certain policies is often used [54] to reduce the degree of freedom in decision variables to make the optimization problem relatively easy to solve. Another application of this strategy can be found in tube-based MPC [11, 50], where a fixed control policy is used for prediction, leading to a sequence of limited sets (known as the "tube") characterizing the deviation between the actual state and the predicted state. Based on this approach, the authors in [8] developed a robust event-triggered MPC scheme by exploiting the fact that, during some open-loop spans, the realized disturbances that may be of insignificant impact will not bring the actual state farther away from the predicted state trajectory than the assumed worst case disturbance with feedback, it is then possible to not calculate and transmit control signals periodically.

Note that the linear feedback control policy used to attenuate the disturbance effect in [8] is static. It is also worth mentioning that a high-gain feedback law, i.e., LQR gain, that provides superior control performance may suffer from a small event-triggering threshold and thus a high sampling rate, while low-gain feedback laws may lead to a larger deviation bound and larger average sampling period with relatively poor control performance. This implies that a constant linear feedback gain may cannot finely balance the control performance and communication cost in robust event-triggered MPC. To solve this important issue, we propose a robust event-triggered MPC method featuring the following: (1) The feedback policy interpolates between low-gain feedback laws and a performance controller and (2) the interpolating coefficients are subject to optimization at triggering time instants to achieve a co-design of the triggering mechanism and the feedback policy. The idea of using interpolating strategy within periodic MPC was originally proposed and explored in [4, 51, 56, 57] for undisturbed linear systems to enlarge the associated feasible region while preserves the control performance; extensions to disturbed linear systems can be found in [52, 60]. However, the proposed control methodology differs from that in [52, 60] in the following two aspects: First, the controllers in [52, 60] solve constrained optimization problems periodically while the proposed controller conducts optimization aperiodically; this poses a challenge to ensuring robust constraint satisfaction and closed-loop stability. Second, compared with the existing control configuration [60] where the closed-loop state trajectory is a convex combination of the disturbed trajectory associated with a performance controller and some undisturbed trajectories governed by low-gain feedback laws, the proposed controller interpolates between multiple disturbed closed-loop state trajectories, and optimizes the interpolating coefficient at each triggering time instant in order to generate an optimized triggering mechanism.

The main contributions of this chapter are two-fold:

- A robust MPC strategy is developed for discrete-time constrained linear systems with bounded additive disturbances, where the feedback policy that helps attenuate the disturbance effect in the prediction process is constructed based on the interpolation technique. To fulfill robust constraint satisfaction, the system constraint sets are properly tightened according to a set of stabilizing feedback gains and the interpolating coefficients between them. The control input and the interpolating coefficients are derived by solving constrained optimization problems where the cost penalizes the weighting factors of the low-gain feedback laws in order to balance the size of attraction region and the control performance.

- An event-triggered control mechanism with dynamic triggering threshold is introduced to the interpolation-based robust MPC strategy such that the controller only needs to solve the constrained optimization problem and transmit the control signals at particular triggering time instants to reduce computation load and communication cost. Rigorous studies on algorithm feasibility and closed-loop stability have been conducted. Simulation examples are provided to validate the theoretical design.

The rest of this chapter is organized as follows. Section 2 formulates the control problem. Section 3 develops the robust event-triggered MPC algorithm. In Section 4, the algorithm feasibility and closed-loop stability are analyzed. Simulation results are provided in Section 5. Finally, Section 6 concludes the chapter.

Notations: In this chapter, we use the notation $\mathbb{R}$, and $\mathbb{N}$ to denote the sets of real and non-negative integers, respectively. $\mathbb{R}^n$ represents the Cartesian product $\underbrace{\mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{n}$. For some $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_{\geq c_1}$, let $\mathbb{R}_{\geq c_1}$ and $\mathbb{R}_{(c_1,c_2]}$ denote the sets $\{t \in \mathbb{R} : t \geq c_1\}$ and $\{t \in \mathbb{R} : c_1 < t \leq c_2\}$, respectively. Given a symmetric matrix S, $S > 0$ ($S \geq 0$) means that the matrix is positive (semi)definite. I_m denotes an identity matrix of size m for some $m \in \mathbb{N}_{>0}$. Given two sets $X, Y \subseteq \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the Minkowski sum of X and Y is $X \oplus Y = \{z \in \mathbb{R}^n : z = x+y, x \in X, y \in Y\}$ and the Pontryagin set difference is $X \ominus Y = \{z \in \mathbb{R}^n : z + y \in X, \forall y \in Y\}$, and $x \oplus X = \{x\} \oplus X$. Given a polytope $Z = \{z \in \mathbb{R}^{n+m} : Az \leq b\}$, $proj(Z, n) = \{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^m$ such that $A \begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \leq b\}$, $proj^*(Z, m) = \{u \in \mathbb{R}^m : \exists x \in \mathbb{R}^n$ such that $A \begin{bmatrix} x^{\mathrm{T}} & u^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \leq b\}$.

2.2 Problem Statement and Preliminaries

Consider the following constrained linear system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^n$ denote the system state, the control input, and unknown, time-varying additive disturbance at discrete time $t \in \mathbb{N}$, respectively. A and B are constant matrices of appropriate dimensions. The system constraints are given by $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$, $w(t) \in \mathcal{W}$, $t \in \mathbb{N}$. It is assumed that $\mathcal{X} \subseteq \mathbb{R}^n$, $\mathcal{U} \subseteq \mathbb{R}^m$, and $\mathcal{W} \subseteq \mathbb{R}^n$ are compact, convex polytopes containing the origin in their interiors. We further assume that the pair (A, B) is controllable and the state information can be measured at any time $t \in \mathbb{N}$.

The objective of this chapter is to stabilize the disturbed constrained system (2.1) asymptotically by using event-triggered MPC, where the control inputs are only required to be calculated and transmitted at some particular time instants $\{t_k : k \in \mathbb{N}\} \in \mathbb{N}$ to save communication and computation resources. In particular, the controller will be scheduled by an event trigger of the form

$$t_k = 0,\ t_{k+1} = t_k + H^*(x(t)), \tag{2.2}$$

where $H^* : \mathbb{R}^n \to \mathbb{N}_{\geq 1}$ is a function. The MPC-based control law becomes

$$u(t) = \mu(x(t_k), t - t_k),\ t \in \mathbb{N}_{[t_k, t_{k+1}-1]}, \tag{2.3}$$

where $\mu : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ is a function to be later designed.

2.3 Robust Event-triggered MPC

2.3.1 Control Policy and Constraint Tightening

Assumption **1.** *$K_p \in \mathbb{R}^{m\times n}, p \in \mathbb{N}_{[0,v]}$ are static feedback gains that render $\Phi_p = A + BK_p$ Schur.*

We consider the following control policy

$$u(t) = \sum_{p=0}^{v} K_p x_p(t),\ v \in \mathbb{N}, \tag{2.4}$$

where variables $x_p(t) = \lambda_p(t)x(t), p \in \mathbb{N}_{[0,v]}$ with the coefficients $\lambda_p(t), p \in \mathbb{N}_{[0,v]}$ satisfying the following:

$$\sum_{p=0}^{v} \lambda_p(t) = 1, \lambda_p(t) \in \mathbb{R}_{[0,1]}. \tag{2.5}$$

The recruitment of control policy in (2.4) in the MPC framework will lead to an enlarged terminal set (convex hull of individual terminal sets that are associated with K_p, $p \in \mathbb{N}_{[0,v]}$ for undisturbed linear systems [51,57]) and therefore a larger attraction region. Note that the parameterization design in control policy may introduce conservativeness, as it essentially reduces the degree of freedom of the decision variables.

Remark 1. *To implement a controller of the form equation* (2.4)*, one should first derive a group of feedback gains K_p that render $\Phi_p = A + BK_p$ stable and then use the coefficients to partition the state; the coefficients can either be fixed or optimized online as done in this chapter. Then the control input can be generated by following equation* (2.4)*.*

Due to disturbance, the original system constraints should be tightened to address any possible disturbance realization, and thus to fulfill robust constraint satisfaction.

Define the following tightened constraint sets

$$
\begin{aligned}
\mathcal{X}_j &= \mathcal{X} \ominus (\oplus_{p=0}^{v} \lambda_p(t) \mathcal{F}_j^p), \\
\mathcal{U}_j &= \mathcal{U} \ominus (\oplus_{p=0}^{v} \lambda_p(t) K_p \mathcal{F}_j^p), \\
\mathcal{F}_j^p &= \oplus_{i=0}^{j-1} (A + BK_p)^i \mathcal{W}.
\end{aligned} \tag{2.6}
$$

Rewrite the prediction policy in (2.4) as

$$
u(t) = K_0 x_0(t) + \sum_{p=1}^{v} K_p x_p(t)
$$

with

$$
x_0(t) = x(t) - \sum_{p=1}^{v} x_p(t).
$$

It follows

$$
u(t) = K_0 x(t) + \sum_{p=1}^{v} (K_p - K_0) x_p(t),
$$

in closed-loop with which the system (2.1) becomes

$$
x(t+1) = \Phi_0 x(t) + B \sum_{p=1}^{v} (K_p - K_0) x_p(t) + w(t).
$$

Consider

$$
x_p(t+1) = \Phi_p x_p(t) + \lambda_p(t) w(t), \; p \in \mathbb{N}_{[1,v]}
$$

and define

$$
\begin{aligned}
z(t) &= \begin{bmatrix} x(t)^{\mathrm{T}} & x_1(t)^{\mathrm{T}} & \cdots & x_v(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\
d(t) &= \begin{bmatrix} w(t)^{\mathrm{T}} & \lambda_1(t) w(t)^{\mathrm{T}} & \cdots & \lambda_v(t) w(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.
\end{aligned}
$$

We then have

$$z(t+1) = \mathit{\Phi} z(t) + Ed(t), \tag{2.7}$$

where

$$\begin{aligned} \mathit{\Phi} &= \begin{bmatrix} \mathit{\Phi}_0 & B(K_1 - K_0) & \cdots & B(K_v - K_0) \\ 0 & \mathit{\Phi}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathit{\Phi}_v \end{bmatrix}, \\ E &= \begin{bmatrix} I_n & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \end{bmatrix}. \end{aligned} \tag{2.8}$$

Let $\mathcal{Z}_f$ be the maximal robust positively invariant (MRPI) set [9] of the system (2.7) with the following constraints

$$\begin{aligned} & x(t) \in \mathcal{X},\ K_0 x(t) + \sum_{p=1}^{v} (K_p - K_0) x_p(t) \in \mathcal{U}, \\ & d(t) \in \underbrace{\mathcal{W} \times \cdots \times \mathcal{W}}_{v+1}. \end{aligned} \tag{2.9}$$

Lemma **1.** *[52] For the system in* (2.7)*, define the cost function* $V(z(t)) = z(t)^{\mathrm{T}} P z(t)$*, where* $P > 0$ *and* $P \in \mathbb{R}^{(v+1)n \times (v+1)n}$*.*

$$\begin{aligned} & V(z(t+1)) - V(z(t)) \\ & \leq -x(t)^{\mathrm{T}} Q x(t) - u(t)^{\mathrm{T}} R u(t) + \sigma d(t)^{\mathrm{T}} d(t), \end{aligned} \tag{2.10}$$

where $Q \geq 0$, $Q \in \mathbb{R}^{n\times n}$, $R > 0$, $R \in \mathbb{R}^{m\times m}$ *and* $\sigma \in \mathbb{R}_{\geq 0}$, *if the following holds*

$$\begin{bmatrix} P - \mathcal{Q} - \mathcal{R} & 0 & (\Phi)^{\mathrm{T}}P \\ 0 & \sigma I_{Nn} & E^{\mathrm{T}}P \\ P\Phi & PE & P \end{bmatrix} \geq 0, \tag{2.11}$$

with

$$\mathcal{Q} = \begin{bmatrix} I_n & 0 \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} I_n & 0 \end{bmatrix}, \mathcal{R} = \begin{bmatrix} K_0 & K \end{bmatrix}^{\mathrm{T}} R \begin{bmatrix} K_0 & K \end{bmatrix},$$

and

$$K = \begin{bmatrix} K_1 - K_0 & \cdots & K_v - K_0 \end{bmatrix}.$$

The proof can be found in [52]; we sketch the proof below for completeness.

