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Efficient total dominating sets in Cayley graphs

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Abstract

A fundamental result of J. Lee relates efficient dominating sets in Cayley graphs to covers of a complete graph. We show that the same methods can be used to relate efficient total dominating sets in Cayley graphs to covers of a reflexive complete graph. A further link is made to efficient dominating sets and efficient total dominating sets in Cayley graphs.

Dedicated to Teresa Haynes on the occasion of her sixtieth birthday.

1 Introduction

Total dominating sets of graphs were introduced by Cockayne, Dawes and Hedetniemi in 1980 [6]. The recent book by Henning and Yeo provides an up to date, comprehensive survey of results that have appeared since then [10] (also see [9]). Efficient total dominating sets were first studied by Gavlas, Schultz and Slater [7] (also see [8], page 115). The problem of deciding if a graph has an efficient total dominating set was shown to be NP-complete in general by Bakker and van Leeuwen [1] (also see [7]), and even when restricted to planar bipartite graphs of maximum degree three by Schaudt [19]. The latter paper also describes polynomial time algorithms for finding efficient total dominating sets in classes of well-structured graphs like

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strongly chordal graphs and claw-free graphs. The concept of efficient total domination in digraphs was introduced by Schaudt [20], who determined the complexity of the existence problem for a variety of classes of directed graphs.

The focus of this paper is efficient total domination in Cayley graphs. Recent results on total domination in Cayley graphs include those of Maheshwari and Maheswari [15], who consider certain circulant graphs, and Tamizh Chelvam and Kalaimurugan [3], who consider Cayley graphs on the dihedral group. Efficient total domination in Cayley graphs arises in the papers of Gavlas, Schultz and Slater [7], who determine the hypercubes with such a set, and Tamizh Chelvam and Mutharasu [4]. The latter paper shows that Lee's proof [14] that a Cayley graph on an Abelian group has an efficient dominating set if and only if it is a covering of a complete graph can be slightly modified to show that a bipartite Cayley graph on an Abelian group has an efficient total dominating set if and only if it is a covering of a complete bipartite graph. We show that if “complete graph” is replaced by “reflexive complete graph”, then Lee's result holds for efficient total domination, using similar proofs that have been adapted for total domination. After discussing efficient total domination and Cayley graphs in the next two sections, the main results of the paper are presented in Section 4. The final section of the paper makes some connections between efficient total dominating sets in certain Cayley graphs and efficient dominating sets in related Cayley graphs.

2 Efficient total domination

A total dominating set of a graph $G$ is a subset $D \subseteq V(G)$ such that for every vertex $x \in V$ there exists a vertex $y \in D$ such that $xy \in E$. The total domination number of a graph $G$ without isolated vertices, denoted $\gamma_t(G)$, is the smallest size of a total dominating set of $G$. A total dominating set is called efficient when every $x \in V$ is adjacent to a unique vertex $y \in D$. A graph $G$ with a total dominating set may, or may not, have an efficient total dominating set, but when such a set exists it has size $\gamma_t(G)$ [7]. In particular, all efficient total dominating sets of a graph $G$ have the same size.

A graph is reflexive if it has a loop at each vertex. We use $K_n^e$ to denote the reflexive complete graph on $n$ vertices, that is, the graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_j : 1 \leq i \leq n, 1 \leq j \leq n\}$.

A covering of a graph $H$ by a graph $G$ is a function $p : V(G) \rightarrow V(H)$
such that, for every vertex \( x \in V(G) \), the restriction of \( p \) to \( N_G(x) \) is a 1-1 correspondence between \( N_G(x) \) and \( N_H(p(x)) \). If \( p \) is \( k \)-to-1 then it is called a \( k \)-fold covering of \( H \) by \( G \).