Proof. Using (2.7), we have

$$\begin{aligned} & V(z(t+1)) - V(z(t)) \\ & = (\Phi z(t) + Ed(t))^{\mathrm{T}} P(\Phi z(t) + Ed(t)) - z(t)^{\mathrm{T}} Pz(t) \\ & = \begin{bmatrix} z(t)^{\mathrm{T}} & d(t)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \Phi^{\mathrm{T}} \\ E^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} \Phi & E \end{bmatrix} \begin{bmatrix} z(t) \\ d(t) \end{bmatrix} - \begin{bmatrix} z(t)^{\mathrm{T}} & d(t)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ d(t) \end{bmatrix}. \end{aligned}$$

We turn to consider

$$\begin{aligned} & -x(t)^{\mathrm{T}} Qx(t) - u(t)^{\mathrm{T}} Ru(t) + \sigma d(t)^{\mathrm{T}} d(t) \\ & = z(t)^{\mathrm{T}} (-\mathcal{Q} - \mathcal{R}) z(t) + \sigma d(t)^{\mathrm{T}} d(t) \\ & = \begin{bmatrix} z(t)^{\mathrm{T}} & d(t)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} -\mathcal{Q} - \mathcal{R} & 0 \\ 0 & \sigma I \end{bmatrix} \begin{bmatrix} z(t) \\ d(t) \end{bmatrix}. \end{aligned}$$

It remains to show that if equation (2.11) holds then

$$\begin{bmatrix} \Phi^{\mathrm{T}} \\ E^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} \Phi & E \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} -\mathcal{Q} - \mathcal{R} & 0 \\ 0 & \sigma I \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} P - \mathcal{Q} - \mathcal{R} & 0 \\ 0 & \sigma I \end{bmatrix} - \begin{bmatrix} \Phi^{\mathrm{T}} \\ E^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} \Phi & E \end{bmatrix} \geq 0.$$

This is true by the positive definiteness of P and the Schur complement. □

2.3.2 Robust Event-triggered MPC Setup

At each triggering time t_k, the controller solves a constrained finite horizon optimization problem, where the decision variable is

$$\Lambda(t_k) = \begin{bmatrix} \lambda_1(t_k) & \cdots & \lambda_v(t_k) \end{bmatrix} \in \mathbb{R}^v. \tag{2.12}$$

The constrained optimization problem is formulated as

$$\min_{\Lambda(t_k)} J(z(t_k), \Lambda(t_k)) \tag{2.13a}$$

$$\text{s.t.} \quad \sum_{p=0}^{v} \lambda_p(t_k) = 1, \lambda_p(t_k) \in \mathbb{R}_{[0,1]}, \tag{2.13b}$$

$$x_p(0, t_k) = \lambda_p(t_k)x(t_k), p \in \mathbb{N}_{[0,v]}, \tag{2.13c}$$

$$x_p(j+1, t_k) = \varPhi_p x_p(j, t_k),\ j \in \mathbb{N}_{[0,N-1]},\quad p \in \mathbb{N}_{[0,v]}, \tag{2.13d}$$

$$u(j, t_k) = \sum_{p=0}^{v} K_p x_p(j, t_k),\ j \in \mathbb{N}_{[0,N-1]}, \tag{2.13e}$$

$$x(j+1, t_k) = Ax(j, t_k) + Bu(j, t_k),\ j \in \mathbb{N}_{[0,N-1]}, \tag{2.13f}$$

$$x(j, t_k) \in \mathcal{X}_j,\ u(j, t_k) \in \mathcal{U}_j,\ j \in \mathbb{N}_{[0,N-1]}, \tag{2.13g}$$

$$[x(N, t_k)^{\mathrm{T}}, x_1(N, t_k)^{\mathrm{T}}, \cdots, x_v(N, t_k)^{\mathrm{T}}]^{\mathrm{T}}$$

$$\in \mathcal{Z}_f \ominus \mathcal{F}_N^{'}(\Lambda(t_k)), \tag{2.13h}$$

where $J(z(t_k), \Lambda(t_k)) = z(t_k)^{\mathrm{T}} P z(t_k) + \Lambda(t_k)^{\mathrm{T}} \Gamma \Lambda(t_k)$ with $\Gamma > 0$ and $\Gamma \in \mathbb{R}^{v \times v}$, $\mathcal{F}_N^{'}(\Lambda(t_k)) = \{(x_0, \cdots, x_v) \in \mathbb{R}^{n(v+1)} : x_0 \in \oplus_{p=0}^{v} \lambda_p(t_k)\mathcal{F}_N^p, x_1 \in \lambda_1(t_k)\mathcal{F}_N^1, \cdots, x_v \in \lambda_v(t_k)\mathcal{F}_N^v\}$.

Let $\mathcal{D}_N(x(t_k)) = \{\Lambda(t_k) \in \mathbb{R}^v : \text{(2.13b) to (2.13h)}\}$ be the set of feasible decision variables for a given state $x(t_k)$. The optimal solution of optimization problem (2.13) is denoted as $\Lambda^*(t_k) = \begin{bmatrix} \lambda_1^*(t_k), \cdots, \lambda_v^*(t_k) \end{bmatrix}$, and the corresponding optimal control input and state are written as $u^*(j, t_k), j \in \mathbb{N}_{[0,N-1]}$ and $x^*(j, t_k), j \in \mathbb{N}_{[0,N]}$, respectively. The optimal cost is denoted by $J^*(z(t_k), \Lambda(t_k))$.

Remark 2. *Note that we use the interpolation technique to construct the control policy and optimize the coefficients online in order to achieve larger region of attraction and better control performance. Due to disturbances and system constraints, real-time tightened constraints must be generated according to the time-varying control* *policy to achieve robust constraint satisfaction. As a limitation, the controller may suffer relatively heavy computation load compared to other standard tube-based MPC schemes where the control policy is fixed, as it needs to perform Pontryagin Difference and Minkowski Sum of polytopes online. Some algorithms for efficiently conducting such set operations have been reported in the literature. Specifically, the Pontryagin Difference can be derived for polytopes by solving a sequence of linear programming problems [34]; the derivation of Minkowski Sum involves a projection operation from* $\mathbb{R}^{2n}$ *down to* $\mathbb{R}^n$ *or vertex enumeration and computation of convex hull [30].*

2.3.3 Triggering Mechanism

In this chapter, we employ an event trigger that is realized by testing whether or not the deviation between the predicted state and the true state exceeds a threshold as in [8, 38, 40, 42]

$$t_0 = 0,\ t_{k+1} = t_k + \min\{i \in \mathbb{N}_{\geq 1} : z(t_k + i) \notin z^*(i, t_k) \oplus \mathcal{T}_i\}, \tag{2.14}$$

where

$$z^*(j, t_k) = \begin{bmatrix} x^*(j, t_k)^{\mathrm{T}} & x_1^*(j, t_k)^{\mathrm{T}} & \cdots & x_v^*(j, t_k)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{2.15}$$

and

$$\mathcal{T}_i = \mathcal{A}^{-1}(\mathcal{F}'_{i+1}(\Lambda^*(t_k)) \ominus (\mathcal{W} \times \lambda_1^*(t_k)\mathcal{W} \times \cdots \lambda_v^*(t_k)\mathcal{W})), \tag{2.16}$$

$i \in \mathbb{N}_{[1,N-1]}$, with $\mathcal{T}_0 = \{0\}$, $\mathcal{T}_N = \emptyset$,

$$\mathcal{A} = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix}, \tag{2.17}$$

and $\mathcal{F}_i^{'}(\Lambda^*(t_k)) = \{(x_0, \cdots, x_v) \in \mathbb{R}^{n(v+1)} : x_0 \in \oplus_{p=0}^{v} \lambda_p^*(t_k)\mathcal{F}_i^p, x_1 \in \lambda_1^*(t_k)\mathcal{F}_i^1, \cdots, x_v \in \lambda_v^*(t_k)\mathcal{F}_i^v\}$.

Remark 3. *The computational complexity of the proposed event-triggered control algorithm mainly results from the test of triggering conditions and the optimization problem in equation* (2.13)*. Testing triggering conditions requires to check whether or not* $\mathcal{A}(z(t_k+i) - z^*(i,t_k))$ *is in the set* $\mathcal{F}_{i+1}^{'}(\Lambda^*(t_k)) \ominus (\mathcal{W} \times \lambda_1^*(t_k)\mathcal{W} \times \cdots \lambda_v^*(t_k)\mathcal{W})$*. Besides, the optimization problem* (2.13) *is a convex quadratic problem, and can be efficiently solved via various optimization packages, e.g., CPLEX and Gurobi.*

2.4 Analysis

Under the event-triggered scheduler (2.14) and controller (2.13), the closed-loop system becomes

$$\begin{aligned} x(t+1) &= Ax(t) + Bu^*(t-t_k, t_k) + w(t),\ t \in \mathbb{N}_{[t_k, t_{k+1}-1]}, \\ t_{k+1} &= t_k + \min\{i \in \mathbb{N}_{\geq 1} : z(t_k+i) \notin z^*(i,t_k) \oplus \mathcal{T}_i\}, \end{aligned} \tag{2.18}$$

where $t, k, t_k \in \mathbb{N}$, $x(0) \in \mathbb{R}^n$, $t_0 = 0$, and $w(t) \in \mathcal{W}$. In this section, recursive feasibility of the proposed control strategy and stability of the closed-loop system (2.18) will be analyzed.

2.4.1 Recursive Feasibility

A useful lemma is presented before proceeding to the main result.

Consider the set $\mathcal{S} = \{(x,u) \in \mathbb{R}^{n+1} : Gx + Hu \leq b\}$, where $G \in \mathbb{R}^{s \times n}$, $H \in \mathbb{R}^s$ and $b \in \mathbb{R}^s_{\geq 0}$. Let $\mathcal{S}_x = proj(\mathcal{S}, n)$ and $\mathcal{S}_u = proj^*(\mathcal{S}, 1)$.

Lemma **2.** *Suppose that convex sets* $\Omega_1 \subseteq \mathcal{S}_x$*,* $\Omega_2 \subseteq \mathcal{S}_u$ *both contain the origin in their* *interiors, and define* $\Omega = \{(x,u) \in \mathbb{R}^{n+1} : x \in \Omega_1, u \in \Omega_2\}$*, then* $proj(\mathcal{S} \ominus \Omega, n) \subseteq (\mathcal{S}_x \ominus \Omega_1)$.

Proof. Following the Fourier-Motzkin elimination method [32], we have

$$\begin{aligned}\mathcal{S}_x = \{x \in \mathbb{R}^n : G^i x \leq b^i, \forall i \in I^0\} \cap \{x \in \mathbb{R}^n : (H^i G^j - H^j G^i)x \\ \leq H^i b^j - H^j b^i, \forall i \in I^+, j \in I^-\},\end{aligned} \tag{2.19}$$

where $I^0 = \{i : H^i = 0\}$, $I^+ = \{i : H^i > 0\}$ and $I^- = \{i : H^i < 0\}$ are subsets of the set $\{1, 2, \cdots, s\}$. Using the support function operation [34], we have

$$\begin{aligned}\mathcal{S} \ominus \Omega = \{(x,u) \in \mathbb{R}^{n+1} : & G^i x + H^i u \leq b^i \\ & - \sup_{(z_1,z_2)\in\Omega}(G^i z_1 + H^i z_2), i \in \mathbb{N}_{[1,s]}\},\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\mathcal{S}_x \ominus \Omega_1 = & \{x \in \mathbb{R}^n : G^i x \leq b^i - \sup_{z\in\Omega_1} G^i z, \forall i \in I^0\} \\ & \cap \{x \in \mathbb{R}^n : (H^i G^j - H^j G^i)x \leq H^i b^j) \\ & - H^j b^i - \sup_{z\in\Omega_1}(H^i G^j - H^j G^i)z, \\ & \forall i \in I^+, j \in I^-\}.\end{aligned} \tag{2.21}$$

Similarly, it can be verified that

$$\begin{aligned}proj(\mathcal{S} \ominus \Omega, n) = & \{x \in \mathbb{R}^n : G^i x \leq b^i - \sup_{(z_1,z_2)\in\Omega} \\ & (G^i z_1 + H^i z_2), i \in I^0\} \\ & \cap \{x \in \mathbb{R}^n : (H^i G^j - H^j G^i)x \\ & \leq H^i(b^j - \sup_{(z_1,z_2)\in\Omega}(G^j z_1 \\ & + H^j z_2)) - H^j(b^i - \sup_{(z_1,z_2)\in\Omega} \\ & (G^i z_1 + H^i z_2)), \forall i \in I^+, j \in I^-\}.\end{aligned} \tag{2.22}$$

Since Ω_2 contains the origin in its interior and $H^i > 0$ and $H^j < 0$, we have

$$\begin{aligned} -H^i \mathrm{sup}_{(z_1,z_2)\in\Omega}(G^j z_1 + H^j z_2) + H^j \mathrm{sup}_{(z_1,z_2)\in\Omega}(G^i z_1 + H^i z_2) \\ \leq - H^i \mathrm{sup}_{z\in\Omega_1} G^j z + H^j \mathrm{sup}_{z\in\Omega_1} G^i z. \end{aligned} \tag{2.23}$$

Consider

$$\begin{aligned} & - \mathrm{sup}_{z\in\Omega_1}(H^i G^j - H^j G^i)z \\ \geq & - \{\mathrm{sup}_{z\in\Omega_1}(H^i G^j z) + \mathrm{sup}_{z\in\Omega_1}(-H^j G^i z)\} \\ = & - H^i \mathrm{sup}_{z\in\Omega_1} G^j z + H^j \mathrm{sup}_{z\in\Omega_1} G^i z. \end{aligned} \tag{2.24}$$

By summarizing (2.23) and (2.24), it readily follows that $proj(\mathcal{S} \ominus \Omega, n) \subseteq (\mathcal{S}_x \ominus \Omega_1)$. $\square$

Lemma **3.** *Given* $\Lambda(t_k)$*, for* $\mathcal{F}_N^p, p \in \mathbb{N}_{[0,v]}$ *defined in* (2.6) *and* $\mathcal{Z}_f$*,* $\mathcal{F}_N^{'}(\Lambda(t_k))$ *defined in* (2.13h)*,* $proj(\mathcal{Z}_f \ominus \mathcal{F}_N^{'}(\Lambda(t_k)), n) \subseteq proj(\mathcal{Z}_f, n) \ominus (\oplus_{p=0}^{v} \lambda_p(t_k) \mathcal{F}_N^p)$ *holds.*

Proof. Based on Lemma 2, Lemma 3 can be proved by following the idea in Lemma 2 in [60]; indeed it reduces to Lemma 2 in [60] by setting $\mathcal{F}_N^{'}(\Lambda(t_k)) = \{(x_0, \mathit{0}) \in \mathbb{R}^{n(v+1)} : x_0 \in \mathcal{F}_N^0\}$. $\square$

The recursive feasibility result is summarized in the following lemma.