**Lemma 2.1** (cf. [14], Lemma 1) Let \( S_1, S_2, \ldots, S_n \) be disjoint efficient total dominating sets of a graph \( G \), and let \( k = |S_1| \). Then the subgraph of \( G \) induced by \( S_1 \cup S_2 \cup \cdots \cup S_n \) is an \( k \)-fold covering of \( K^R_n \).

**Proof.**

Let \( H \) denote the subgraph of \( G \) induced by \( S_1 \cup S_2 \cup \cdots \cup S_n \). Define \( p : V(H) \rightarrow V(K^R_n) \) by \( p(s) = v_i \) for each \( s \in S_i \), \( i = 1, 2, \ldots, n \). We will show that \( p \) is a covering.

Let \( x \in V(H) \), and suppose there exist \( a, b \in N_H(x) \) such that \( p(a) = p(b) = v_j \) for some \( 1 \leq j \leq n \). Then by the definition of \( p \), it follows that \( a, b \in S_i \). Since \( S_i \) is an efficient total dominating set, it must be that \( a = b \). Thus, \( p \) restricted to \( N_H(x) \) is injective.

Note that since every \( x \in V(H) \) is adjacent to exactly one vertex in each \( S_i \), \( |N_H(x)| = n \) for every vertex \( x \) in \( V(H) \). Clearly, \( |N_{K^R_n}(p(x))| = n \) also. Since the restriction of \( p \) to \( N_H(x) \) is an injective mapping from \( N_H(x) \) to \( N_{K^R_n}(p(x)) \) and the two sets have the same finite cardinality, it follows that \( p \) is also surjective.

Thus, \( p \) is a covering. Moreover, \( p \) is \( k \)-to-1 because all \( k \) elements of each \( S_i \) map to \( v_i \) for \( i = 1, 2, \ldots, n \), and nothing else maps to \( v_i \).

\[ \square \]

**Lemma 2.2** (cf. [14], Lemma 2) Let \( p : G \rightarrow H \) be a covering. If \( S \) is an efficient total dominating set of \( H \), then \( p^{-1}(S) \) is an efficient total dominating set of \( G \).

**Proof.**

Suppose \( S \) is an efficient total dominating set of \( H \), and \( p : V(G) \rightarrow V(H) \) is a covering.

First, we show that \( p^{-1}(S) \) is a total dominating set of \( G \). Let \( x \in V(G) \), and suppose \( x \) is not adjacent to any \( y \in p^{-1}(S) \). Then \( p(x) \) is not adjacent to any \( p(y) \) in \( S \), contradicting \( S \) being an efficient total dominating set of \( H \). Thus, \( x \) is adjacent to at least one element of \( p^{-1}(S) \), and \( p^{-1}(S) \) is a total dominating set.
Now, we show that $x$ can be adjacent to at most one element of $p^{-1}(S)$. Suppose there exist $y, z \in p^{-1}(S)$ such that $x$ is adjacent to both $y$ and $z$. Then $p(x)$ is adjacent to both $p(y)$ and $p(z)$, and both $p(y)$ and $p(z)$ belong to $S$, an efficient total dominating set. Thus, $p(y) = p(z)$. Since $p$ maps $N_G(x)$ bijectively to $N_H(x)$, it follows that $y = z$.

Thus, $p^{-1}(S)$ is an efficient total dominating set of $G$. 

\[\square\]

**Theorem 2.3** (cf. [14], Theorem 1) A graph $G$ is a covering of $K_n^R$ if and only if $V(G)$ admits a partition $\{S_1, S_2, \ldots, S_n\}$ such that each set $S_i$ is an efficient total dominating set of $G$.

**Proof.**

The backwards implication follows from Lemma 2.1. We prove the forwards implication.

Suppose $p : V(G) \to V(K_n^R)$ is a covering. Since $\{v_i\}$ is an efficient total dominating set of $K_n^R$ for any $1 \leq i \leq n$, by Lemma 2.2, $p^{-1}(\{v_i\})$ is an efficient total dominating set of $G$ for each $i = 1, 2, \ldots, n$. We will show that this collection forms a partition of $V(G)$.

Since every vertex of $G$ maps to some $v_i$, clearly \( \bigcup_{i=1}^{n} p^{-1}(\{v_i\}) = V(G) \).

We now show that all sets are pairwise disjoint. Let $v_i, v_j \in V(K_n^R)$ and suppose $p^{-1}(\{v_i\}) \cap p^{-1}(\{v_j\}) \neq \emptyset$. Let $x \in p^{-1}(\{v_i\}) \cap p^{-1}(\{v_j\})$. Then $p(x) = v_i$ and $p(x) = v_j$, so it must be that $v_i = v_j$. Thus, $\{p^{-1}(\{v_1\}), p^{-1}(\{v_2\}), \ldots, p^{-1}(\{v_n\})\}$ is a partition of $V(G)$ with each set in the partition forming an efficient total dominating set of $G$.

\[\square\]

3 Cayley graphs

Let $\mathcal{G}$ be a finite group, and $S \subseteq \mathcal{G}$ be such that $x \in S$ if and only if $x^{-1} \in S$. The Cayley graph on the group $\mathcal{G}$ with distance set $S$ is the graph $\text{Cay}(\mathcal{G}, S)$ with vertex set $\mathcal{G}$ and $u$ adjacent to $v$ if and only if $w = uv$ for some $s \in S$. If the identity element $e \in S$, then $\text{Cay}(\mathcal{G}, S)$ has a loop at each vertex; otherwise it is a simple graph.

It follows from the definition that the Cayley graph $\text{Cay}(\mathcal{G}, S)$ is connected if and only if $S$ generates $\mathcal{G}$, and that left multiplication by $x \in \mathcal{G}$
is an automorphism of \( \text{Cay}(G, S) \). Right multiplication by \( x \) is not always an automorphism. The following proposition is included for completeness. Its easy proof is omitted.

**Proposition 3.1** Let \( x \) be an element of the group \( G \). Right multiplication by \( x \) is an automorphism of \( \text{Cay}(G, S) \) if and only if \( Sx = xS \).

## 4 Efficient total dominating sets in Cayley graphs

**Lemma 4.1** (cf. [14], Lemma 3) Let \( D \) be an efficient total dominating set of \( \text{Cay}(G, S) \). Then

1. \( |D \cap S| = 1 \).
2. For each \( x \in S \), the set \( xD \) is an efficient total dominating set of \( \text{Cay}(G, S) \).
3. \( \{Dx : x \in S\} \) is a partition of \( G \).

**Proof.**

**(Proof of 1.)**

Since \( D \) is an efficient total dominating set, every \( x \in G \) is adjacent to exactly one element of \( D \). In particular, there exists \( d \in D \) so that \( e \in D \) adjacent to \( d \). That is, \( d = es \) for some \( s \in S \). But then \( d = s \), so \( d \in S \). Therefore \( |D \cap S| \geq 1 \).

Let \( d_1, d_2 \in D \cap S \). Since \( d_1 = ed_1 \) and \( d_2 = ed_2 \) and \( d_1, d_2 \in S \), \( e \) is adjacent to both \( d_1 \) and \( d_2 \). Since \( D \) is an efficient total dominating set, this implies \( d_1 = d_2 \), and \( |D \cap S| \leq 1 \).

Thus, \( |D \cap S| = 1 \).

**(Proof of 2.)**

Fix \( x \in S \).

Let \( y \in G \). Then \( x^{-1}y \in G \), so \( x^{-1}y \) is dominated by an element of \( D \). That is, there exists \( d \in D \) so that \( d = x^{-1}ys \) for some \( s \in S \). Then \( xd = yxs \), so \( y \) is dominated by \( xd \in XD \). Therefore, \( XD \) is a total dominating set.