Lemma **4.** *For the system* (2.1) *with initial state* $x(t_0)$*, if* $\mathcal{D}_N(x(t_0)) \neq \emptyset$ *and the time series* $\{t_k\}$*,* $k \in \mathbb{N}$ *is determined by the triggering mechanism* (2.14)*, then* $\mathcal{D}_N(x(t_k)) \neq \emptyset$*,* $k \in \mathbb{N}$ *holds.*

Proof. We make use of the induction principle to prove the optimization problem (2.13) is recursively feasible. Assume that $\mathcal{D}_N(x(t_k)) \neq \emptyset$ for some t_k. Based on $\Lambda^*(t_k)$ at time t_k, a decision variable candidate can be constructed as follows

$$\tilde{\Lambda}(t_{k+1}) = \Big[\lambda_1^*(t_k), \cdots, \lambda_v^*(t_k)\Big]; \tag{2.25}$$

the satisfaction of constraint (2.13b) follows. Due to $x(t_k+i) = Ax(t_k+i-1) + B\sum_{p=0}^{v} K_p x_p(i-1,t_k) + w(t_k+i-1)$, $i \in \mathbb{N}_{[1,t_{k+1}-t_k]}$, the constraint (2.13c) can be satisfied by choosing

$$\begin{aligned}\tilde{x}_p(0,t_{k+1}) =& x_p(t_{k+1}-t_k,t_k) + \sum_{j=0}^{t_{k+1}-t_k-1} \lambda_p^*(t_k) A^j w(t_{k+1}-1-j) \\ =& \lambda_p^*(t_k)(x(t_{k+1}-t_k,t_k) + \sum_{j=0}^{t_{k+1}-t_k-1} A^j w(t_{k+1}-1-j)), p \in \mathbb{N}_{[0,v]}.\end{aligned} \tag{2.26}$$

From the prediction dynamics (2.13d) and the definition of decision variable candidate $\tilde{\Lambda}(t_{k+1})$, one gets, for $j \in \mathbb{N}_{[0,N]}$, $p \in \mathbb{N}_{[0,v]}$,

$$\tilde{x}_p(j,t_{k+1}) = \Phi_p^j(\tilde{x}_p(t_{k+1}) - x_p(t_{k+1}-t_k,t_k)) + x_p(t_{k+1}-t_k+j,t_k), \tag{2.27}$$

with, for $j \in \mathbb{N}_{[N+t_k-t_{k+1}+1,N]}$,

$$x_p(t_{k+1}-t_k+j,t_k) = \Phi_p^{t_{k+1}-t_k+j-N} x_p(N,t_k), p \in \mathbb{N}_{[0,v]}. \tag{2.28}$$

It follows, for $j \in \mathbb{N}_{[0,N]}$,

$$\begin{aligned}\tilde{x}(j,t_{k+1}) =& \sum_{p=0}^{v} \Phi_p^j(\tilde{x}_p(t_{k+1}) - x_p(t_{k+1}-t_k,t_k)) + x(t_{k+1}-t_k+j,t_k), \\ \tilde{u}(j,t_{k+1}) =& \sum_{p=0}^{v} K_p\Phi_p^j(\tilde{x}_p(t_{k+1}) - x_p(t_{k+1}-t_k,t_k)) + u(t_{k+1}-t_k+j,t_k),\end{aligned} \tag{2.29}$$

which implies that constraints (2.13e)-(2.13f) are satisfied.

Note that no event was triggered during time period $t \in \mathbb{N}_{[t_k+1,t_{k+1}-1]}$, which means that

$$x_p(t_k+j+1) - x_p^*(j+1,t_k) = A(x_p(t_k+j) - x_p(j,t_k)) + \lambda^*(t_k)w(t_k+j)$$

holds for $j \in \mathbb{N}_{[0,t_{k+1}-t_k-2]}$. By induction, we have

$$\begin{aligned} x(t_{k+1}) - x(t_{k+1}-t_k, t_k) &\in \oplus_{p=0}^{v} \lambda_p^*(t_k) \mathcal{F}_{t_{k+1}-t_k}^{p}, \\ \tilde{x}_p(0, t_{k+1}) - x_p(t_{k+1}-t_k, t_k) &\in \lambda_p^*(t_k) \mathcal{F}_{t_{k+1}-t_k}^{p}, p \in \mathbb{N}_{[0,v]}. \end{aligned} \tag{2.30}$$

Considering that

$$x(t_{k+1}-t_k+j, t_k) \in \mathcal{X}_{t_{k+1}-t_k+j}, j \in \mathbb{N}_{[0,N+t_k-t_{k+1}]}, \tag{2.31}$$

and

$$\begin{aligned} &\mathcal{X}_{t_{k+1}-t_k+j} \oplus (\oplus_{p=0}^{v} \lambda_p^*(t_k) \Phi_p^j \mathcal{F}_{t_{k+1}-t_k}^{p}) \\ =&\mathcal{X} \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k) \mathcal{F}_{t_{k+1}-t_k+j}^{p}) \oplus (\oplus_{p=0}^{v} \lambda_p^*(t_k) \Phi_p^j \mathcal{F}_{t_{k+1}-t_k}^{p}) \\ \subseteq&\mathcal{X} \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k) \mathcal{F}_{j}^{p}),\ j \in \mathbb{N}_{[0,N+t_k-t_{k+1}]}, \end{aligned} \tag{2.32}$$

and similarly,

$$\begin{aligned} &\mathcal{U}_{t_{k+1}-t_k+j} \oplus (\oplus_{p=0}^{v} \lambda_p^*(t_k) K_p \Phi_p^j \mathcal{F}_{t_{k+1}-t_k}^{p}) \\ \subseteq&\mathcal{U} \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k) K_p \mathcal{F}_{j}^{p}),\ j \in \mathbb{N}_{[0,N+t_k-t_{k+1}]}, \end{aligned} \tag{2.33}$$

it follows, for $j \in \mathbb{N}_{[0,N+t_k-t_{k+1}]}$,

$$\tilde{x}(j, t_{k+1}) \in \mathcal{X}_j,\ \tilde{u}(j, t_{k+1}) \in \mathcal{U}_j. \tag{2.34}$$

Since $\mathcal{Z}_f$ is a robustly positively invariant set of the system (2.7), one gets, for $j \in$

$\mathbb{N}_{[N+t_k-t_{k+1}+1,N]}$,

$$
\begin{aligned}
&[(\sum_{p=0}^{v} x_p(t_{k+1}-t_k+j,t_k)+\sum_{i=0}^{t_{k+1}-t_k+j-1} \Phi_p^i \lambda_p^*(t_k)w(i))^{\mathrm{T}},\\
&(x_1(t_{k+1}-t_k+j,t_k)+\sum_{i=0}^{t_{k+1}-t_k+j-1} \Phi_1^i \lambda_1^*(t_k)w(i))^{\mathrm{T}},\cdots,\\
&(x_v(t_{k+1}-t_k+j,t_k)+\sum_{i=0}^{t_{k+1}-t_k+j-1} \Phi_v^i \lambda_v^*(t_k)w(i))^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{Z}_f,
\end{aligned} \tag{2.35}
$$

and

$$
\begin{aligned}
&u(t_{k+1}-t_k+j,t_k) = K_0(x(t_{k+1}-t_k+j,t_k)+y)\\
&+\sum_{p=1}^{v}(K_p-K_0)(x_p(t_{k+1}-t_k+j,t_k)+y^p) \in \mathcal{U},
\end{aligned} \tag{2.36}
$$

where $y \in \oplus_{p=0}^{v} \lambda_p^*(t_k)\mathcal{F}_N^p$, $y^p \in \lambda_p^*(t_k)\mathcal{F}_{t_{k+1}-t_k+j}^p, p \in \mathbb{N}_{[1,v]}$. It follows

$$
\begin{aligned}
&[x(t_{k+1}-t_k+j,t_k)^{\mathrm{T}}, x_1(t_{k+1}-t_k+j,t_k)^{\mathrm{T}},\cdots,\\
&x_v(t_{k+1}-t_k+j,t_k)^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{Z}_f \ominus \mathcal{F}'_{t_{k+1}-t_k+j}(\Lambda_{t_k}^*),\\
&j \in \mathbb{N}_{[N+t_k-t_{k+1}+1,N]}.
\end{aligned} \tag{2.37}
$$

Considering (2.29) and (2.36), one gets

$$
\tilde{u}(j,t_{k+1}) \in \mathcal{U} \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k)K_p\mathcal{F}_j^p),\ j \in \mathbb{N}_{[N+t_k-t_{k+1}+1,N]}. \tag{2.38}
$$

By application of Lemma 3, one gets $x(t_{k+1}-t_k+j,t_k) \in \mathcal{X}_f \ominus \oplus_{p=0}^{v} \lambda_p^*(t_k)\mathcal{F}_{t_{k+1}-t_k+j}^p$ where $\mathcal{X}_f$ denotes the projection of $\mathcal{Z}_f$ onto x space. Due to

$$
\begin{aligned}
&\mathcal{X}_f \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k)\mathcal{F}_{t_{k+1}-t_k+j}^p) \oplus (\oplus_{p=0}^{v} \lambda_p^*(t_k)\Phi_p^j\mathcal{F}_{t_{k+1}-t_k}^p)\\
&\subseteq \mathcal{X}_f \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k)\mathcal{F}_j^p),
\end{aligned} \tag{2.39}
$$

we have

$$\tilde{x}(j, t_{k+1}) \in \mathcal{X}_f \ominus (\oplus_{p=0}^{v} \lambda_p^*(t_k) \mathcal{F}_j^p), j \in \mathbb{N}_{[N+t_k-t_{k+1}+1, N]}. \tag{2.40}$$

By summarizing (3.2), (2.38) and (3.3) and considering $\mathcal{X}_f \subseteq \mathcal{X}$, we have that constraint (2.13g) is satisfied.

By letting $j = N$ in (2.35) and considering (2.29), we have

$$[\tilde{x}(N, t_{k+1})^{\mathrm{T}}, \tilde{x}_1(N, t_{k+1})^{\mathrm{T}}, \cdots, \tilde{x}_v(N, t_{k+1})^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{Z}_f \ominus \mathcal{F}_N^{'}(\Lambda^*(t_k)), \tag{2.41}$$

implying that the satisfaction of constraint (2.13h) can be achieved by $\tilde{\Lambda}(t_{k+1})$. The proof is completed.

□

2.4.2 Stability

The closed-loop stability result is presented in the following theorem.

Theorem **1.** *For the system* (2.1) *with initial state* $x(t_0)$*, if* $\mathcal{D}_N(x(t_0)) \neq \{\emptyset\}$ *and the time series* $\{t_k\}$*,* $k \in \mathbb{N}$ *is determined by the triggering mechanism* (2.14)*, then the closed-loop system in* (2.18) *is ISS.*

Proof. Without loss of generality, the following two cases are considered to prove the theorem. First, if the event is not triggered at time instant $t_k + 1$, from Lemma 1 we have

$$\begin{aligned} &J(z(t_k+1), \Lambda^*(t_k)) - J(z(t_k), \Lambda^*(t_k)) \\ &\leq V(z(t_k+1)) - V(z(t_k)) \\ &\leq -x(t_k)^{\mathrm{T}} Q x(t_k) - u(t_k)^{\mathrm{T}} R u(t_k) + \sigma d(t_k)^{\mathrm{T}} d(t_k). \end{aligned} \tag{2.42}$$

Second, if the event is triggered at time instant $t_{k+1} = t_k + 1$, from Lemma 4 we have that $\tilde{\Lambda}(t_k+1) = \Lambda^*(t_k)$ is a feasible solution of the optimization problem (2.13). Similarly, we consider

$$\begin{aligned} & J(z(t_k+1), \Lambda^*(t_{k+1})) - J(z(t_k), \Lambda^*(t_k)) \\ & \leq J(z(t_k+1), \Lambda^*(t_k)) - J(z(t_k), \Lambda^*(t_k)) \\ & \leq V(z(t_k+1)) - V(z(t_k)) \\ & \leq -x(t_k)^{\mathrm{T}} Q x(t_k) - u(t_k)^{\mathrm{T}} R u(t_k) + \sigma d(t_k)^{\mathrm{T}} d(t_k). \end{aligned} \tag{2.43}$$

Therefore, $J(z(t), \Lambda(t))$ is an ISS Lyapunov function of the closed-loop system (2.18), implying that the closed-loop system (2.18) is ISS. This completes the proof. □

2.5 Simulation

Consider the following linear system [11, 60]

$$x(t+1) = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) + w(t), \tag{2.44}$$

where the constraint sets are given by $\mathcal{X} = [-30, 30] \times [-10, 10]$, $\mathcal{U} = [-5, 5]$ and $\mathcal{W} = [-0.2, 0.2] \times [-0.2, 0.2]$. Set $v = 1$. $K_0 = \begin{bmatrix} -0.4991 & -0.9546 \end{bmatrix}$ is derived by using the LQR technique by setting (Q, R) to be $(I_2, 1)$; a low-gain feedback is chosen as $K_1 = \begin{bmatrix} -0.0333 & -0.4527 \end{bmatrix}$. Set $N = 5$ and $x(0) = [-30; 10]$. The weighting matrix

$$P = \begin{bmatrix} 1980.1 & 522.5 & -1947.4 & -398.4 \\ 522.5 & 1517.3 & -494.9 & -1368.3 \\ -1947.4 & -494.9 & 1953.3 & 495.4 \\ -398.4 & -1368.3 & 495.4 & 1842.4 \end{bmatrix} \tag{2.45}$$

and $\sigma = 8186.2$ are derived by solving the following optimization problem:

$$\min_{P>0} \sigma \quad \text{s.t.} \quad \text{Eq.}\,(2.11), \tag{2.46}$$

where Q and R are chosen as identity matrices of appropriate dimensions. Set $\Gamma = 20000$.

By using Multi-Parametric Toolbox 3.0 [30], the terminal regions for K_0, K_1 and the proposed control strategy are plotted in Fig. 2.1; it can be seen that the proposed strategy enjoys a much larger terminal region compared with that for static feedback gains K_0 and K_1. To highlight the advantages of the proposed control strategy, its periodic counterpart is also executed. The additive disturbances in this simulation are randomly chosen, but keep the same for both event-triggered and periodic control cases. The optimization problems are solved by using YALMIP [45]. The results are reported as follows. Table 2.1 compares the average sampling period and the closed-loop performance of these two cases, where the performance indices $J_{\mathrm{p}} = \frac{\sum_{t=0}^{T_{\mathrm{sim}}-1} x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} R u(t)}{T_{\mathrm{sim}}}$ with $T_{\mathrm{sim}} = 1000$ being the simulation time. It can be seen that the proposed control strategy significantly reduces the sampling frequency while preserves the closed-loop control performance. Note that J_{p} in event-triggered control is even smaller than that in periodic case; it may be because that there is a gap between the cost function to be optimized and J_{p}. To clearly illustrate the simulation results, only for the first 30 steps the closed-loop behavior is plotted. It is worth mentioning that the number of triggering in the first 30 steps is 17. Specifically, Fig. 2.2 shows the evolution of the system state, Fig. 2.3 depicts the control input trajectory, and Fig. 2.4 illustrates the change of λ_1 over time.

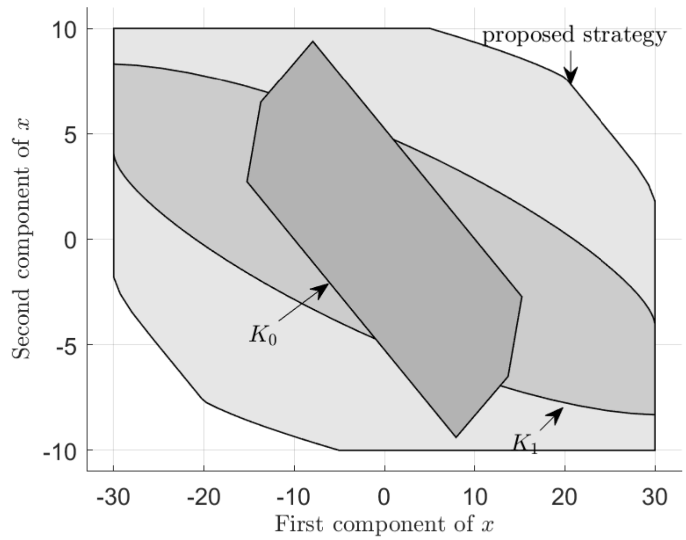

Figure 2.1: Comparison of terminal regions.

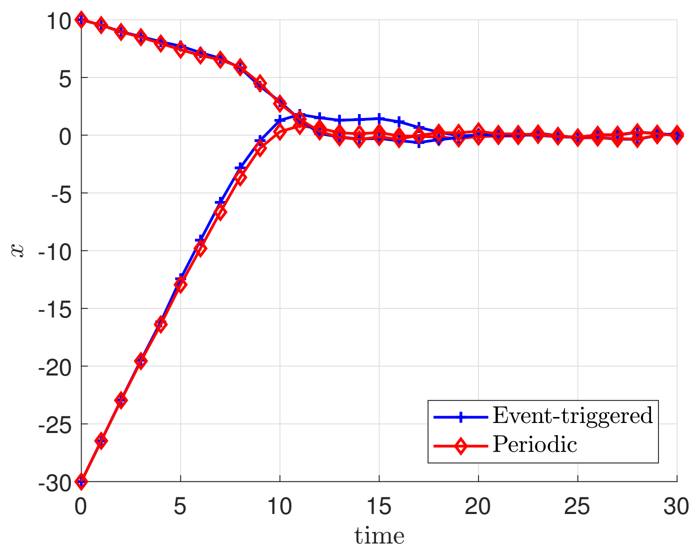

Figure 2.2: Trajectories of system state.

	Average sampling time	J_p
Periodic	1.0000	3.8873
Event-triggered	1.2019	3.8599

Table 2.1: Performance comparison

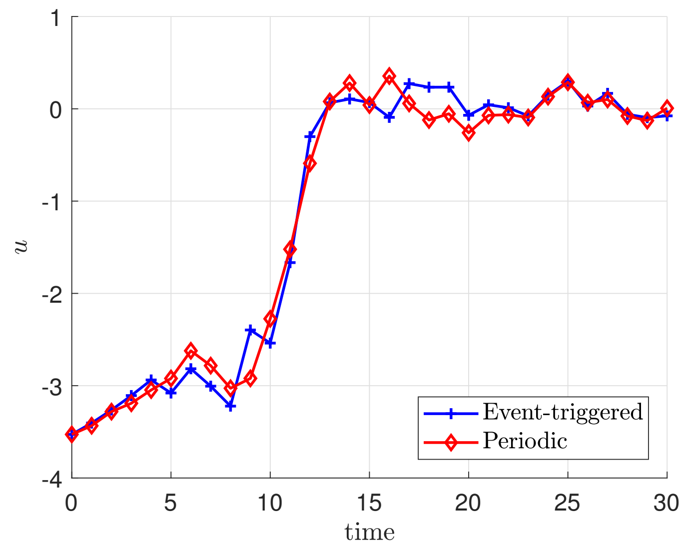

Figure 2.3: Trajectories of control input.

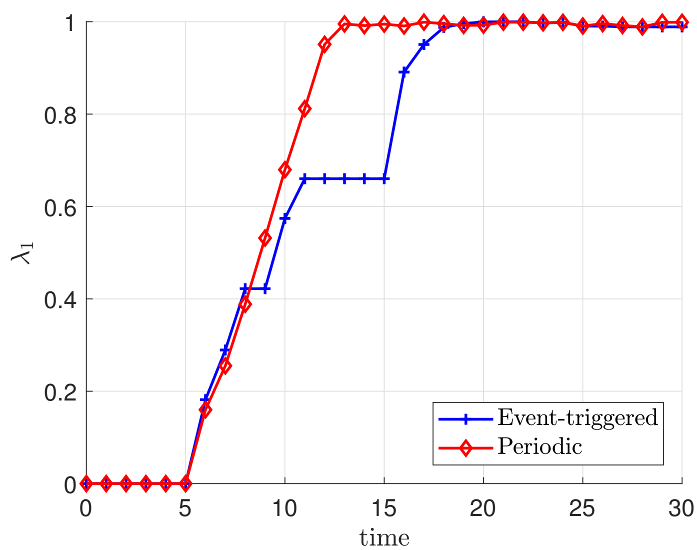

Figure 2.4: Trajectories of λ_1.

2.6 Conclusion

We have studied the robust event-triggered MPC problem for discrete-time constrained linear systems with bounded additive disturbances. A novel robust event-triggered MPC strategy has been developed, where the robust constraint satisfaction is guaranteed by taking advantage of an interpolation-based feedback policy within the MPC framework and appropriately tightening the original constraint sets. At each triggering time instant, by solving a constrained optimization problem the controller generates a sequence of control inputs and a set of interpolating coefficients that characterizes the triggering threshold of the event trigger. The recursive feasibility and closed-loop stability have been rigorously analyzed. A simulation example has been provided to illustrate the effectiveness of the proposed approach.

Chapter 3

Self-triggered Min-max MPC for Uncertain Constrained Nonlinear Systems

3.1 Introduction

Self-triggered MPC for uncertain systems is of particular importance as uncertainties are not avoidable in practice, which is also the focus of this chapter. Among the results of self-triggered MPC, [15, 16, 25, 38] use nominal models to formulate the optimization problems, the stability is ensured by exploring the inherent robustness of MPC and the original system constraints are tightened to achieve robust constraint satisfaction. In these cases, the closed-loop stability is usually established by exploiting the system inherent robustness. Unfortunately, this method suffers from very small attraction regions, especially for unstable linear systems and nonlinear systems with relatively large Lipschitz constants, due to the constraint tightening procedure. To enlarge attraction region, the authors in [3, 6] recently investigated the robust self-triggered MPC problem for discrete-time linear systems based on the idea of tube-based MPC [18, 50], where a pre-stabilizing linear feedback controller is introduced into the prediction model to attenuate disturbance impacts. In contrast to robust self-triggered MPC using a nominal model, self-triggered MPC with a tube-based strategy has less conservative tightened constraints, therefore offering relatively large regions of attraction.

It is worth noting that the existing results of self-triggered MPC might not be able to handle systems with generic parameter uncertainties, though model uncertainties are almost unavoidable in system modeling. Besides, enlarging the region of attraction is always preferred for MPC design. Motivated by these facts, this chapter proposes a robust self-triggered min-max MPC approach to constrained nonlinear systems with both parameter uncertainties and disturbances, leading to an enlarged region of attraction in comparison with [6].

The main contributions of this chapter are two-fold:

- A self-triggered min-max MPC algorithm is designed for generic constrained nonlinear system with both parameter uncertainties and disturbances. The designed algorithm is proved to be recursively feasible and the closed-loop system is ISpS at triggering time instants in its region of attraction. Compared with existing self-triggered MPC strategies where nominal models are used for prediction, we take advantage of the worst case of all possible uncertainty realizations in the self-triggered control, ensuring robust constraint satisfaction in presence of parametric uncertainties and external disturbances.

- More specific results are developed for linear systems with parameter uncertainties and external disturbances. In particular, we show that for linear systems with additive disturbances, the approximate closed-loop prediction strategy [21, 36, 47, 54] can be adopted to facilitate the self-triggered min-max linear

MPC design to yield an enlarged attraction region, the feasibility and stability conditions reduce to an LMI, which can be solved easily.

The rest of the chapter is organized as follows. Section 2 introduces some preliminaries and formulates the control problem. The robust self-triggered feedback min-max MPC strategy is developed in Section 3. The feasibility and stability analyses are conducted in Section 4. The extension to linear case is further presented in Section 5. Simulations and comparison studies are provided in Section 6, and the conclusions are given in Section 7.

The notations adopted in this chapter are as follows. Let $\mathbb{R}$, and $\mathbb{N}$ denote by the sets of real and non-negative integers, respectively. $\mathbb{R}^n$ denotes the Cartesian product $\underbrace{\mathbb{R} \times \mathbb{R} \cdots \times \mathbb{R}}_{n}$. We use the notation $\mathbb{R}_{\geq c_1}$ and $\mathbb{R}_{(c_1,c_2]}$ to denote the sets $\{t \in \mathbb{R} | t \geq c_1\}$ and $\{t \in \mathbb{R} | c_1 < t \leq c_2\}$, respectively, for some $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}_{\geq c_1}$. The notation $\|\cdot\|$ is used to denote an arbitrary p-norm. Given a matrix S, $S \succ 0$ ($S \prec 0$) means that the matrix is positive (negative) definite. A scalar function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, positive definite and strictly increasing. It belongs to class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\alpha(s) \to +\infty$ as $s \to +\infty$. A scalar function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a $\mathcal{KL}$-function if for fixed $k \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is non-increasing with $\lim_{k \to \infty} \beta(s, k) = 0$. For $m, n \in \mathbb{N}_{>0}$, $I_{m \times m}$ denotes an identity matrix of size m and $0_{m \times n}$ represents an $m \times n$ matrix whose entries are zero.

3.2 Preliminaries and Problem Statement

3.2.1 Preliminaries

Consider the discrete-time perturbed nonlinear system given by

$$x_{t+1} = g(x_t, d_t), \tag{3.1}$$

where $x_t \in \mathbb{R}^n$, $d_t = [w_t^{\mathrm{T}}, v_t^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{D} \subset \mathbb{R}^d$ are the system state, unknown time-varying model uncertainties, respectively, at discrete time $t \in \mathbb{N}$. More specifically, $w_t \in \mathcal{W} \subset \mathbb{R}^w$ denotes parametric uncertainties and $v_t \in \mathcal{V} \subset \mathbb{R}^v$ stands for additive disturbances. $\mathcal{W}$ and $\mathcal{V}$ are compact sets, and contain the origin in their interiors. $g : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ is a nonlinear function satisfying $g(0, 0) = 0$.

Definition 1. *(RPI).* A set Ω is a robust positively invariant (RPI) set for the system (3.1) if $g(x_t, d_t) \in \Omega$, $\forall x_t \in \Omega, d_t \in \mathcal{D}$.

Definition 2. *(Regional ISpS). The system in* (3.1) *is said to be ISpS in $\mathcal{X}$ if there exist a $\mathcal{KL}$-function β, a $\mathcal{K}$-function γ and a number $\tau \geq 0$ such that, for all $x_0 \in \mathcal{X}$, all $\mathbf{w}_t = \left[w_0^{\mathrm{T}}, \cdots, w_{t-1}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{W}^t$, $\mathbf{v}_t = \left[v_0^{\mathrm{T}}, \cdots, v_{t-1}^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathcal{V}^t$, the state of* (3.1) *satisfies*

$$\|x_t\| \leq \beta(\|x_0\|, t) + \gamma(\|\mathbf{v}_{t-1}\|) + \tau, \forall t \in \mathbb{N}_{\geq 1}.$$

We recall a useful lemma from [36], which provides sufficient conditions for ISpS.