Suppose \( y \) is adjacent to \( xd_1 \) and \( xd_2 \) in \( XD \). Then \( xd_1 = y_{s_1}, xd_2 = y_{s_2} \) for some \( s_1, s_2 \in S \). Thus, \( d_1 = x^{-1}ys_1 \) and \( d_2 = x^{-1}ys_2 \), so \( x^{-1}y \)
is adjacent to both \( d_1, d_2 \in D \). Since \( D \) is efficient, this implies \( d_1 = d_2 \). Therefore, \( xd_1 = xd_2 \), and \( xD \) is an efficient total dominating set of \( \text{Cay}(G, S) \).

(Proof of 3.)
Let \( s_1, s_2 \in S \), and suppose \( Ds_1 \cap Ds_2 \neq \emptyset \). Let \( y \in Ds_1 \cap Ds_2 \). Then \( y = d_1s_1 = d_2s_2 \) for some \( d_1, d_2 \in D \), so \( y \) is adjacent to both \( d_1 \) and \( d_2 \). \( D \) is an efficient total dominating set, so \( d_1 = d_2 \). Then \( d_1s_1 = d_1s_2 \), so \( s_1 = s_2 \). Thus, the elements of \( \{ Dx : x \in S \} \) are pairwise disjoint.

Let \( g \in G \). \( g \) is dominated by some \( d \in D \), so \( d = gs \) for some \( s \in S \). Then \( g = ds^{-1} \), and \( s^{-1} \in S \), so \( g \in Ds^{-1} \). This implies \( \bigcup_{s \in S} Ds = G \).

Thus, \( \{ Dx : x \in S \} \) is a partition of \( G \).

Corollary 4.2 (cf. [14], Corollary 1) A set \( D \) satisfying \( xD = Dx \) for each \( x \in S \) is an efficient total dominating set of \( \text{Cay}(G, S) \) if and only if there is a covering \( p \) of the reflexive complete graph on \( |S| \) vertices by \( \text{Cay}(G, S) \) such that, for every \( v \in V(K_{|S|}^R) \), there exists \( s \in S \) for which \( p^{-1}(\{v\}) = Ds \).

Proof.

(\( \Rightarrow \)) Let \( D \) be an efficient total dominating set of \( \text{Cay}(G, S) \), and let \( x \in S \). Since \( xD = Dx \), by Lemma 4.1, \( \{ sD : s \in S \} \) is a partition of \( G \) into \( |S| \) efficient total dominating sets.

Index the elements of \( S \) so that \( S = \{ s_1, s_2, \ldots, s_{|S|} \} \). Define \( p : G \to V(K_{|S|}^R) \) by \( p(x) = v_i \) where \( x \in s_iD \). The function \( p \) is well-defined because each vertex belongs to exactly one \( s_iD \). As in the proof of Lemma 2.1, \( p \) is a covering.

By construction, \( p^{-1}(\{v_i\}) = s_iD \) for every \( v_i \in V(K_{|S|}^R) \), as desired.

(\( \Leftarrow \)) Let \( v \in V(K_{|S|}^R) \). Then \( p^{-1}(\{v\}) = Ds \) for some \( s \in S \). The set \( \{v\} \) is an efficient total dominating set of \( K_{|S|}^R \), so by Lemma 2.2, \( p^{-1}(\{v\}) = Ds = sD \) is an efficient total dominating set of \( \text{Cay}(G, S) \). Then \( (s^{-1}s)D = D \) is an efficient total dominating set by Lemma 4.1.

Corollary 4.3 (cf. [14], Corollary 1) Let \( G \) be an Abelian group. \( D \) is an efficient total dominating set of \( \text{Cay}(G, S) \) if and only if there is a covering.
p of $K_{|S|}^R$ by $\text{Cay}(G, S)$ such that for every $v \in V(K_{|S|}^R)$, there exists an $s \in S$ for which $p^{-1}(\{v\}) = D + s$.