Lemma **5.** *Given an RPI set $\mathcal{X}$ with $\{0\} \subset \mathcal{X}$ for the system* (3.1)*, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function such that,*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + \tau_1, \tag{3.2a}$$

$$V(g(x, d)) - V(x) \leq -\alpha_3(\|x\|) + \sigma(\|v\|) + \tau_2, \tag{3.2b}$$

for all $x \in \mathcal{X}, d = [w^{\mathrm{T}}, v^{\mathrm{T}}] \in \mathcal{D}$, where $\alpha_1(s) \triangleq as^\lambda$, $\alpha_2(s) \triangleq bs^\lambda$ and $\alpha_3(s) \triangleq cs^\lambda$ with $a, b, c, \tau_1, \tau_2, \lambda \in \mathbb{R}_{>0}$ and $c \leq b$, and σ is a $\mathcal{K}$-function, then the system (3.1) *is ISpS in $\mathcal{X}$ with respect to v.*

Proof. By $V(x) \leq \alpha_2(\|x\|) + \tau_1$ for all $x \in \mathcal{X}$, one gets, for all $x \in \mathcal{X} \setminus \{0\}$,

$$V(x) - \alpha_3(\|x\|) \leq \frac{\alpha_3(\|x\|)}{\alpha_2(\|x\|)}(V(x) - \tau_1) = \rho V(x) + (1 - \rho)\tau_1$$

where $\rho \triangleq 1 - \frac{c}{b} \in \mathbb{R}_{[0,1)}$. It can be verified that if $x = 0$ the preceding inequality also holds since

$$V(0) - \alpha_3(0) = V(0) = \rho V(0) + (1-\rho)V(0) \leq \rho V(0) + (1-\rho)\tau_1.$$

Further, this inequality in conjunction with equation (3.2b) gives

$$V(g(x,d)) \leq \rho V(x) + \sigma(\|v\|) + (1-\rho)\tau_1 + \tau_2$$

for all $x \in \mathcal{X}, d \in \mathcal{D}$. By recursion, one obtains

$$V(x_{t+1}) \leq \rho^{t+1} V(x_0) + \sum_{i=0}^{t} \rho^i \big(\sigma(\|v_{t-i}\|) + (1-\rho)\tau_1 + \tau_2\big)$$

for all $x \in \mathcal{X}$ and any uncertainty realizations, i.e., $\mathbf{w}_t = \begin{bmatrix} w_0^{\mathrm{T}}, \cdots, w_t^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{W}^{t+1}$, $\mathbf{v}_t = \begin{bmatrix} v_0^{\mathrm{T}}, \cdots, v_t^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{V}^{t+1}$. Considering equation (3.2a), $\sigma(\|v_i\|) \leq \sigma(\|\mathbf{v}_t\|)$, and $\sum_{i=0}^{t} \rho^i = \frac{1-\rho^{t+1}}{1-\rho}$, we have

$$\begin{aligned}
V(x_{t+1}) &\leq \rho^{t+1}\alpha_2(\|x_0\|) + \rho^{t+1}\tau_1 + \sum_{i=0}^{t} \rho^i \big(\sigma(\|v_{t-i}\|) + (1-\rho)\tau_1 + \tau_2\big) \\
&\leq \rho^{t+1}\alpha_2(\|x_0\|) + \tau_1 + \frac{1-\rho^{t+1}}{1-\rho}\sigma(\|\mathbf{v}_t\|) + \frac{1-\rho^{t+1}}{1-\rho}\tau_2 \\
&\leq \rho^{t+1}\alpha_2(\|x_0\|) + \tau_1 + \frac{1}{1-\rho}\sigma(\|\mathbf{v}_t\|) + \frac{1}{1-\rho}\tau_2
\end{aligned}$$

for all $x_0 \in \mathcal{X}$, $\mathbf{w}_t = \begin{bmatrix} w_0^{\mathrm{T}}, \cdots, w_t^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{W}^{t+1}$, $\mathbf{v}_t = \begin{bmatrix} v_0^{\mathrm{T}}, \cdots, v_t^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathcal{V}^{t+1}$. Define

$\xi = \tau_1 + \frac{1}{1-\rho}\tau_2$ and α_1^{-1} as the inverse of α_1. We have

$$\begin{aligned} \|x_{t+1}\| &\leq \alpha_1^{-1}(V(x_{t+1})) \\ &\leq \alpha_1^{-1}\big(\rho^{t+1}\alpha_2(\|x_0\|) + \xi + \frac{\sigma(\|\mathbf{v}_t\|)}{1-\rho}\big), \end{aligned} \tag{3.3}$$

which in conjunction with

$$\alpha_1^{-1}(z + y + s) \leq \alpha_1^{-1}(3z) + \alpha_1^{-1}(3y) + \alpha_1^{-1}(3s)$$

gives

$$\|x_{t+1}\| \leq \alpha_1^{-1}\big(3\rho^{t+1}\alpha_2(\|x_0\|)\big) + \alpha_1^{-1}(3\xi) + \alpha_1^{-1}\big(3\frac{\sigma(\|\mathbf{v}_t\|)}{1-\rho}\big)$$

for all $x_0 \in \mathcal{X}$, $\mathbf{w}_t = \Big[w_0^{\mathrm{T}}, \cdots, w_t^{\mathrm{T}}\Big]^{\mathrm{T}} \in \mathcal{W}^{t+1}$, $\mathbf{v}_t = \Big[v_0^{\mathrm{T}}, \cdots, v_t^{\mathrm{T}}\Big]^{\mathrm{T}} \in \mathcal{V}^{t+1}$. Two cases are considered in order.

- $\rho \neq 0$. Define $\beta(s,t) = \alpha_1^{-1}(3\rho^t\alpha_2(s))$. Since $\rho \in \mathbb{R}_{(0,1)}$, $\beta(s,t)$ is a $\mathcal{KL}$-function. Let $\gamma(s) = \alpha_1^{-1}(\frac{3\sigma(s)}{1-\rho})$. We then have $\gamma(s) \in \mathcal{K}$ since $\frac{1}{1-\rho} > 0$, $\alpha_1^{-1} \in \mathcal{K}_\infty$ and $\sigma(s) \in \mathcal{K}$. $\xi \geq 0$ by definition and therefore $\alpha_1^{-1}(3\xi) \geq 0$.

- $\rho = 0$. From equation (3.3), one gets that

  $$\|x_t\| \leq \alpha_1^{-1}(3\xi) + \alpha_1^{-1}(3\sigma(\|\mathbf{v}_{t-1}\|)) \leq \beta(\|x_0\|, t) + \alpha_1^{-1}(3\xi) + \alpha_1^{-1}(3\sigma(\|\mathbf{v}_{t-1}\|))$$

  holds for any $\beta \in \mathcal{KL}$ and $\forall t \in \mathbb{N}_{\geq 1}$.

This completes the proof. $\square$

3.2.2 Problem Statement

Consider a discrete-time perturbed nonlinear system given by

$$x_{t+1} = f(x_t, u_t, d_t), \tag{3.4}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $d_t = [w_t^{\mathrm{T}}, v_t^{\mathrm{T}}] \in \mathcal{D} \subset \mathbb{R}^d$ are the system state, the control input, unknown, possibly time-varying model uncertainties, respectively, at discrete time $t \in \mathbb{N}$. More specifically, $w_t \in \mathcal{W} \subset \mathbb{R}^w$ represents parametric uncertainties and $v_t \in \mathcal{V} \subset \mathbb{R}^v$ stands for additive disturbances. $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^n$ is a nonlinear function satisfying $f(0, 0, 0) = 0$. It is assumed that the system is subject to state and input constraints given by $x_t \in \mathcal{X}$, $u_t \in \mathcal{U}$, where $\mathcal{X}$ and $\mathcal{U}$ are compact sets containing the origin in their interiors. Throughout the chapter, we assume that $\mathcal{W}$ and $\mathcal{V}$ are compact sets and contain the origin in their interiors. We further assume that the state is available as a measurement at any time instant.

The control objective of this chapter is to design a self-triggered MPC strategy to robustly asymptotically stabilize the system (3.4) while satisfying the system constraints. Let the sequence $\{t_k | k \in \mathbb{N}\} \in \mathbb{N}$ where $t_{k+1} > t_k$ be the time instants when optimization problem needs to be solved. In particular, the control law is of the form

$$u_t = \mu(x_{t_k}, t - t_k),\ t \in \mathbb{N}_{[t_k, t_{k+1}-1]}, \tag{3.5}$$

where $\mu : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ is a function, and $\{t_k | k \in \mathbb{N}\} \in \mathbb{N}$ are sampling instants that are determined by using a self-triggering scheduler, i.e.

$$t_0 = 0,\ t_{k+1} = t_k + H^*(x_{t_k}),\ k \in \mathbb{N}, \tag{3.6}$$

where $H^* : \mathbb{R}^n \to \mathbb{N}_{\geq 1}$ is a function.

3.3 Robust Self-triggered Feedback Min-max MPC

3.3.1 Min-max Optimization

For a given prediction horizon $N \in \mathbb{N}_{\geq 1}$ and $H \in \mathbb{N}_{[1,N]}$, the cost function at time $t_k \in \mathbb{N}$ is formulated as

$$J_N^H(x_{t_k}, \mathbf{u}_{t_k,N}, \mathbf{d}_{t_k,N}) \triangleq \sum_{j=0}^{H-1} \frac{1}{\beta} L(x_{j,t_k}, u_{j,t_k}) + \sum_{j=H}^{N-1} L(x_{j,t_k}, u_{j,t_k}) + F(x_{N,t_k}),$$

where $\beta \in \mathbb{R}_{\geq 1}$ is a fixed constant, x_{j,t_k} denotes the predicted state for system (3.4) at time $j \in \mathbb{N}_{[0,N-1]}$ initialized at $x_{0,t_k} = x_{t_k}$ with the control input sequence

$$\mathbf{u}_{t_k,N} = \left[u_{0,t_k}^{\mathrm{T}}, \cdots, u_{N-1,t_k}^{\mathrm{T}} \right]^{\mathrm{T}}$$

and the disturbance sequence

$$\mathbf{d}_{t_k,N} = \left[d_{0,t_k}^{\mathrm{T}}, \cdots, d_{N-1,t_k}^{\mathrm{T}} \right]^{\mathrm{T}}.$$

We assume that L and F are continuous functions. Specifically, the stage cost is given by $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ with $L(0,0) = 0$, and the terminal cost is given by $F : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ with $F(0) = 0$.

We make use of the min-max MPC strategy to achieve robust constraint satisfaction in this chapter. In particular, the control input is derived by solving the following min-max optimization problem.

$$
\begin{aligned}
V_N^H(x_{t_k}) = & \min_{u_{0,t_k}\in\mathcal{U},\cdots,u_{H-1,t_k}\in\mathcal{U}} \Big\{ \max_{d_{0,t_k}\in\mathcal{D},\cdots,d_{H-1,t_k}\in\mathcal{D}} \Big\{ \sum_{j=0}^{H-1} \frac{1}{\beta} L(x_{j,t_k}, u_{j,t_k}) + V_{N-H}(x_{H,t_k}) \Big\} \\
& \quad \text{such that } x_{H,t_k} \in \mathcal{X}_{N-H}, \forall d_{0,t_k} \in \mathcal{D}, \cdots, d_{H-1,t_k} \in \mathcal{D} \Big\}, \\
& \text{s.t.} \quad x_{0,t_k} = x_{t_k}, \quad x_{j,t_k} \in \mathcal{X},\ j \in \mathbb{N}_{[0,H-1]}, \\
& x_{j+1,t_k} = f(x_{j,t_k}, u_{j,t_k}, d_{j,t_k}),\ j \in \mathbb{N}_{[0,H-1]},
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
V_i(x_{i,t_k}) = \min_{u_{i,t_k}\in\mathcal{U}} \Big\{ \max_{d_{i,t_k}\in\mathcal{D}} \Big\{ L(x_{i,t_k}, u_{i,t_k}) + V_{i-1}(f(x_{i,t_k}, u_{i,t_k}, d_{i,t_k})) \Big\} \\
\text{such that } f(x_{i,t_k}, u_{i,t_k}, d_{i,t_k}) \in \mathcal{X}_{i-1}, \forall d_{i,t_k} \in \mathcal{D} \Big\},
\end{aligned} \tag{3.8}
$$

where $i \in \mathbb{N}_{[1,N-H]}$ and $\mathcal{X}_i \subseteq \mathcal{X}$ denotes the set of states that can be robustly controlled into the terminal set $\mathcal{X}_f$ in i steps by using feedback laws. The optimization problem is defined for $i = 1, \cdots, N$ with the boundary conditions

$$
\begin{aligned}
V_0(x) &\triangleq F(x), \\
\mathcal{X}_0 &\triangleq \mathcal{X}_f.
\end{aligned}
$$

The optimal solution of optimization problem (3.7) is denoted as

$$
\mathbf{u}^*_{t_k,N} = [u^{*\mathrm{T}}_{0,t_k}, \cdots, u^{*\mathrm{T}}_{N-1,t_k}]^{\mathrm{T}},
$$

and the optimal predicted model uncertainty is written as

$$
\mathbf{d}^*_{t_k,N} = [d^{*\mathrm{T}}_{0,t_k}, \cdots, d^{*\mathrm{T}}_{N-1,t_k}]^{\mathrm{T}}.
$$

In the sequel, we particularly denote, for the optimization problem in (3.7) with $\beta = 1$ and $H = 1$, the cost function by $J_N(x_{t_k}, \mathbf{u}_{t_k,N}, \mathbf{d}_{t_k,N})$, the corresponding optimal cost by $V_N(x_{t_k})$, and the initial feasible region by $\mathcal{X}_N$.

Remark 4. *It is worth noting that, we formulate a new cost function* $J_N^H(.)$ *in min-max optimization in order to design a self-triggered strategy. The solution of optimization problem in* (3.7) *is a combination of a sequence of control values* u^*_{j,t_k}, $j \in \mathbb{N}_{[0,H-1]}$ *(generated by open-loop min-max strategy) and a sequence of control policies* u^*_{j,t_k}, $j \in \mathbb{N}_{[H,N-1]}$ *(generated by feedback min-max strategy). This configuration is necessarily formulated to facilitate the self-triggered design as the state information is not available to construct feedback laws during triggering time instants in self-triggered control; it will reduce to the conventional one in standard feedback min-max MPC by letting* $H = 1$ *and* $\beta = 1$*, and recovers the standard open-loop min-max MPC framework [36, 47, 54] by setting* $H = N$ *and* $\beta = 1$*. Also note that the proposed optimization problem can conveniently incorporate the sparsity of control inputs,* $u_{j,t_k} = 0, j \in \mathbb{N}_{[1,H-1]}$ *or* $u_{j,t_k} = u_{0,t_k}, j \in \mathbb{N}_{[1,H-1]}$ *as in [3, 5, 6, 20], if necessary.*

3.3.2 Self-triggering in Optimization

At some sampling time instant $t \in \mathbb{N}$, the control input is defined as

$$u_t^{ST}(x_{t_k}) \triangleq u^*_{t-t_k,t_k}, \; t \in \mathbb{N}_{[t_k,t_{k+1}-1]}, \tag{3.9}$$

where $u^*_{t-t_k,t_k}$, $t \in \mathbb{N}_{[t_k,t_{k+1}-1]}$ represents the optimal solution of optimization problem (3.7). It can be observed that the control input u_t^{ST} is open-loop for $t \in \mathbb{N}_{[t_k+1,t_{k+1}-1]}$ since it only depends on the state at the last sampling time instant t_k.

In the standard scheme of self-triggered MPC, both the control input and the next triggering time need to be decided at each sampling time instant. In general, the triggering time instants are derived by checking whether or not the optimal cost is deceasing. In this chapter, the triggering time instants are determined as follows:

$$\begin{aligned} &t_{k+1} = t_k + H^*(x_{t_k}), \\ &H^*(x_{t_k}) \triangleq \max\{H \in \mathbb{N}_{[1,H_{\max}]} | V_N^H(x_{t_k}) \leq V_N^1(x_{t_k})\}, \end{aligned} \tag{3.10}$$

where $H_{\max} \in \mathbb{N}_{[1,N]}$ denotes the maximal length of the open-loop phase.

The self-triggered min-max MPC strategy is formulated in Algorithm 1.

Remark 5. *It is worth noting that, the triggering condition in [3, 6] leads to a separate design of feedback control and triggering time instant, but the triggering condition in (3.10) with the min-max framework provides a co-design of the feedback control and triggering time instant, and the model uncertainty is explicitly considered in the co-design. Specifically, the co-design is realized by the self-triggering scheduler that involves comparing min-max costs with different open-loop spans. As a result, for linear systems the proposed strategy will provide a larger attraction region and achieve a better trade-off between average sampling time and the control performance, and it involves min-max optimization over all possible uncertainty realizations that is computationally more expensive than general min-max optimization used in [6].*

Algorithm 1 Self-triggered min-max MPC algorithm

Require: Prediction horizon N; design parameters β and $H_{\max}$.

```
Set t = t_k = k = 0;
while The control action is not stopped do
    Measure the current state x_{t_k} of system (3.4);
    Solve the optimization problems in (3.7) and (3.10), obtain u*(x_{t_k}) and H*(x_{t_k});
    while t ≤ t_k + H*(x_t) − 1 do
        Apply u*_{t−t_k,t_k} to the system;
        Set t = t + 1;
    end while
    Set k = k + 1, t_k = t;
end while
```

3.4 Feasibility and Stability Analysis

By applying Algorithm 1 to system (3.4), the closed-loop system becomes

$$x_{t+1} = f(x_t, u_t^{ST}, d_t), \tag{3.11a}$$

$$u_t^{ST} = u^*_{t-t_k,t_k},\ t \in \mathbb{N}_{[t_k,t_{k+1}-1]}, \tag{3.11b}$$

$$t_{k+1} = t_k + H^*(x_{t_k}). \tag{3.11c}$$

To approach the feasibility and stability problem for the closed-loop system (3.11), we first make the following assumptions.

Assumption **2.** There exist a function $\kappa_f : \mathbb{R}^n \to \mathbb{R}^m$ with $\kappa_f(0) = 0$, a $\mathcal{K}$-function σ, and $\alpha_l, \alpha_f, \alpha_F, \lambda \in \mathbb{R}_{>0}$ with $\alpha_l \leq \alpha_F$ such that:

1) $\mathcal{X}_f \subseteq \mathcal{X}$ and $0 \in \text{int}(\mathcal{X}_f)$;

2) $\mathcal{X}_f$ is an RPI set for system (3.4) in closed-loop with $u = \kappa_f(x)$;

3) $L(x, u) \geq \alpha_l \|x\|^\lambda$ for all $x \in \mathcal{X}$ and $u \in \mathcal{U}$;

4) $\alpha_f \|x\|^\lambda \leq F(x) \leq \alpha_F \|x\|^\lambda$ for all $x \in \mathcal{X}_f$;

5) $F(f(x, \kappa_f(x), d)) - F(x) \leq -L(x, \kappa_f(x)) + \overline{\sigma}(\|v\|)$ for all $x \in \mathcal{X}_f$ and $d \in \mathcal{D}$.

In Assumption 2, the terminal cost $F(x)$ serves as a local ISS Lyapunov function for the closed-loop system $x_{t+1} = f(x_t, \kappa_f(x_t), d_t)$. In the literature regarding robust MPC, some methods for deriving ISS Lyapunov functions satisfying Assumption 2 have been proposed in [35] for linear systems, and [54] for nonlinear systems.

Before proceeding to the main result, we first present two useful lemmas.

Lemma **6.** *For all $x_0 \in \mathcal{X}_f$ and any realization of the disturbances $d_t \in \mathcal{D}$ with $t \in \mathbb{N}$, if Assumption 2 holds for system* (3.4)*, then*

$$F(x_m) - F(x_0) \leq -\sum_{t=0}^{m-1} (L(x_t, \kappa_f(x_t)) - \overline{\sigma}(\|v_t\|)), \tag{3.12}$$

where x_m is derived by applying the local stabilizing law κ_f to system (3.4)*, and $m \in \mathbb{N}_{[1,N]}$.*

Proof. According to Assumption 2, there exists a feedback law κ_f for system (3.4) such that

$$F(x_{t+1}) - F(x_t) \leq -L(x_t, \kappa_f(x_t)) + \overline{\sigma}(\|v_t\|), \tag{3.13}$$

for all $x_t \in \mathcal{X}_f$. Since $\mathcal{X}_f$ is an RPI set for system (3.4) in closed-loop with κ_f, by summing (3.13) from $t = 0$ to $t = m - 1$, we obtain the inequality (3.12). $\square$

Lemma **7.** *For the optimization problem defined in* (3.7)*,*

$$V_N^1(x_{t_k}) \leq V_N(x_{t_k}). \tag{3.14}$$

Proof. Without loss of generality, assume the solutions corresponding to $V_N(x_{t_k})$ are

$\mathbf{u}^*_{t_k,N} = [u^{*\mathrm{T}}_{0,t_k}, \cdots, u^{*\mathrm{T}}_{N-1,t_k}]^{\mathrm{T}}$, $\mathbf{d}^*_{t_k,N} = [d^{*\mathrm{T}}_{0,t_k}, \cdots, d^{*\mathrm{T}}_{N-1,t_k}]^{\mathrm{T}}$. Due to optimality, we have

$$\begin{aligned}
&V^1_N(x_{t_k})\\
\leq&\max_{\mathbf{d}_{t_k,N}} J^1_N(x_{t_k}, \mathbf{u}^*_{t_k,N}, \mathbf{d}_{t_k,N})\\
\leq&\max_{\mathbf{d}_{t_k,N}} J_N(x_{t_k}, \mathbf{u}^*_{t_k,N}, \mathbf{d}_{t_k,N}) + \frac{1-\beta}{\beta} L(x_{0,t_k}, u^*_{0,t_k})\\
=&V_N(x_{t_k}) + \frac{1-\beta}{\beta} L(x_{0,t_k}, u^*_{0,t_k}).
\end{aligned} \tag{3.15}$$

Since $L(x_{0,t_k}, u^*_{0,t_k}) \geq 0$ and $\beta \in \mathbb{R}_{\geq 1}$, we can obtain the inequality in (3.14). □

The main results on the algorithm feasibility and closed-loop stability are summarized in the following theorem.

Theorem **2.** *For the perturbed nonlinear system* (3.4) *with* $x_0 \in \mathcal{X}_N$*, suppose that Assumption 2 holds, then Algorithm 1 is recursively feasible, system* (3.4) *in closed-loop with the self-triggered feedback min-max MPC control* (3.9) *and* (3.10) *is ISpS with respect to* v *in* $\mathcal{X}_N$ *at triggering time instants.*

Proof. We sketch the proof in two steps. First, we show that $\mathcal{X}_N$ is an RPI set for closed-loop system (3.11) to prove the recursive feasibility of the optimization problem (3.7). Second, we prove that the min-max MPC optimal cost function $V(\cdot)$ is an ISpS Lyapunov function for the closed-loop system at triggering time instants.

Without loss of generality, we assume that $x_t = x_{t_k} \in \mathcal{X}_N$ and the calculated span of open-loop phase is $H^*(x_{t_k})$ at time t_k. Due to Assumption 2-2), a vector of feedback control polices can be constructed as a feasible solution for the optimization problem (3.7) at time t_{k+1} as follows

$$(u^*_{H^*(x_{t_k}),t_k}, \cdots, u^*_{N-1,t_k}, \kappa_f(x_{N,t_k}), \cdots, \kappa_f(x_{N+H^*(x_{t_k})-1,t_k})), \tag{3.16}$$

implying that $\mathcal{X}_N$ is an RPI set for system (3.4) in closed-loop with the proposed self-triggered min-max MPC law. Note that each element of the vector in (3.16) is a feedback law, i.e., its value depends on the actual disturbance realization.

Then we will derive lower and upper bounds on the min-max MPC optimal cost function at triggering time instants. From the definition of the optimization problem (3.7), for all $x_{t_k} \in \mathcal{X}_N$ we have

$$\begin{aligned} V_N^{H^*(x_{t_k})}(x_{t_k}) &= J_N^{H^*(x_{t_k})}(x_{t_k}, \mathbf{u}^*_{t_k,N}, \mathbf{d}^*_{t_k,N}) \\ &\geq \min_{\mathbf{u}_{t_k,N}} J_N^{H^*(x_{t_k})}(x_{t_k}, \mathbf{u}_{t_k,N}, \mathbf{0}) \\ &\geq \frac{\alpha_l}{\beta}\|x_{t_k}\|^{\lambda}. \end{aligned} \tag{3.17}$$

For all $x_{t_k} \in \mathcal{X}_N$, we consider

$$\begin{aligned} &J^1_{N+1}(x_{t_k}, \tilde{\mathbf{u}}_{t_k,N+1}, \mathbf{d}_{t_k,N+1}) \\ =&\big(- F(x_{N,t_k}) + F(x_{N+1,t_k}) + L(x_{N,t_k}, \kappa_f(x_{N,t_k}))\big) \\ &+ J^1_N(x_{t_k}, \mathbf{u}^*_{t_k,N}, \mathbf{d}_{t_k,N}), \end{aligned} \tag{3.18}$$

where

$$\tilde{\mathbf{u}}_{t_k,N+1} = [\mathbf{u}^{*\mathrm{T}}_{t_k,N}, \kappa_f(x_{N,t_k})^{\mathrm{T}}]^{\mathrm{T}}. \tag{3.19}$$

By application of point 5 of Assumption 2 and sub-optimality of the control input sequence $\tilde{\mathbf{u}}_{t_k,N+H^*(x_{t_k})}$, it follows, for all $x_{t_k} \in \mathcal{X}_N$,

$$\begin{aligned} V^1_{N+1}(x_{t_k}) &\leq \max_{\mathbf{d}_{t_k,N+1}} \big(J^1_{N+1}(x_{t_k}, \tilde{\mathbf{u}}_{t_k,N+1}, \mathbf{d}_{t_k,N+1})\big) \\ &\leq V^1_N(x_{t_k}) + \max_v \overline{\sigma}(\|v\|). \end{aligned} \tag{3.20}$$

Analogously, we have

$$\begin{aligned} & V_N^1(x_{t_k}) \leq V_1^1(x_{t_k}) + (N-1)\max_v \overline{\sigma}(\|v\|) \\ & \leq F(x_{t_k}) + N\max_v \overline{\sigma}(\|v\|) + \frac{1-\beta}{\beta} L(x_{0,t_k}, \kappa_f(x_{0,t_k})) \\ & \leq \alpha_F \|x_{t_k}\|^\lambda + N\max_v \overline{\sigma}(\|v\|) \end{aligned} \tag{3.21}$$

for all $x_{t_k} \in \mathcal{X}_f$. Recalling the triggering mechanism in (3.10), it follows

$$V_N^H(x_{t_k}) \leq \alpha_F \|x_{t_k}\|^\lambda + N\max_v \overline{\sigma}(\|v\|) \tag{3.22}$$

for all $x_{t_k} \in \mathcal{X}_f$. For $x_{t_k} \in \mathcal{X}_N \backslash \mathcal{X}_f$, one can establish the upper bound of $V_N^H(x_{t_k})$ by following the idea in [41] (Lemma 1) as follows. Define a set

$$\mathcal{B}_r = \{x \in \mathbb{R}^n | \|x\| \leq r\} \subseteq \mathcal{X}_f,$$

where $r > 0$. Following the compactness of $\mathcal{X}$, $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{V}$, there always exists a finite $\overline{J}_N > 0$ such that $V_N^H(x_{t_k}) \leq \overline{J}_N$ for all $x_{t_k} \in \mathcal{X}_N$. Define $\theta = \max(\alpha_F, \frac{\overline{J}_N}{r^\lambda})$. It follows

$$V_N^H(x_{t_k}) \leq \theta \|x_{t_k}\|^\lambda + N\max_v \overline{\sigma}(\|v\|)$$

for all $x_{t_k} \in \mathcal{X}_N$.