If $G$ is a group and $X, Y \subseteq G$, then the set denoted by $XY$ is $\{xy : x \in X, y \in Y\}$.

**Theorem 4.4** (cf. [4], Theorem 8, and also [14], Theorem 2) Let $D$ be a normal subset of a group $G$. The following are equivalent:

1. $D$ is an efficient total dominating set of $\text{Cay}(G, S)$.

2. There exists a covering $p : G \rightarrow V(K_{|S|}^R)$ such that $p^{-1}(\{v_i\}) = D$ for some $1 \leq i \leq |S|$.

3. $|D| = \frac{|G|}{|S|}$ and $D \cap (D(SS \setminus \{e\})) = \emptyset$.

**Proof.**

The equivalence of statements 1 and 3 is known [4]. The assertion $2 \Rightarrow 1$ follows from Lemma 2.2. We prove the remaining implication.

Suppose $D$ is an efficient total dominating set of $\text{Cay}(G, S)$. By Lemma 4.1, $\{xD : x \in S\}$ is a partition of $G$ into efficient total dominating sets. Fix $s' \in S$. Since left multiplication by $s$ is an automorphism of $\text{Cay}(G, S)$, $\{s'xD : x \in S\}$ is also a partition of $G$ into efficient total dominating sets, and since $S$ is closed with respect to inverses, $D \in \{s'xD : x \in S\}$.

Index $S$ so that $S = \{s_1, s_2, \ldots, s_{|S|}\}$. Define $p : G \rightarrow V(K_{|S|}^R)$ by $p(y) = v_i$, where $y \in s's_iD$. As in the proof of Lemma 2.1, $p$ is a covering. Since $D \in \{s'xD : x \in S\}$, clearly $p$ has the desired property.

**Theorem 4.4** gives conditions under which there is a covering $p : G \rightarrow V(K_{|S|}^R)$ such that one of the vertex fibres $p^{-1}(\{v\}) = D$. This also happens for the covering in Corollary 4.2 if $Ds = D$ for some $s \in S$. In the latter case, the efficient total dominating set $D$ has a special structure.

Note that we can always assume that an efficient total dominating set $D$ of $\text{Cay}(G, S)$ contains the identity element $e$. If $x \in D$, then $x^{-1}D$ is an efficient total dominating set by 4.1, and clearly $e \in x^{-1}D$. Since left multiplication is an automorphism, $x^{-1}D$ is an efficient total dominating set.
Proposition 4.5 Let $D$ be an efficient total dominating set of $\text{Cay}(G, S)$ with $e \in D$, $xD = Dx$ for every $x \in S$, and $p$ be the covering in Corollary 4.2. If one of the vertex fibres $p^{-1}(\{v\}) = D$, then there exists $s \in D \cap S$ such that $Ds = D$. Further, $s^2 = e$ and $D$ is a union of left cosets of $\langle s \rangle$.

Proof.
Since $D$ is an efficient total dominating set, $|D \cap S| = 1$. The vertex fibres with respect to the covering $p$ in Corollary 4.2 are $\{Dx : x \in S\}$. Suppose one of these is $D$, say $Ds = D$. Since $e \in D$ we have $s \in Ds = D$, and $s^{-1} \in D = Ds$. Since, also, $s^{-1} \in S$, it must be that $s = s^{-1}$. Thus, $s^2 = e$. Since $Ds = D$, we have $d \in D$ if and only if $ds \in D$. Thus, $D$ is a union of left cosets of $\langle s \rangle$.

$\square$

5 A link to efficient domination

An efficient dominating set of a graph $G$ is a subset $D \subseteq V(G)$ such that $|N[x] \cap D| = 1$ for every $x \in V(G)$. The study of efficient dominating sets originates with Biggs [2] and Kratochvíl [11] (also see [8], Chapter 4). Besides Lee’s paper [14], there has been other recent work on efficient domination in Cayley graphs [3, 5, 12, 13, 15, 16, 17, 18].