Next, we will show that $V_N^{H^*(x_{t_k})}(x_{t_k})$ satisfies condition (3.2b). From the triggering mechanism in (3.10), we have

$$\begin{aligned}
&V_N^{H^*(x_{t_{k+1}})}(x_{t_{k+1}}) - V_N^{H^*(x_{t_k})}(x_{t_k}) \\
\leq & V_N^1(x_{t_{k+1}}) - V_N^{H^*(x_{t_k})}(x_{t_k}) \\
\leq & V_N^1(x_{t_{k+1}}) - \max_{d_{0,t_k}\in\mathcal{D},\cdots,d_{H-1,t_k}\in\mathcal{D}} \Big\{ \sum_{j=0}^{H^*(x_{t_k})-1} \frac{1}{\beta} L(x_{j,t_k}, u^*_{j,t_k}) + V_{N-H^*(x_{t_k})}(x_{H,t_k}) \Big\} \\
\leq & V_N^1(x_{t_{k+1}}) - V_{N-H^*(x_{t_k})}(x_{t_{k+1}}) - \sum_{j=0}^{H^*(x_{t_k})-1} \frac{1}{\beta} L(x_{t_k+j}, u^*_{j,t_k}), \forall x_{t_k} \in \mathcal{X}_N.
\end{aligned} \tag{3.23}$$

By using Lemma 6 and an analogous reasoning as in (3.18) – (3.21), one can get

$$V_N(x_{t_{k+1}}) - V_{N-H^*(x_{t_k})}(x_{t_{k+1}}) \leq H^*(x_{t_k}) \max_v \overline{\sigma}(\|v\|), \tag{3.24}$$

for $x_{t_{k+1}} \in \mathcal{X}_{N-H^*(x_{t_k})}$. Considering Lemma 7 and plugging (3.24) into (3.23), we have

$$\begin{aligned}
&V_N^{H^*(x_{t_{k+1}})}(x_{t_{k+1}}) - V_N^{H^*(x_{t_k})}(x_{t_k}) \\
\leq & -\sum_{j=0}^{H^*(x_{t_k})-1} \frac{1}{\beta} L(x_{t_k+j}, u^*_{j,t_k}) + H^*(x_{t_k}) \max_v \overline{\sigma}(\|v\|) \\
\leq & -\frac{1}{\beta}\alpha_l \|x_{t_k}\|^\lambda + H^*(x_{t_k}) \max_v \overline{\sigma}(\|v\|),
\end{aligned} \tag{3.25}$$

for all $x_{t_k} \in \mathcal{X}_N$.

By now, we have shown that $V_N^{H^*(x_{t_k})}(x_{t_k})$ is an ISpS Lyapunov function at triggering time instants. With the aid of Lemma 5, we can conclude that the closed-loop system (3.11) is ISpS in $\mathcal{X}_N$ with respect to v at triggering time instants. □

Remark 6. *Note that Theorem 1 investigates the stability of the closed-loop system at triggering time instants. For system states at time instants in between, one can* *ensure $x_t \in \mathcal{X}$. However, if the states in between are expected in a smaller set, one could tighten the state constraints in* (3.7) *to achieve the goal, or if the asymptotic stability of the closed-loop system is desired, one could utilize the dual-mode strategy to satisfy the requirement.*

Remark 7. *Theorem 2 indicates that the closed-loop system* (3.11) *is ISpS in $\mathcal{X}_N$. From the derivations, we can see that there is a trade-off between the frequency of optimization and the size of the convergence set with respect to the control parameter β. That is, a larger β lowers the average sampling rate to alleviate the computational load, however it enlarges the size of the convergence set (This argument will be elaborated by means of numerical simulations in the sequel).*

3.5 The Case of Linear Systems with Additive Disturbances

In this section, we develop more explicit results for linear cases.

Consider the following uncertain linear system

$$x_{t+1} = A(w_t)x_t + B(w_t)u_t + v_t, \tag{3.26}$$

where the pair $(A(w_t), B(w_t))$ is assumed controllable for all $w_t \in \mathcal{W}$.

In this case, the feedback control law can adopt the following linear structure for prediction [44]:

$$\mathbf{u}_{t_k,N} \triangleq \mathbf{c}_{t_k,N} + \mathbf{M}_N^H \mathbf{v}_{t_k,N}, \tag{3.27}$$

where $\mathbf{c}_{t_k,N} = [c_{0,t_k}, \cdots, c_{N-1,t_k}]^{\mathrm{T}}$ with $c_{\cdot,t_k} \in \mathbb{R}^m$, $\mathbf{v}_{t_k,N}$ denotes disturbance sequence, and

$$\mathbf{M}_N^H = \begin{bmatrix} 0_{Hm\times n} & 0_{Hm\times n} & \cdots & 0_{Hm\times n} & 0_{Hm\times n} \\ M_{H,0} & \cdots & M_{H,H-1} & 0_{m\times n} & 0_{m\times n} \\ \vdots & \ddots & \ddots & \vdots & 0_{m\times n} \\ M_{N-1,0} & \cdots & \cdots & M_{N-1,N-2} & 0_{m\times n} \end{bmatrix}$$

with $M \in \mathbb{R}^{m\times n}$.

Note that the disturbance parameterization min-max MPC introduces conservatism, as the inputs to be optimized are not completely free.

In what follows, we consider a particular case, namely, the system matrices A and B are static and known, which is also the system studied in [6]. For this particular case, we develop an optimization-based method to calculate the control parameters satisfying Assumption 2. The results are summarized in the following corollary.

Corollary 1. *For the perturbed linear system* (3.26) *with fixed* w_t *and* $x_0 \in \mathcal{X}_N$*, consider the stage cost* $L(x,u) = x^{\mathrm{T}}C^{\mathrm{T}}Cx + u^{\mathrm{T}}D^{\mathrm{T}}Du$ *with* $C^{\mathrm{T}}C \succ 0$*,* $D^{\mathrm{T}}D \succ 0$*,* $\overline{\sigma}(\|v\|) = \gamma v^{\mathrm{T}}v$ *with* $\gamma \in \mathbb{R}_{>0}$*,* $\kappa_f(x) = Kx$ *with* K *being a matrix, and the terminal cost* $F(x) = x^{\mathrm{T}}Px$ *with* $P \succ 0$*. If matrices* Q*,* R*,* P *and* K *are designed by solving the following optimization problem*

$$\begin{aligned} &\min \gamma \\ &\text{s.t.} \\ &\begin{bmatrix} P & 0_{n\times n} & (P(A+BK))^{\mathrm{T}} & C^{\mathrm{T}} & K^{\mathrm{T}}D^{\mathrm{T}} \\ 0_{n\times n} & \gamma I_{n\times n} & P & 0_{n\times n} & 0_{n\times m} \\ P(A+BK) & P & P & 0_{n\times n} & 0_{n\times m} \\ C & 0_{n\times n} & 0_{n\times n} & I_{n\times n} & 0_{n\times m} \\ DK & 0_{m\times n} & 0_{m\times n} & 0_{m\times n} & I_{m\times m} \end{bmatrix} \succ 0, \end{aligned} \tag{3.28}$$

then Algorithm 1 is recursively feasible, and the system (3.26) *in closed-loop with the self-triggered min-max MPC control* (3.9) *and* (3.10) *is ISpS with respect to* v *in* $\mathcal{X}_N$.

Proof. Assumption 2-3) and 2-4) hold since the quadratic cost is used. By pre- and post-multiplying (3.28) by $\text{diag}\{I, I, P^{-1}, I, I\}$ and using the Schur complement lead to

$$\begin{bmatrix} P & 0_{n\times n} \\ 0_{n\times n} & \gamma I_{n\times n} \end{bmatrix} - \begin{bmatrix} (A+BK)^{\mathrm{T}} & C^{\mathrm{T}} & K^{\mathrm{T}}D^{\mathrm{T}} \\ I_{n\times n} & 0_{n\times n} & 0_{n\times m} \end{bmatrix} \times \begin{bmatrix} P & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_{n\times n} \end{bmatrix} \begin{bmatrix} (A+BK) & I_{n\times n} \\ C & 0_{n\times n} \\ DK & 0_{m\times n} \end{bmatrix} \succ 0. \tag{3.29}$$

It follows

$$((A+BK)x+v)^{\mathrm{T}}P((A+BK)x+v) < x^{\mathrm{T}}Px - x^{\mathrm{T}}Qx - x^{\mathrm{T}}K^{\mathrm{T}}RKx + \gamma v^{\mathrm{T}}v,$$

implying the satisfaction of Assumption 2-5). $A + BK$ being stable ensures the existence of set $\mathcal{X}_f$. Therefore, Assumption 2-1) and Assumption 2-2) hold true. Furthermore, the corresponding RPI set $\mathcal{X}_f$ can be calculated as [55]. The recursive feasibility of Algorithm 1, stability of the closed-loop system can be analogously analyzed as that in Theorem 2. □

Remark 8. *The results in Corollary 1 can be directly extended to a more general case, i.e.,* $[A(w_t), B(w_t)] = \sum_{s=1}^{S} \chi_s [A(w^s), B(w^s)], \sum_{s=1}^{S} \chi_s = 1$ *with* $s \in \mathbb{N}$, χ^s *being nonnegative reals and* w^s *being the vertices of* $\mathcal{W}$, *by following the similar lines of [35] (pp: 151-158) provided that* $\mathcal{W}$ *is a polyhedral set.*

Remark 9. *In comparison with conventional min-max MPC, Algorithm 1 might need less computational load. This is because, though the additional optimization problems* *(at most $H_{\max}$ quadratic programs) needs to be solved at each triggering time instant, the optimization frequency is greatly reduced due to the triggering strategy. Also note that for linear case with quadratic cost, the min-max optimization problem* (3.7) *can be solved as the conventional min-max MPC in [9, 17, 21, 44].*

3.6 Simulation and Comparisons

3.6.1 Comparison of $\mathcal{X}_N$ with [6]

Consider an unstable system

$$x_{t+1} = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.2 \end{bmatrix} x_t + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u_t + v_t, \tag{3.30}$$

where for all $t \in \mathbb{N}$, $v_t \in \mathcal{V}$ with $\mathcal{V} = [-0.2, 0.2] \times [-0.2, 0.2]$. The state and input constraints are given by $\mathcal{X} = [-10, 10] \times [-10, 10]$ and $\mathcal{U} = [-2, 2]$. We set the prediction horizon as $N = 5$. The stage cost is designed as $L(x, u, v) = x^{\mathrm{T}} C^{\mathrm{T}} C x + u^{\mathrm{T}} D^{\mathrm{T}} D u$ where $C = \mathrm{diag}(1, 1)$ and $D = 1$, respectively, and the terminal cost is set as $F(x) = x^{\mathrm{T}} P x$. Set $K = \begin{bmatrix} -0.8286 & -1.5013 \end{bmatrix}$. Note that the constant γ and the matrix P should be designed to satisfy Assumption 2 . We take advantage of the optimization toolbox YALMIP [45] to solve the optimization problem (3.28), and get $\gamma = 6.4547$, $P = \begin{bmatrix} 4.4888 & 2.2715 \\ 2.2715 & 3.6604 \end{bmatrix}$. Finally, the RPI set for system (3.30) in closed-loop with $u = Kx$ is derived by using Multi-Parametric Toolbox 3.0 [30].

To illustrate the advantages of the proposed self-triggered min-max MPC with disturbance parameterization strategy over the event-triggering strategy with tube MPC [6], we plotted their regions of attraction in Fig. 3.1, where the darkest shade denotes the RPI set of the system (3.30) in closed-loop with $u = Kx$, the lighter one stands for the region of attraction of the strategy in [6], and the lightest one represents that of the proposed strategy.

It can be seen that the proposed control strategy gives a larger region of attraction. This is primarily because that the feedback gains are jointly optimized with the next sampling time depending on the current state, whereas the feedback gain in [6] was fixed. Considering that the inputs are not completely free but parameterized as in (3.27), only a suboptimal solution of the min-max problem in (3.7) is obtained.

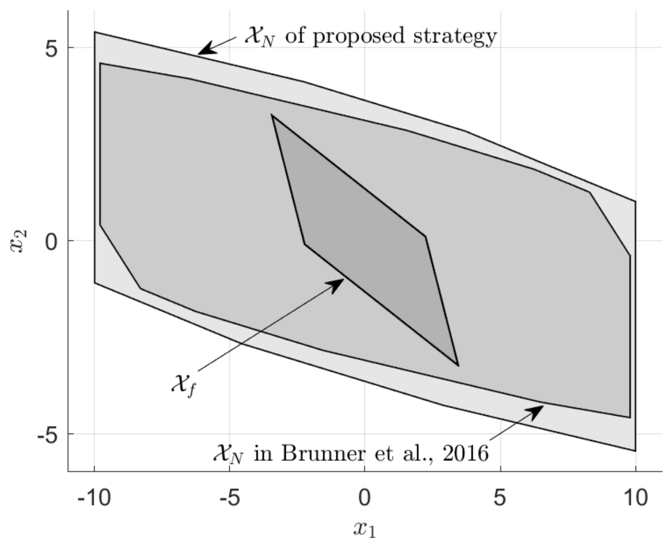

Figure 3.1: Comparison of regions of attraction.