Let $\mathcal{H}$ be a normal subgroup of the group $\mathcal{G}$. The quotient graph of the Cayley graph $\text{Cay}(\mathcal{G}, S)$ is defined to be the Cayley graph, $\text{Cay}(\mathcal{G}/\mathcal{H}, S/\mathcal{H})$, on the quotient group $\mathcal{G}/\mathcal{H}$ with distance set $\mathcal{S}/\mathcal{H} = \{hs : s \in S\}$. Observe that if $\mathcal{H} \cap S \neq \emptyset$, then the quotient graph is reflexive.

The following proposition is included partly for the sake of interest, as it strongly resembles the Fundamental Homomorphism Theorem, and partly because it is used in the proof of Theorem 5.2. Recall that a homomorphism from a graph $G$ to a graph $H$ is a function $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$.

Proposition 5.1 Let $\mathcal{H}$ be a normal subgroup of the group $\mathcal{G}$. The function $f : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ defined by $f(x) = \mathcal{H}x$ is a graph homomorphism from $\text{Cay}(\mathcal{G}, S)$ to $\text{Cay}(\mathcal{G}/\mathcal{H}, S/\mathcal{H})$.

Observe that the vertex fibres of the homomorphism in Proposition 5.1 are the cosets of the normal subgroup $\mathcal{H}$.
For example, consider the graph \( \text{Cay}(\mathbb{Z}_{12}, \{1, 6, 11\}) \). The subgroup \( \langle 6 \rangle = \{0, 6\} \) is normal and \( f(x) = \langle 6 \rangle x \) is a homomorphism to the reflexive graph \( \text{Cay}(\mathbb{Z}_{12}/\langle 6 \rangle, \{1, 6, 11\}/\langle 6 \rangle) \cong \text{Cay}(\mathbb{Z}_6, \{0, 1, 5\}) \).

**Theorem 5.2** Let \( D \) be an efficient total dominating set of \( \text{Cay}(G, S) \) with \( e \in D \), and let \( s \) be the unique element belonging to \( D \cap S \). If \( D_\sigma = D \) and \( \langle s \rangle \) is a normal subgroup of \( G \), then the quotient graph \( \text{Cay}(G/\langle s \rangle, S/\langle s \rangle) \) has an efficient total dominating set.

**Proof.**

By Proposition 4.5, we have \( s^2 = e \) and the efficient total dominating set \( D \) is a union of cosets of \( \langle s \rangle \). Thus, \( d \in D \) if and only if \( ds \in D \).

Let \( D/\langle s \rangle = \{\langle s \rangle d : d \in D\} \). We claim that \( D/\langle s \rangle \) is an efficient total dominating set of the reflexive graph \( \text{Cay}(G/\langle s \rangle, S/\langle s \rangle) \).

Let \( f \) be the homomorphism in Proposition 5.1, so that vertex fibres \( f^{-1}(\{\langle s \rangle x\}) \) are the cosets of \( \langle s \rangle \). In particular, \( f(D) = D/\langle s \rangle \). Hence the subgraph of \( \text{Cay}(G/\langle s \rangle, S/\langle s \rangle) \) induced by \( f(D) \) is a disjoint union of loops. Since homomorphisms preserve edges, \( f(D) \) is a total dominating set. It remains to show that each vertex is adjacent to exactly one element of this set.

Suppose \( \langle s \rangle x \in G/\langle s \rangle \) is adjacent to the elements \( \langle s \rangle d_1 \) and \( \langle s \rangle d_2 \) of \( D/\langle s \rangle \). Then the vertex \( x \) is adjacent in \( \text{Cay}(G, S) \) to an element of \( \langle s \rangle d_1 = \{sd_1 \} \subseteq D \), and to an element of \( \{d_2, sd_2 \} \subseteq D \). Since \( D \) is an efficient total dominating set, and the cosets of \( \langle s \rangle \) partition \( G \), these two sets must be equal.

Therefore, \( D/\langle s \rangle \) is an efficient total dominating set of \( \text{Cay}(G/\langle s \rangle, S/\langle s \rangle) \). \( \square \)

Again, consider the graph \( \text{Cay}(\mathbb{Z}_{12}, \{1, 6, 11\}) \). It has the efficient total dominating set \( D = \{0, 3, 6, 9\} \). We have \( 6 \in D \) and \( D = D + 6 \). Since \( \langle 6 \rangle \) is a normal subgroup, it follows that the quotient graph \( \text{Cay}(\mathbb{Z}_6, \{0, 1, 5\}) \) has an efficient total dominating set, namely \( \{0, 3\} \). If the loops are deleted, that is, \( 0 \) is removed from the distance set, then \( \{0, 3\} \) is an efficient dominating set of the resulting simple graph.

Let \( G \) be a graph. Recall that the **underlying simple graph** of \( G \) is the graph \( G' \) with \( V(G') = V(G) \) and an edge joining \( x \) and \( y \) if and only if \( x \) and \( y \) are adjacent vertices of \( G \) with \( x \neq y \). If \( H \) is a normal subgroup of \( G \), then the underlying simple graph of \( \text{Cay}(G/H, S/H) \) is the Cayley graph on
$G/H$ with distance set $S/H = \{hs : s \in S - H\}$, that is, $\text{Cay}(G/H, S/H - H)$.

**Corollary 5.3** Let $D$ be an efficient total dominating set of $\text{Cay}(G, S)$ with $e \in D$, and let $s$ be the unique element belonging to $D \cap S$. If $Ds = D$ and $(s)$ is a normal subgroup of $G$, then

1. $\text{Cay}(G/(s), S/(s))$ has an efficient total dominating set, and
2. $\text{Cay}(G/(s), S/(s) - (s))$ has an efficient dominating set.

**Corollary 5.4** Let $G$ be an Abelian group. If $\text{Cay}(G, S)$ has an efficient total dominating set $D$ with $0 \in D$ such that $D + s = D$ for the unique element $s \in D \cap S$, then

1. $\text{Cay}(G/(s), S/(s))$ has an efficient total dominating set, and
2. $\text{Cay}(G/(s), S/(s) - (s))$ has an efficient dominating set.

We conclude the paper with a sort of converse to the above results.

Let $G$ be a group with $s \in G$ such that $s^2 = e$ and $(s)$ is a normal subgroup of $G$. Let $H$ be a subgroup of $G$ isomorphic to $G/(s)$. In this case, $G = H \cup Hs$.

The set $T$ can be partitioned into $T_1$ and $T_2$, where $T_1$ is the set of all $t \in T$ such that the inverse of $t$ taken in $G$ is also in $T$, and $T_2 = T - T_1$. Because $T$ is closed with respect to inverses as a subset of $H$, $T_2$ can be partitioned into 2-element subsets $\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{2k-1}, x_{2k}\}$, where $x_{2i-1}$ is inverses in $H$ for $1 \leq i \leq k$. Since the inverse of $x_{2i-1}$ in $G$ is in the same coset of $(s)$ as $x_{2i}$, the element $x_{2i-1}x_{2i}s$ is the identity element of $G$.

**Theorem 5.5** Let $G$ be a group with $s \in G$ such that $s^2 = e$ and $(s)$ is a normal subgroup of $G$. Let $H$ be a subgroup of $G$ isomorphic to $G/(s)$. If $\text{Cay}(H, T)$ has an efficient dominating set, then $\text{Cay}(G, S)$ has an efficient total dominating set, for any distance set $S$ such that $S = \{s\} \cup T_1 \cup \{x_{2i-1}, x_{2i}: 1 \leq i \leq k\}$, where $T_1$ and the $x_{2i-1}, x_{2i}$'s are as defined above.