3.6.2 Comparison with Periodic Min-max MPC

Consider the discrete-time nonlinear system [54] as follows

$$\begin{aligned} x_{t+1}(1) =& x_t(1) + T x_t(2) \\ x_{t+1}(2) =& -\frac{lT}{m} e^{-x_t(1)} x_t(1) + \frac{m - hT}{m} x_t(2) + \frac{T}{m} u_t - \frac{T}{m} w_t x_t(2) + \frac{T}{m} v_t, \end{aligned} \tag{3.31}$$

where the system parameters are given by: $m = 1$ kg; $l = 0.33$ N/m; $h = 1.1$ Ns/m; $T = 0.4$ s. The model uncertainties are limited by

$$-0.1 \leq w_t \leq 0.1$$
$$-0.2 \leq v_t \leq 0.4.$$

The system constraints are set as

$$-4.5 \text{ N} \leq u_t \leq 4.5 \text{ N}$$
$$-2 \text{ m} \leq x_t(1) \leq 2 \text{ m}.$$

The prediction horizon is chosen as $N = 5$. Set $H_{\max} = 4$. The cost function is set as

$$L(x, u) = x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u$$

with

$$Q = \begin{bmatrix} 0.64 & 0 \\ 0 & 0.64 \end{bmatrix}, R = 1.$$

By following the method for deriving min-max MPC parameters developed in [54], the local stabilizing law and terminal stage cost are derived as

$$\kappa_f(x) = \begin{bmatrix} -0.7797 - 1.1029 \end{bmatrix} x,$$

and

$$F(x) = x^{\mathrm{T}} P x$$

with $P = \begin{bmatrix} 4.5678 & 3.2018 \\ 3.2018 & 4.3500 \end{bmatrix}$, respectively. The terminal region is numerically chosen as

$$\mathcal{X}_f = \{x : x^{\mathrm{T}} P x \leq 3.8\}.$$

The policies

$$u(x) = a\kappa_f(x) + b(x_1^2 + x_2^2) + c,$$

where $a, b, c \in \mathbb{R}$, are used for prediction from the prediction horizon $N - H$ to N. The initial state is given by $x_0 = [0.5, 0.4]$.

The simulation is conducted by following the self-triggered min-max MPC Algorithm 1, where the MATLAB subroutine `fminimax` is employed to solve constrained min-max optimization problems. We consider two configurations in the simulation, that is, $\beta = 1.2$ and $\beta = 3$. Besides, the periodic min-max robust MPC is also executed in the simulation with the same system parameters. In the simulation, the chosen trajectories of uncertainties are plotted in Fig. 3.5. The results are reported as follows. Fig. 3.2-3.3 show the evolutions of system states, and Fig. 3.4 depicts the control input. To further illustrate the difference of control performance, the performance indices

$$J_{\mathrm{p}} = \frac{\sum_{t=0}^{T_{\mathrm{sim}}-1} x_t^{\mathrm{T}} Q x_t + u_t^{\mathrm{T}} R u_t}{T_{\mathrm{sim}}}$$

and the average sampling instants are presented in Table 3.6.2, where T_{sim} is the simulation time. It can be observed from Table 3.6.2 that the self-triggered min-max MPC strategy reduces the computation load while achieves comparable control performance as the periodic one. It can also be seen that the proposed self-triggered strategy is feasible and the closed-loop system is stable, and with a larger β, the controller has a lower optimization frequency but also a larger convergence set.

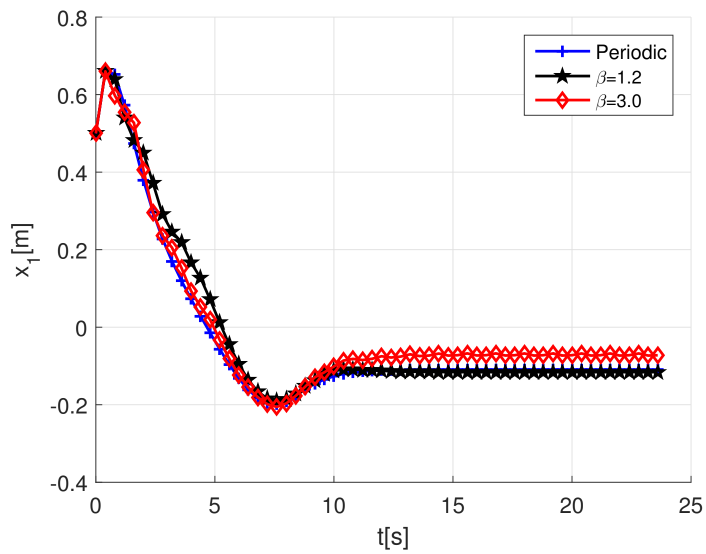

Figure 3.2: Trajectories of system state x_1.

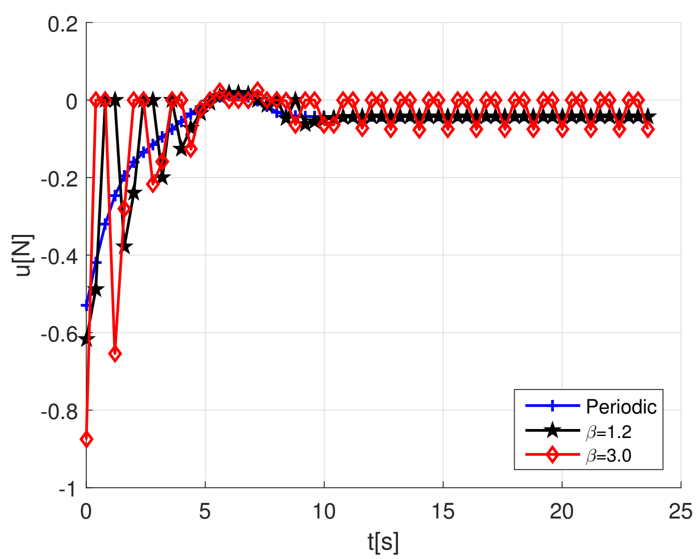

Figure 3.4: Trajectories of control input u.

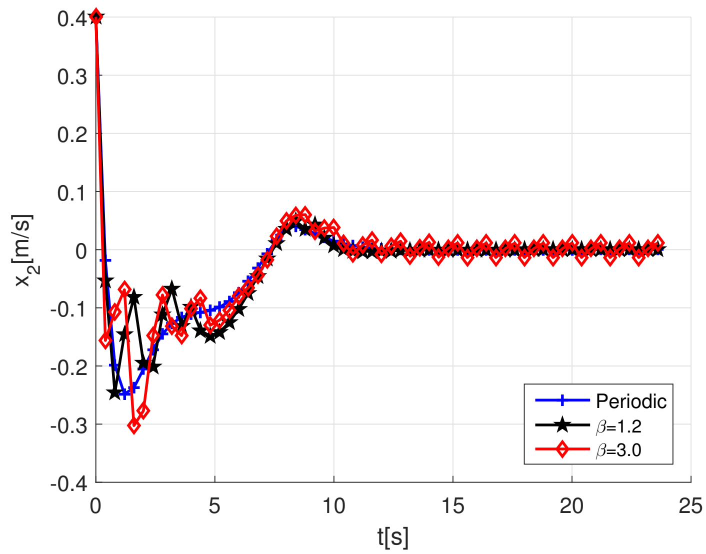

Figure 3.3: Trajectories of system state x_2.

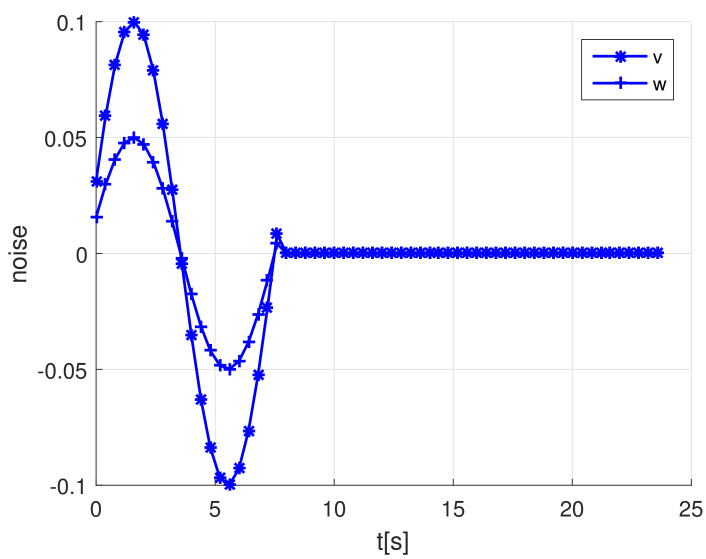

Figure 3.5: Trajectories of disturbances.

	Average sampling time	J_{p}
Periodic	1.0000	0.0477
$\beta = 1.2$	1.2000	0.0519
$\beta = 3.0$	3.3333	0.0560

Table 3.1: Performance comparison

3.7 Conclusion

We have studied the robust self-triggered min-max MPC problem for constrained uncertain discrete-time nonlinear systems. A self-triggered control scheduler has been proposed to maximize the inter-sampling time of feedback min-max MPC, and the algorithm feasibility and closed-loop ISpS at triggering time instants have been proved. Numerical simulations and comparison studies have verified the effectiveness and advantages of the proposed results.

Chapter 4

Conclusions and Future Work

4.1 Conclusions

In this thesis, the co-design problem of robust MPC and scheduling for networked CPSs has been investigated.

The co-design problem of event trigger and robust tube-based MPC for constrained linear systems with additive disturbances has been studied in **Chapter 2**. Based on multiple stabilizing feedback gains, the interpolation technique is used to construct a feedback policy where the interpolating coefficient is determined via optimization. According to the dynamic interpolation, the original constraints are properly tightened to achieve robust constraint satisfaction and a sequence of threshold sets that characterize the maximum endurable deviation between the predicted and actual system states are generated, leading to a co-design between the event trigger and the tube-based MPC. The recursive feasibility and closed-loop stability have been rigorously studied. Numerical results have been provided to verify the design.

A robust self-triggered min-max MPC strategy for constrained nonlinear systems with both additive disturbances and parametric uncertainties has been proposed in

Chapter 3, where a new cost function that relaxes the stage cost penalty for a prediction horizon period during which the controller will not be invoked is designed. The triggering instant is then determined by solving the optimization problem in the min-max MPC framework. Note that the constraints in presence of both additive disturbances and parametric uncertainties can be well handled as min-max MPC considers the worst case of all possible uncertainty realizations. The closed-loop system has been proved to be ISpS in the attraction region under some standard conditions. Extensions have been made to linear systems; the feasibility and stability conditions reduce to an LMI. Numerical simulations and comparison studies have been conducted to verify our theoretical findings.

4.2 Future Work

There are many interesting issues that are worth exploring in the future. Here, two main research branches are listed.

- The thesis is concerned with the centralized control of CPSs. When the dimension of the system state becomes very large, which is always the case in CPSs, the controller should collect all the state measurements, generate and distribute control signals to multiple modules at triggering time instants. This sort of paradigm might be undesirable as it requires one-to-all communication and the breakdown of one communication channel completely disables the controller. It would be much more beneficial if the overall system can be firstly decoupled into multiple subsystems and then a number of controllers that work cooperatively to stabilize the overall system based on peer-to-peer communication are assigned to the subsystems. Generally speaking, this distributed framework is not only more flexible and reliable, but also lowers down the requirement on computation for each controller, which can be seen as a byproduct.

Although this is a quite promising solution to the control of CPSs, how to achieve a co-design of distributed MPC and scheduling is very challenging. This is mainly because that the aperiodic scheduling of subsystems naturally introduces asynchrony which makes the cooperation pattern between subsystems unreliable and thus threats the closed-loop stability.

- Another research direction would be how to alleviate the computation burden in self-triggered min-max MPC designed in Chapter 3. The min-max optimization problem with general feedback policies is intractable, the introduction of parameter H and the modification of the standard optimization problem make the computation even more computationally expensive. There are two possible research avenues to tackle this. The first one is to fix the state feedback policy, e.g., disturbance feedback as done in Chapter 3.6 for linear systems with additive disturbances or state feedback, used for the prediction in min-max MPC. The second one is to devise efficient optimization algorithms for constrained optimization problems.

Appendix A

Publications

- **Refereed Journal Papers**

    1. **C. Liu**, H. Li, Y. Shi, and D. Xu. Distributed event-triggered gradient method for constrained convex minimization. *IEEE Transactions on Automatic Control*, 2019, DOI: 10.1109/TAC.2019.2916985
    2. **C. Liu**, H. Li, Y. Shi, and D. Xu. Co-design of event trigger and feedback policy in robust model predictive control. *IEEE Transactions on Automatic Control*, 2019, DOI: 10.1109/TAC.2019.2914416
    3. **C. Liu**, H. Li, J. Gao, and D. Xu. Robust self-triggered min-max model predictive control for discrete-time nonlinear systems. *Automatica*, 89: 333-339, 2018

- **Journal Papers Under Review**

    1. **C. Liu**, H. Li, and Y. Shi. Distributed subgradient method for convex constrained optimization: Non-ergodic convergence guarantees. Submitted to *Automatica*, under the second-round review, 2019.

2. **C. Liu**, H. Li, Y. Shi, and D. Xu. Distributed event-triggered model predictive control of coupled nonlinear systems. Submitted to *SIAM Journal on Control and Optimization*, under the second-round review, 2019.

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