**Proof.**

Note that $|S| = 1 + |T|$. Hence, $\text{Cay}(G, S)$ is regular of degree one greater
than $\text{Cay}(\mathcal{H}, T)$. Let $D$ be an efficient dominating set of $\text{Cay}(\mathcal{H}, T)$. We claim that $D \cup D_s$ is an efficient total dominating set of $\text{Cay}(\mathcal{G}, S)$. Since $\text{Cay}(\mathcal{G}, S)$ has twice as many vertices as $\text{Cay}(\mathcal{H}, T)$, it suffices to show that every vertex is adjacent to an element of $D \cup D_s$.

Let $x \in \mathcal{H}$ be a vertex of $\text{Cay}(\mathcal{G}, S)$. If $x \in D$, then $x$ is adjacent to a vertex of $D \cup D_s$. Suppose $x \notin D$ is adjacent in $\text{Cay}(\mathcal{H}, T)$ to $d \in D$. Then there exists $t \in T$ such that $xt = d$, where the operation is in $\mathcal{H}$. If $t \in T_1$, then $x$ is adjacent to $d$ in $\text{Cay}(\mathcal{G}, S)$. If $t \in T_2$, then either $t, st^{-1} \in S$ or $t^{-1}, st \in S$, with inverses taken in $\mathcal{H}$. In the former case, $x$ is adjacent to $d$ in $\text{Cay}(\mathcal{G}, S)$. In the latter case, $x$ is adjacent to $ds \in D_s$.

Now let $x \in \mathcal{H} s$ be a vertex of $\text{Cay}(\mathcal{G}, S)$. If $x \in D_s$, then $x$ is adjacent to a vertex of $D \cup D_s$. Suppose $x \notin D_s$. Then $xs$ is adjacent in $\text{Cay}(\mathcal{H}, T)$ to $d \in D$. As above, $xs$ is adjacent to $d$ or $ds$. Consequently $xs^2 = x$ is adjacent to $ds$ or $d$.

It follows that $D \cup D_s$ is an efficient total dominating set of $\text{Cay}(\mathcal{G}, S)$.

\[ \square \]

Proposition 5.6 The graph $\text{Cay}(\mathcal{G}, S)$ has an efficient dominating set $D$ with $e \in D$ if and only if $\text{Cay}(\mathcal{G} \times \mathbb{Z}_2, (S \times \{0\}) \cup \{(e, 1)\})$ has an efficient total dominating set $D'$, with $(e, 1) \in D'$, satisfying $(x, 0) \in D' \iff (x, 1) \in D'$.

\textbf{Proof.}

$(\Rightarrow)$ Suppose $\text{Cay}(\mathcal{G}, S)$ has an efficient dominating set, $D$, with $e \in D$. Then $D' = (D \times \{0\}) \cup (D \times \{1\})$ is the required efficient total dominating set.

$(\Leftarrow)$ Suppose $\text{Cay}(\mathcal{G} \times \mathbb{Z}_2, (S \times \{0\}) \cup \{(e, 1)\})$ has an efficient total dominating set $D'$. Note that $(e, 1)$ is a normal subgroup of $\mathcal{G} \times \mathbb{Z}_2$, $D'(e, 1) = D'$, $(\mathcal{G} \times \mathbb{Z}_2)/\langle (e, 1) \rangle \cong \mathcal{G}$ and $[(S \times \{0\}) \cup \{(e, 1)\}]/\langle (e, 1) \rangle \cong S$. Thus, by 5.3, $\text{Cay}(\mathcal{G}, S)$ has an efficient dominating set $D$. If $x \in D$, then the set $x^{-1}D$ is an efficient dominating set of $\text{Cay}(\mathcal{G}, S)$ containing $e$.

\[ \square \]

The forwards implication in the previous proof can also be seen to follow from Theorem 5.5, but a direct argument is much simpler.
References


